
Zbl 294.33006
Erdős, Paul; Freud, G.
On orthogonal polynomials with regularly distributed zeros. (In English)
Proc. Lond. Math. Soc., III. Ser. 29, 521-537 (1974). [0024-6115]

Let $p_n(d\alpha;x) = \gamma_n(d\alpha)x^n + \dots (n = 0,1,\dots)$ be the sequence of orthonormal polynomials with respect to the nonnegative measure $d\alpha$, $x_{kn}(d\alpha)$ $(k = 1,2,\dots,n)$ be the zeros of $p_n(d\alpha;x)$ in decreasing order. Let $N_n(d\alpha;t)$ be the number of the $x_{kn}(d\alpha)$ satisfying $x_{kn}(d\alpha) - x_{nn}(d\alpha) \geq t[x_{1n}(d\alpha) - x_{nn}(d\alpha)]$. We say that $d\alpha$ is arc-sine iff

$$\lim_{n\to\infty} n^{-1}N_n(d\alpha;t) = \frac{1}{2} - \frac{1}{\pi}\arcsin(2t-1).$$

[*J. L. Ullman*, Proc. London math. soc., III. Ser. 24, 119-148 (1972; Zbl 232.33007)]. By well-known properties of the zeros of the classical orthogonal polynomials, $(1-x)^\beta(1+x)^\gamma dx(-1 < x < 1)$ is arc- sine for $\beta,\gamma > -1$ but neither $e^{-x^2}dx(-\infty < x < \infty)$ is arc-sine nor is $x^\rho e^{-x}dx(0 < x < \infty)$ arc-sine for any $\rho > -1$.

A class of absolutely continuous arc-sine measures with non-compact support was discovered by *P. Erdős* [Proc. Conf. construct. Theory Functions (Approximation Theory) 1969, 145-150 (1972; Zbl 234.33014)]. The authors prove that we have for arbitrary $d\alpha$

$$\lim_{n\to\infty} \sqrt[n-1]{\gamma_{n-1}(d\alpha)}[x_{1n}(d\alpha) - x_{nn}(d\alpha)] \geq 4$$

and that the relation

$$(*) \qquad \lim_{n\to\infty} \sqrt[n-1]{\gamma_{n-1}(d\alpha)}[x_{1n}(d\alpha) - x_{nn}(d\alpha)] = 4$$

implies that $d\alpha$ is arc-sine. (*) is not only sufficient but also necessary if $d\alpha = wdx$ is absolutely continuous and either it has compact support or $w(x) = \exp\{-2Q(|x|)\}$ $(-\infty < x < \infty)$ where $Q(x)$ $(x \geq 0)$ is a positive increasing differentiable function for which $x^\rho Q'(x)$ is increasing for some $\rho < 1$. An example is constructed of an absolutely continuous arc-sine measure $d\alpha$ for which (*) does not hold.

Following *J.L.Ullman*, loc. cit. we say that $A \subset [-1,1]$ is a determining set if every absolutely continuous $d\alpha = w(x)dx$ which satisfies $A \subseteq \{x : w(x) > 0\} \subseteq [-1,1]$ is arc-sine on $[-1,1]$. We give a proof of the conjecture of *P.Erdős* that a measurable set A is a determining set if and only if it has minimal logarithmic capacity $\frac{1}{2}$.

Classification:
33A65 33A65
42C05 General theory of orthogonal functions and polynomials