PAUL ERDOS, THE OTHER WOMAN AND THE THEOREM OF PENTA-CHORDS

Preamble

God has the Big Book; the beautiful proofs of math theorems are listed here.
Paul Erdos
Lucifer has a Little Black Book; all beautiful theorems for which God has no proofs are listed here.

Andrew Vazsonyi

An Urgent Phone Call

I was sitting in my office at North American Aviation, in the early 60s, at the time of the Cuban Missile Crisis, when my secretary buzzed, and told me I had an urgent phone call. It was Paul Erdos. "Itt vagyok, — I am here — at UCLA, in Boelter Hall. Szervusz, Vazsonyi, how is Beatrice, your boss child? I must see you right away," he says. (If you don't know who the late Paul Erdos was, read the end of this paper.)

I hop in my car, drive to UCLA and find Erdos. He is playing Go with Dr. Po. Erdos introduces me: "This is Vazsonyi, he is an executive at North American Aviation, his Erdos number is 1, he is dead. —— Wait, Vazsonyi, I have no time for you." Dr. Po looks at me. "My Vazsonyi number is 2," he brags.

As I am sitting idly, waiting for Erdos, I overhear an animated discussion by two young mathematicians of the diagram at the blackboard. "What is this about?" I ask. "Quiet", says Erdos. "You are supposed to be dead." (I stopped proving mathematical theorems, so in Erdos vocabulary I was dead.)

The young mathematicians are more tolerant and explain. This is the theorem (I call it the Theorem of Penta-Chords, TPC) for which there is no known geometric proof.

The Theorem of Penta-Chords

Consider a circle and the arbitrary A1-A2 chord. (Exhibit 1.) M is the midpoint on the chord. Draw two other arbitrary chords, B1-B2, C1-C2 through M. Draw the two chords B1-C2 and B2-C1. Find the points D1, and D2, the intersections of chords A1-A2 and chords B1-C2 and C1-B2. The theorem says that M is the midpoint between D1 and D2.

[Aside. If you do not have the exhibit you can make your own. Draw a circle and a horizontal diameter. Mark point M on the diameter, somewhere on the right, off the center of the circle. Chord One of the circle, is through M, is perpendicular to the first diameter, and is marked on the top A1, on the bottom A2. Chord Two, is through M, slants backward at about 45 degrees, and is marked on the top B1, on the bottom B2. Chord Three, is through M, slants forward, is marked C1 on the bottom, and C2 on the top. End of Aside.]

"Doesn't look too hard", I say. "Not for an industrial tycoon", they sneer. Before I have a chance to get mad, Erdos says: "I must go to Cal Tech to see XYZ. You don't have to drive me around anymore". He raises his right thumb and points backwards. "What does this thumb-pointing mean?" I ask. Without turning his head he says: "She drives me around." I look in the corner of the room, and the woman sitting there, I will refer to as the Other Woman, OW. (This was before Erdos' mother started to travel with him.)

"I will not be available in the next morning, because I have to preach. But I will come to see you Sunday, you can cook me shishkebab," he says. "You do not have to drive my old bones around anymore," he repeats, and points with his thumb to the corner again.

Anyway, it was time for me to go home for dinner.

Time Sharing on the Sand Diego Freeway.

The drive on the San Diego freeway was horrible. I had nothing else to do but to think about TPC. When I got out of my car in Manhattan Beach I had the heuristics of a new, geometric proof. I called Erdos on the phone, all excited. I stuttered on the phone, tried to put the proof into words, without paper. (I have never used paper for math, not even for my dissertation.) Erdos mumbled and said the proof was no good. Finally he was convinced. "Supreme Fascist! (My God.) Straight from the Big Book. You must publish this. Dead men do prove theorems." I felt real good, I removed the theorem from Lucifer's Little Black Book. (Later Laura, my wife told me that Erdos said: "Strange, Vazsonyi has been dead for years, but never lost the touch".)

How did I find the proof? When I was still alive, I was a Grand Master of projective geometry. When I look at a problem in geometry I automatically use the heuristics: *Can this problem be generalized as a problem in projective geometry?* This was the question I posed and answered with respect to TPC in the traffic jam of the San Diego Freeway.

Often the more general problem is easier to solve, as George Polya says in his book *How to Solve It*. This sounds paradoxical, but it worked well for TPC. My main achievement in solving the problem was to discover the general problem. After this only odds-and-ends remained. Now, there is a need to pause for a moment and discuss projective geometry.

The Ten-Minute Primer on Projective Geometry.

Projective geometry deals with theorems that are invariant with respect to projections. (Aleksandrov et al, 1963, Coxeter, 1987, and Felix Klein, 1939, give excellent introductions to projective geometry.) Consider two planes, a source plane A, and a target plane B. Consider a center of projection C, not in A or B. What theorems and properties are invariant against projective transformations?

When a "conic" is projected, it stays as a conic. A circle does not stay a circle, an ellipse may turn into a hyperbola. In fact any conic can be transformed by projection into a circle. Cross ratios are invariant under projective transformations.

Blaise Pascal, in the first half of the 17th century, at the age of sixteen, discovered an important theorem in projective geometry. Consider the three intersections of the opposing sides of a hexagon that is inscribed into a conic. Pascal discovered that the three points are collinear, that is, lie on a straight line.

Five points uniquely determine a conic. The Pascal theorem allows the construction, with the aid of a straight edge, of any number of points on the conic (Mobius net).

"Duality," is an important principle in projective geometry. The heuristic is: replace points with straight lines, and straight lines with points. A famous example is the Brianchon theorem, the dual of the Pascal theorem, concerning the hexagon of tangents circumscribed about a conic. Consider the three diagonals of the hexagon. The three are concurrent, that, is the three lines intersect at the same points. The projective plane has interesting properties. It contains a straight line in infinity. All circles intersect this straight line in the very same pair of imaginary points.

A conic is uniquely determined by five points. A circle by three points, because all circles pass through the same pair of imaginary points in the infinite.

A projective plane can be projected into itself. The equations of the transformation are *fractional linear functions*. Namely, if points x, y are transformed into points x', y', then

$$x' = (a1*x + b1*y + c1) / (a3*x + b3*y + c3)$$
$$y' = (a2*x + b2*y + c2) / (a4*x + b4*y + c4)$$

Aleksandrov et al give an interesting example of how to use projective geometry:

Given two straight lines a and b (Exhibit 2) and a point M. The intersection of lines a and b are off the paper. Construct the straight line c, passing through points M, and the intersection of a and b.

This problem is clearly invariant against projective transformations; it is a problem in projective geometry. Aleksandrov et al give the solution using Desargue's theorem:

Solution: Draw through the point M two straight lines 1 and 2 and then the lines 3 and 4 through their points of intersection with lines a and b. Obtain the intersection K. Draw lines 5, 6, and 7. Determine L. The line c, passing through points M and L is the desired line.

Note that solution is given by using only a straight edge, and nothing else.

Complicated? Send the intersection of a and b, and point K, both to infinity. The solution becomes trivial.

Affine Transformations.

A special type of projective transformation is the affine transformation, where the center of projection is in infinity. Here the projecting rays are parallel. This means that parallel lines stay parallel; the line of infinity, with all its points, stays in infinity. An ellipse stays an ellipse, a hyperbola a hyperbola, a parabola a parabola. Exhibit 3 shows how Aleksandrov et al "contract" a circle into an ellipse. This technique can be used to construct geometrically the intersection of a straight line and an ellipse.

How do you see the validity of Pascal's theorem? Send the straight line to infinity. Now the opposing sides are parallel. "Expand" the ellipse into a circle. Now the theorem is obvious.

After this interruption let's return to my proof of TPC.

Proof of TPC: My Triumph

When I was driving home on the San Diego Freeway I asked myself the question: Can this theorem be generalized into a theorem in projective geometry? The circle will turn into a conic. But what about the midpoint M? It will not remain a midpoint. This is not a theorem invariant against projective transformations. I could try to generalize the theorem by replacing the midpoint with cross ratios, bringing the infinite into the finite, but this looks like going too far. What about affine transformations? Midpoints stay midpoints. The TPC must hold for any ellipse, hyperbola, or parabola. Eureka, I got it!

Don't try to use the dumb tools of metric geometry, such as congruent triangles, equal distances, or angles, etc. The tools must be projective, such as linear transformations, ratios, and properties of conics. From now on the proof should be easy.

Shift focus from the circle in Exhibit 1, concentrate on the four points of B1, B2, C1, C2. Five points uniquely determine a conic, but there are infinitely many conics, a family of conics, passing through the four points. Take an arbitrary point X1 on the straight line s, and pass a conic through X1, B1, B2, C1,C2. It will cut s in a second point X2. For example, the circle (a special type of conic) in Exhibit 1 intersects the straight line s, in A1 and A2. The family of conics establishes the pairs of points X1 and X2, and thereby defines the transformation T that maps s into itself. The mapping T is unique and reversible and therefore is expressible by a fractional linear function. (Klein, 1939) T must be of the form

$$x' = (ax + b) / (cx + d)$$

where x and x' are measured on line s, using M as the origin.

Consider X1 at M and move it up from M: X2 will move down from M. This means that the infinite point cannot be brought into the finite, and so T is not a fractional but a linear function. It must be of the form

$$x' = ax + b$$

So T is an affine transformation. Concentrate on the three members of the family of conics. The first is the degenerate (please no sexual innuendoes) hyperbola represented by the straight lines defined by the points B1, B2, and C1, C2. This "hyperbola" intersects line s, at the double point M. Point M is the fixed point of transformation T. The mid-ray above M is mapped into the mid-ray below M by a linear function T, of the form

$$x' = ax$$

The second conic in the family is the circle (or ellipse) that maps A1 into A2. The ratio of the distances A1 to M and A2 to M equals unity, because M is the midpoint. The transformation T is now uniquely defined

$$x' = -x$$

H will be the midpoint for every member of the family of conics.

The third conic of the family is the degenerate hyperbola represented by the straight lines defined by B1, C2 and B2, C1. This hyperbola intersects line s in the points D1 and D2. As for every member of the family of conics, M must be a midpoint. So M is the midpoint of D1, and D2. QED.

The idiotic reason is that the transformation is a good old symmetry, with respect to point M. (Aside. When John von Neumann used this term in 1938, at the Eotvos Lorand Matematikai es Fizikai Tarsulat, the audience froze. End of aside.)

Little orphan hyperbola

There is still a fourth hyperbola, the pair of straight lines going through B1-C1, and B2-C2. They cut the line s in a pair of points, and the midpoint is still M. The theorem should really be called the Theorem of Seven Chords. But it does not sound so good.

Generalizations

When driving on the San Diego Freeway I thought of trying more general mappings. Instead of starting with the four points of the chord, the midpoint, and the point in infinity, start with four points in the finite and work with harmonic associates, and cross ratios. I don't know if you just get baroque curls, or something interesting. You could transform the problem into something real simple, like a trapezoid in a circle, and make the theorem trivial in the transformed plane.

(An Aside. When I was a student at the Pazmany Peter University, in Budapest, in 1936, the Reverend Sutak taught geometry. A highlight of his lectures was when he got to projective geometry. To a standing-room-only audience he showed the interval AB, and the midpoint M. Sutak said that M was looking for his harmonic associate but could not find it. At this point the Meltosagos Ur (His Excellency, the official title of a professor) galloped around the class, in his gown, looking for the associate. When Sutak got back to the blackboard, he banged it with all his might and announced that M got to infinity, and M cried out: "Te vagy az en haromonikus tarsam!" — You are my harmonic associate! — . End of Aside.)

The Other Woman

Erdos indeed came Sunday to Don Diablo Drive, my home, and indeed, OW was the driver. I had no idea what was the extent, or length of the relationship between Erdos and OW. It was certainly Platonic.

One weekend we went to Laguna Beach. We stopped on the way to see a mission, but OW would not pay the fee to visit the place, and stayed outside. She was an adamant Protestant, and refused to give money to the Catholics. Erdos had no hang-ups, so he entered, and happily fed the pigeons with Beatrice (Bobbi), my daughter.

When we arrived at Laguna Beach, a crisis developed. They had one room for our party, and another one for the Erdos party. Where will OW sleep? The manager suggested they both sleep in the only room she had available. Erdos got visibly disturbed and yelled: "Impossible!" The impending catastrophe was somehow resolved.

Erdos would have caused some notice anywhere else but in California. He sat on the beach on a rock, his feet dangling in the ocean, an open black umbrella in his left hand against the sun, and an open math journal in the right hand.

One day OW told Laura that she was finished with Erdos. She was tired of being his chauffeur. That was it. Some time later I heard that she departed. Anyway, the matter would have ended because Erdos' mother appeared on the scene.

Comment about Polya

I was having lunch with George Polya, at the Faculty Club at Stanford, some time later in the eighties and scribbled my proof of TPC on the napkin. "Zsenialis! Hogy az ordogbe gondolt erre?" ("Genius! How the devil did you think of this?", he said.

This was the last time I saw Polya. The first time I saw him, in December 1938, just after the Munich Agreement, in Zurich, a year after he published his celebrated Necklace Theorem. I was escaping from Hungary, as a young mathematician, and wanted to pay homage to the author of the famous book by George Polya and Gabor Szego, both Hungarian mathematicians. Polya loaned me 20 Swiss francs, to help in my voyage, a loan I paid back in dollars in the eighties. He also told me that his main intellectual pleasure was to read his old papers on his projecting device. "I used to be pretty smart," he remarked. So did I, when looking back at my long ago proof of TPC.

Who was Paul Erdös?

Paul Erdös was one of the most important mathematicians of the 20th century. He has worked in dozens of fields, and has written over 1,000 mathematical papers. He was renowned for his ability to identify new problems in math and new directions math should take. It was his custom to offer prizes for the solutions to problems he poses. He died on September 20, 1996, in Warsaw. Or to use Erdos' own language he "left" at the age of 83

Captions for Exhibits

Exhibit 1: The Theorem of Penta-Chords, TPC
Exhibit 2. Line c goes through the off-paper intersection of lines a and b

Exhibit 3. “Contracting” a circle into an ellipse

References

Aleksandrov, A. D., et al, *Mathematics, Its Contents, Methods, and Meaning*, MIT Press, 1963.

Coxeter, H. S. M, *Projective Geometry*, Springer Verlag, 1987.

Klein, Felix, Elementary Mathematics from an Advanced Standpoint, Geometry, p.91, Dover Publications, Inc., 1963.

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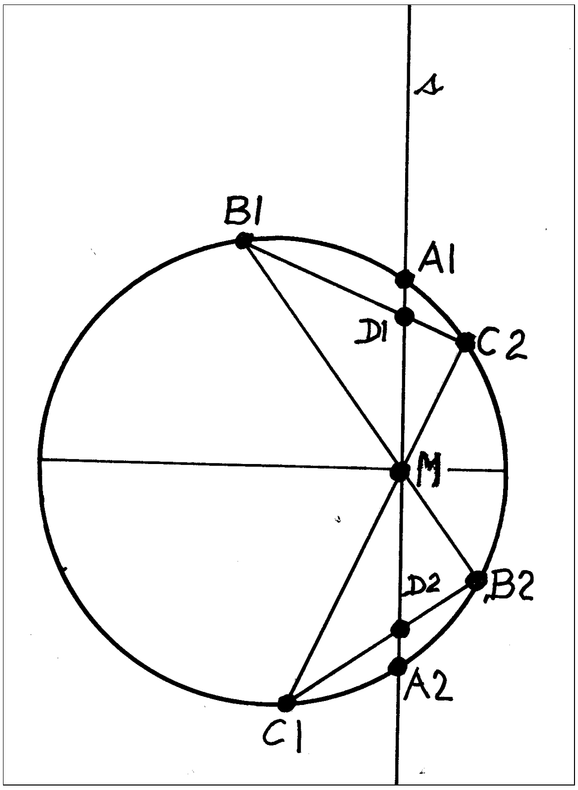
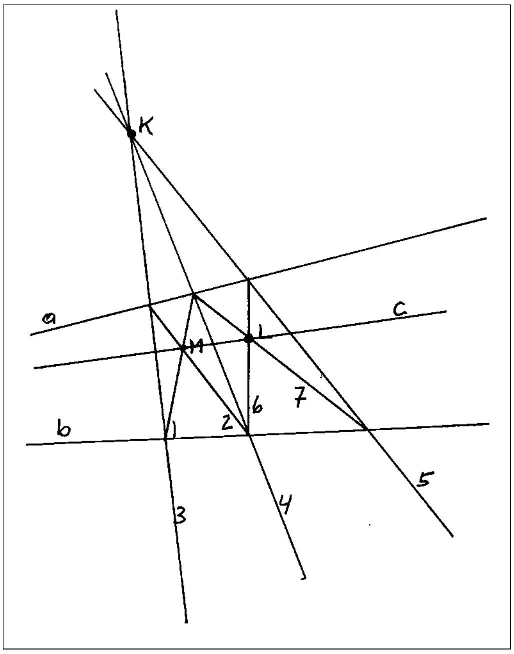


EXHIBIT 1. VAZSONYI



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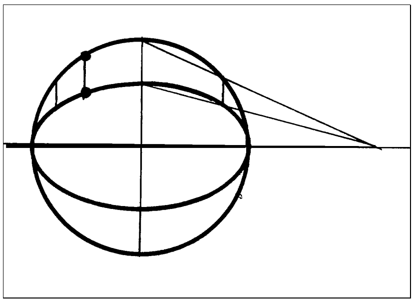

EXHIBIT 3. VAZSONYI