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Burr, Stefan A.; Erdős, Paul; Lovász, László

On graphs of Ramsey type. (In English)

Ars combinat. 1, 167-190 (1976).

Let F be a graph and $\mathfrak{G}$ and $\mathfrak{H}$ sets of graphs. (All graphs will be finite and simple.) A colouring of F will be a sequence of subgraphs of H — each having the same vertex-set as F — such that their edge-sets form a disjoint partition of the edge-set of F. $F \to (\mathfrak{G}, \mathfrak{H})$ means that in any 2-colouring of F either the first part contains a subgraph isomorphic to a member of $\mathfrak{G}$, or the second part contains a subgraph isomorphic to a member of $\mathfrak{H}$. When $\mathfrak{G} = \mathfrak{H}$, the notation is simplified to $F \to \mathfrak{G}$. When $\mathfrak{G}$ or $\mathfrak{H}$ contain but a single graph, the notation is adjusted so that the name of the unique member replaces the name of the set. For any graph G, $\kappa(G)$ denotes the connectivity of G; for any positive integer m, mG denotes the disjoint union of m copies of G. The set of homomorphic images of G is denoted by $\hom G$. The direct product $G \times H$ of two graphs G and H is the graph with vertex-set $V(G) \times V(H)$, otherwise known as the conjunction. The chromatic Ramsey number $r_c(\mathfrak{G}, \mathfrak{H})$ is the least integer m for which there exists a graph F such that $F \to (\mathfrak{G}, \mathfrak{H})$ and $\chi(F) = m$; the Ramsey number $r(\mathfrak{G}, \mathfrak{H})$ is the least integer n such that $K_r \to (\mathfrak{G}, \mathfrak{H})$. Theorem 1: For any classes $\mathfrak{G}$ and $\mathfrak{H}$, $r_c(\mathfrak{G}, \mathfrak{H}) = r(\hom \mathfrak{G}, \hom \mathfrak{H})$. Theorem 2: Let G and H be graphs of chromatic number r, and suppose that every vertex of H is contained in a complete $(r-1)$-graph. Then $G \times H$ has chromatic number r. Two conjectures are stated: Conjecture 1: $\min r_c(G) = (r-1)^2 + 1$, where the minimum is taken over all r-chromatic graphs. Conjecture 2: $\chi(G \times H) = \min(\chi(G), \chi(H))$. (A weakened version of Conjecture 2 follows from Theorem 2. The truth of Conjecture 2 would imply that of Conjecture 1.) Theorem 3: Conjecture 1 is valid for $r = 4$. Define $\delta(F)$ and $\Delta(F)$ to be the minimum and maximum degrees of vertices of F. F is (G,H)-irreducible if $F \to (G,H)$ but no proper subgraph of F has this property. Theorem 4: $2^{r/2} \leq \min \Delta(G) \leq r(K_r) - 1$, where the minimum is taken over all G for which $G \to K_r$. Theorem 5: $\min \delta(G) = (r-1)(s-1)$, where the minimum is taken over all (K_r, K_s)-irreducible graphs. Theorem 6: If $r, s \geq 3$, there are infinitely many non-isomorphic (K_r, K_s)-irreducible graphs. Theorem 7: If $r, s \geq 3$, then there exists (K_r, K_s)-irreducible graphs with arbitrarily large Δ. Theorem 8: If $r, s \geq 3$, then $\min K(G) = 2$ or 3 according as $r \neq s$ or $r = s$, where the minimum is taken over all (K_r, K_s)-irreducible graphs G. Theorem 9: A necessary and sufficient condition that $G \to K_{1,n}$ is that $\Delta(G) \geq 2n-1$, or, if n is even, that G has a component which is regular of degree $2n-2$ and which has an odd number of vertices. Theorem 10: $G \to 2K_2$ if and only if G contains three disjoint edges or a 5-cycle. Theorem 11: For any positive integers m and n, the number of (mK_2, nK_2)-irreducible graphs is finite.

W.G.Brown

Classification:

05C35 Extremal problems (graph theory)

05C15 Chromatic theory of graphs and maps