Several authors have investigated the question of approximating certain functions by reciprocals of polynomials under the uniform norm on the positive real axis. In this paper, the authors survey many of the known results in this direction. They also provide simplified proofs in some cases and obtain several new results as well. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with non-negative real $a_k(a_0 > 0)$. If π_n denotes the class of all ordinary polynomials of degree n then for $P \in \pi_n$, set

$$\psi(f, P) = \left| \frac{1}{f(x)} - \frac{1}{P(x)} \right|_{L^\infty[0,\infty]}, \lambda_{0,n}(f) = \inf_{P \in \pi_n} \psi(f, P).$$

The results reviewed and obtained in this paper can be broadly classified in problems of the following type: (i) Assuming various growth restrictions on the entire function f, obtain upper and lower bounds for $\lambda_{0,n}(f)$, $\lambda_{0,n}(f(x)/x)$, $\liminf_{n \to infty}(\lambda_{0,n}(f))^{1/n}$, $\limsup_{n \to \infty}(\lambda_{0,n}(f))^{1/n}$ etc, and conversely given bounds for these quantities, obtain restrictions on f. (ii) Given f continuous or entire satisfying certain growth restrictions, prove the existence of polynomials $P \in \pi_n$ for which $\psi(f, P)$, $\psi(f(x)/x, P)$ etc. are very small and conversely given sequence $\{P_n\}$, $P_n \in \pi_n$, such that these quantities are small, find the conditions which f must satisfy. (iii) Obtain sharper results corresponding to previous problems for certain special functions such as e^{-x}, $(x+1)^{-n}$ etc. (iv) Under stringent conditions on the entire function f, extend the above results to complex polynomials of complex variable z under the unifom norm on some subset of the complex plane. The results reviewed and generalized by the authors are mainly due to Erdős, Meinardus, Newman, Reddy, Shisha, Taylor, Varga and others. The results are too numerous to be stated in detail here (The paper contains statements of 50 theorems!). The paper also includes nine open problems. It is a very good, readable acccount of the results obtained in this field.

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Classification:
30E10 Approximation in the complex domain
30-02 Research monographs (functions of one complex variable)
41A20 Approximation by rational functions