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SOME QUALITATIVE ASPECTS OF A FREE BOUNDARY PROBLEM FOR THE p-LAPLACE OPERATOR

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Abstract. Given an L^∞-function $\mu \geq 0$, with compact support in $\mathbf{R}^n$, and a bounded domain Ω $(\supset \operatorname{supp}(\mu))$. Suppose there exists a function u, satisfying the following overdetermined boundary value problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\partial u/\partial \nu = 1 & \text{on } \partial\Omega, \end{cases}$$

where ν is the outward unit normal vector on $\partial\Omega$ (the boundary of Ω). We assume that $\partial\Omega$ is C^2, $u \in C^2(\overline{\Omega} \setminus \operatorname{supp}(\mu)) \cap C^1(\overline{\Omega})$, and $1 < p < \infty$.

In this paper we study uniqueness, uniform bounds and some geometric properties for solutions (u, Ω) of the above overdetermined (free) boundary value problem. We show, among other things, that if (u_i, Ω_i) $(i = 1, 2)$ are two solutions to the above problem for a fixed μ, with Ω_1 convex, then $\Omega_2 \subset \Omega_1$. Consequently if both Ω_1 and Ω_2 are convex, then they coincide. Using an argument due to J. Serrin, we also prove certain types of symmetry and monotonicity, along lines, for u.

0. Introduction

The mean value property for harmonic functions

$$\int_{\partial B(x^0,r)} h\, d\sigma = c_n r^{n-1} h(x^0), \tag{0.1}$$

is well known and has a central role in classical potential theory. Here σ is the surface element, and c_n is the area of the unit sphere in $\mathbf{R}^n$ $(n \geq 2)$. Now (0.1) can be written as

$$\int_{\partial B(x^0,r)} h\, d\sigma = \int h\, d\mu, \tag{0.2}$$

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where $\mu = c_n r^{n-1}\delta_{x^0}$, is a constant multiple of the Dirac measure. Hence, letting

$$h_y(x) = \begin{cases} (1/c_2)\log|x-y|^{-1} & n=2, \\ (1/(n-2)c_n)|x-y|^{2-n} & n \geq 3, \end{cases}$$

where y is fixed, and defining

$$u(y) = \int_{\partial B(x^0,r)} h_y(x)\,d\sigma - \int h_y(x)\,d\mu \qquad \text{in } B(x^0,r),$$

we have (by (0.2)) $u=0$, and $-\partial u/\partial\nu = 1$ on $\partial B(x^0,r)$ (see [K]). Here ν is the outward normal vector on $\partial B(x^0,r)$. The function u, thus defined, satisfies (in the sense of distributions)

$$\begin{cases} \Delta u = -\mu & \text{in } B(x^0,r), \\ u = 0 & \text{on } \partial B(x^0,r), \\ -\partial u/\partial\nu = 1 & \text{on } \partial B(x^0,r). \end{cases} \tag{0.3}$$

In general, one may consider the following question: For a given (positive Radon) measure (in this paper, apart from Examples 1–6, we only consider positive L^∞ functions) with compact support, find a domain Ω ($\supset \operatorname{supp}(\mu)$), and a function u satisfying (in some weak sense)

$$\begin{cases} \Delta_p u = -\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\partial u/\partial\nu = 1 & \text{on } \partial\Omega, \end{cases} \tag{0.4}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, is the p-Laplace operator and $1 < p < \infty$. This is an overdetermined (free) boundary value problem and hence not always well-posed (see Examples 4 and 5 below). The existence depends, strongly, on the concentration of the measure μ. For $p=2$ there are some results on the existence for certain types of μ; see [GS] and [Sh1]. As for other values of p, there are no results known to the authors.

One may also change the problem slightly by letting μ be identically zero, and instead having nonzero Dirichlet boundary conditions, working in a subdomain of $\mathbf{R}^n$; see [AC] for $p=2$, and [HS] for $1<p<\infty$, and convex subdomains.

In this paper, we will not consider the question of existence or regularity of solutions to (0.4). We only consider some qualitative aspects of the solutions, assuming they already exist. Our results concern uniqueness (Theorem 1), uniform bounds (Theorem 2), monotonicity along lines (Theorem 3), and starshapeness (Theorem 4).

Rules. Throughout this paper the following rules apply: Ω denotes a bounded domain, with C^2 boundary; except from Examples 1–6, μ always denotes a nonnegative L^∞-function with

$$\operatorname{supp}(\mu) \subset \Omega;$$

ν denotes the unit normal vector on the boundary, and pointing outward, of a domain in consideration.

Any solution u to (0.4) is extended continuously to entire $\mathbf{R}^n$, with $u \equiv 0$ in $\mathbf{R}^n \setminus \Omega$; we also assume that if (u, Ω) is a solution to (0.4) then $u \in C^1(\overline{\Omega})$ and it is C^2 in an interior neighborhood of $\partial\Omega$, and has a C^2 extension across $\partial\Omega$. It is also noteworthy that since u is constant on $\partial\Omega$ and positive in Ω, we will have

$$|\nabla u| = -\partial u/\partial\nu \qquad \text{on } \partial\Omega.$$

1. Examples

In this section we give a few explicit examples, where μ will be a non-negative measure. For this purpose we define $a_1 = (p-1)/(p-n)$, $a_2 = p/(p-1)$, $a_3 = -n/(p-1)$, and $a_4 = a_1 + r^{1/a_1}(1/a_2 - a_1)$, for a fixed r.

Example 1. Let

$$u(x) = \begin{cases} a_1(1 - |x|^{1/a_1}), & p \neq n, \\ \log |x|^{-1}, & p = n. \end{cases}$$

Then on $\partial B(0,1)$, $u = 0$, and $|\nabla u| = 1$, and in $B(0,1)$ we have $\Delta_p u = -c_n \delta_0$, where c_n is as in (0.1).

Example 2. Let $p \neq n$, and set $f(r) = r - r^{1-1/a_1}$. Then it is not hard to see that for each $0 < r_1 < 1$ there is $r_2 > 1$ such that $f(r_1) = -f(r_2)$. Hence $r_1 + r_2 = r_1^{1-1/a_1} + r_2^{1-1/a_1}$. This can be used to show that the function u defined as

$$u(x) = \begin{cases} a_1 r_1 \big(\big(|x|/r_1\big)^{1/a_1} - 1 \big), & r_1 \le |x| \le 1, \\ a_1 r_2 \big(1 - \big(|x|/r_2\big)^{1/a_1} \big), & 1 \le |x| \le r_2. \end{cases}$$

is Lipschitz. Now define $\Omega = \{x : r_1 < |x| < r_2\}$. Then (u, Ω) solves (0.4) with $d\mu = (r_1^{n-1} + r_2^{n-1}) \chi_{\partial B(0,1)} d\sigma$.

Example 3. Let $p \neq n$, $0 < r < 1$, and define u as

$$u(x) = \begin{cases} (-r^{a_3}/a_2)|x|^{a_2} + a_4, & |x| \le r, \\ a_1(1 - |x|^{1/a_1}), & r < |x| \le 1. \end{cases}$$

Then $\Delta_p u = -n r^{-n} \chi_{B(0,r)}$ and $u = 0$ and $|\nabla u| = 1$ on $\partial B(0,1)$.

For the case $p = n$, in Examples 2 and 3, one may define a similar function using the log-function.

Example 4 (Non-existence). Here we give a simple example of an L^∞-function μ, such that (0.4) has no solutions. Fix μ and let (u, Ω) solve (0.4). Then, integration by parts gives

$$\int d\mu = -\int_\Omega \operatorname{div}(|\nabla u|^{p-2} \nabla u) = -\int_{\partial\Omega} |\nabla u|^{p-2} \nabla u \cdot \nu = \int_{\partial\Omega} d\sigma. \tag{1.1}$$

Now let μ be defined as $d\mu = C\chi_D dx$, where

$$C = n^{(n-1)/n}\left(\frac{c_n}{\text{volume}\,(D)}\right)^{1/n},$$

and c_n is as in (0.1). Then (1.1) reduces to

$$n^{(n-1)/n}\cdot\left(\frac{c_n}{\text{volume}\,(D)}\right)^{1/n}\cdot\text{volume}\,(D) = \text{area}\,(\partial\Omega),$$

and hence, by the (strict) inclusion $D \subset \Omega$,

$$c_n^{1/n}\big(n\,\text{volume}\,(\Omega)\big)^{(n-1)/n} > c_n^{1/n}\big(n\,\text{volume}\,(D)\big)^{(n-1)/n} = \text{area}\,(\partial\Omega).$$

This obviously contradicts the well-known isoperimetric inequality [B]. Therefore for μ as above there cannot exist a solution to (0.4).

Example 5 (Non-existence). Let $d\mu = f(x)\,dx$, and suppose (u,Ω) solves (0.4). Set $M = \sup f$, and let $B(x^0, r_\Omega)$ be the smallest ball containing Ω. Then we claim

$$n < Mr_\Omega. \tag{1.2}$$

This provides us with a test for non-existence.

To prove (1.2), we define

$$v(x) = \left(\frac{p-1}{p}\right)\frac{r_\Omega^{p/(p-1)} - |x-x^0|^{p/(p-1)}}{r_\Omega^{1/(p-1)}}.$$

Then

$$\Delta_p v = \frac{-n}{r_\Omega}.$$

Now if (1.2) fails, then

$$\Delta_p u = -f \ge -M \ge \frac{-n}{r_\Omega} = \Delta_p v \qquad \text{in } \Omega.$$

Since also $u = 0 \le v$ on $\partial\Omega$, we may apply Lemma 1 (a) and (c) (Section 3) to deduce that $u < v$ in Ω. (Or at least in some interior neighborhood of $\partial\Omega$; see the remark following Lemma 1.) Now let $y \in \partial\Omega$ correspond to the largest distance to x^0, i.e. $|y - x^0| = r_\Omega$, and observe that the unit outward normal vector ν at y equals $(y-x^0)/|y-x^0|$ and that $u(y) = v(y) = 0$. Invoking part (b) of Lemma 1 we conclude

$$-1 = \frac{\partial u}{\partial\nu}(y) > \frac{\partial v}{\partial\nu}(y) = -1, \tag{1.3}$$

which is a contradiction. Hence (1.2) holds.

Example 6 (Perimeter of Ω). Let (u,Ω) solve (0.4), and let us drop the assumption that $\partial\Omega$ is C^2. The boundary condition $\partial u/\partial\nu = -1$ is to be interpreted as (replaced by) $|\nabla u| = 1$, in the sense that for any small $\varepsilon > 0$ there is a small neighborhood N_ε of $\partial\Omega$ such that

$$\big||\nabla u| - 1\big| < \varepsilon, \qquad \text{in } \Omega \cap N_\varepsilon. \tag{1.4}$$

We will show that Ω has finite perimeter; see [Z] for a definition. Let us define $v = u^{p/(p-1)}$ (observe that $u \geq 0$). Then one may easily verify that

$$\Delta_p v = C_p(u\Delta_p u + |\nabla u|^p), \qquad C_p = \left(\frac{p}{p-1}\right)^{p-1}.$$

Since $\Delta_p u = 0$ in $\{0 < u < \varepsilon\}$, for ε small, we have

$$\Delta_p v = C_p |\nabla u|^p \geq \tfrac{1}{2} C_p.$$

Using this and that $|\nabla v| = 0$ on $\{u = 0\}$, we have (by (1.4))

$$\begin{aligned} \tfrac{1}{2} C_p \int_{\{0<u<\varepsilon\}} dx &\leq \int_{\{0<u<\varepsilon\}} \Delta_p v \, dx = \int_{\{u=\varepsilon\}} |\nabla v|^{p-2} \nabla v \cdot \nu \, d\sigma \\ &= C_p \int_{\{u=\varepsilon\}} |\nabla u|^{p-1} u \, d\sigma = C_p \varepsilon \int_{\{u=\varepsilon\}} |\nabla u|^{p-1} \, d\sigma = C_p \varepsilon \int d\mu. \end{aligned}$$

This shows that

$$\chi_{\{u>\varepsilon\}} \to \chi_{\{u>0\}} \qquad \text{in } L^1(\mathbf{R}^n). \tag{1.5}$$

Next, by (1.4), for any $\varepsilon > 0$ there is $\delta_\varepsilon \searrow 0$ such that $|\nabla u| > 1 - \delta_\varepsilon$ in $\{u \leq \varepsilon\}$. Hence

$$\int d\mu = -\int_\Omega \Delta_p u \, dx = \int_{\partial\{u>\varepsilon\}} |\nabla u|^{p-1} \, d\sigma \geq (1-\delta_\varepsilon)^{p-1} \operatorname{Perimeter}(\{u > \varepsilon\}).$$

Letting $\varepsilon \to 0$ and using (1.5) and the lower-semicontinuity of the variation measure (perimeter) we obtain

$$\operatorname{Perimeter}(\Omega) = \operatorname{Perimeter}(\{u > 0\}) \leq \int d\mu.$$

2. Main results

In this section we state our main results, which will be proven in Section 4. These results are of qualitative nature (uniqueness and geometric features) and for $p = 2$ they where proven in [GS] and [Sh2]. To avoid technical difficulties we assume, throughout this section, that u is C^1 on $\overline{\Omega}$, it is C^2 in an interior neighbourhood of $\partial\Omega$ and it has a C^2 extension across $\partial\Omega$. Also once again we stress that $\operatorname{supp}(\mu) \subset \Omega$, and μ is a bounded L^∞-function with compact support.

Theorem 1. *Let (u_j, Ω_j) $(j = 1, 2)$ be solutions to (0.4), with $\operatorname{supp}(\mu) \subset \Omega_1 \cap \Omega_2$. Then the following hold.*
(1) *If Ω_1 is convex then $\Omega_2 \subset \Omega_1$.*
(2) *If $\Omega_1 \cap \Omega_2$ is convex, then $\Omega_1 \equiv \Omega_2$.*
(3) *If both Ω_1 and Ω_2 are convex, then $\Omega_1 \equiv \Omega_2$ and $u_1 \equiv u_2$.*

Theorem 2. *Let $1 < p \leq n$ and (u, Ω) be a solution to (0.4), with Ω bounded. Then there exists $R = R(\mu, p, n)$ such that*

$$\Omega \subset B(0, R).$$

For the next result we need some new definitions and notations.

For a fixed unit vector $a \in \mathbf{R}^n$ and for $\lambda \in \mathbf{R}$ set

$$T_\lambda = T_{a,\lambda} := \{x \cdot a = \lambda\}, \qquad T_\lambda^- := \{x \cdot a < \lambda\}, \qquad T_\lambda^+ := \{x \cdot a > \lambda\}.$$

For $x \in \mathbf{R}^n$ let x^λ denote the reflected point with respect to T_λ and set $u^\lambda(x) = u(x^\lambda)$. Also let

$$\Omega_\lambda = \Omega \cap T_\lambda^+ = \text{ the cap cut off by } T_\lambda,$$
$$\widetilde{\Omega}_\lambda = \{x^\lambda : x \in \Omega_\lambda\} = \text{ the reflection of } \Omega_\lambda \text{ in } T_\lambda.$$

Theorem 3. *Assume that for some unit vector $a \in \mathbf{R}^n$ and some $\lambda_0 \in \mathbf{R}$ we have*

$$\operatorname{supp}(\mu) \cap T_\lambda^+ = \emptyset,$$

for all $\lambda \geq \lambda_0$. Then for any solution u to (0.4) the following hold.

$$\widetilde{\Omega}_\lambda \subset \Omega \qquad \text{for all } \lambda \geq \lambda_0, \tag{2.1}$$
$$u \leq u^\lambda \qquad \text{in } \Omega_\lambda \text{ for all } \lambda > \lambda_0, \tag{2.2}$$
$$a \cdot \nabla u \leq 0 \qquad \text{in } \Omega_{\lambda_0}. \tag{2.3}$$

Corollary 1. *Let (u, Ω) be a solution to (0.4) and suppose $\operatorname{supp}(\mu) \subset K$, a compact convex set. Then for any $x \in \partial\Omega \setminus K$ the inward normal ray L_x, of $\partial\Omega$ at x intersects K. Moreover, $\partial\Omega \setminus K$ is Lipschitz.*

Proof. If for $x \in \partial\Omega \setminus K$ we have $L_x \cap K = \emptyset$ then one can find $a \in \mathbf{R}^n$ and $\lambda_0 \in \mathbf{R}$ such that $K \subset T^-_{a,\lambda_0}$, $L_x \subset T^+_{a,\lambda_0}$. The first inclusion implies that the assumptions of Theorem 3 are satisfied and hence (2.1) holds. This in particular means that $\partial\Omega \cap T^+_{a,\lambda_0}$ is a graph, seen from T_{a,λ_0}, and therefore the outward normal vector ν_y at every point $y \in \partial\Omega \cap T^+_{a,\lambda_0}$ makes an angle $0 \le \theta < \frac{1}{2}\pi$ with a, i.e.

$$\nu_y \cdot a > 0 \qquad \text{for all } y \in \partial\Omega \cap T^+_{a,\lambda_0} \qquad \text{(scalar product).} \tag{2.4}$$

Now the second inclusion ($L_x \subset T^+_{a,\lambda_0}$) implies that ν_x (outward normal at x) points into T^-_{a,λ_0}. Hence the angle between a and ν_x is greater than $\frac{1}{2}\pi$, i.e. $\nu_x \cdot a < 0$. Since $\partial\Omega$ is C^1 we may choose $y \in \partial\Omega \cap T^+_{a,\lambda_0}$ near x with ν_y pointing into $T^-_{a,\lambda}$, i.e. $\nu_y \cdot a < 0$. This contradicts (2.4). Hence the first statement of the corollary follows. The second statement follows easily by varying a and λ_0 such that $K \subset T^-_{a,\lambda_0}$. □

For the proof of Theorem 3 we shall use some reflection methods related to the "moving plane method" which has previously been used in similar problems in [Se], [GNN], [BN], [Sh2], [GS].

Theorem 4. *Let (u, Ω) solve (0.4) and suppose there is $\alpha \le -1$ such that*

$$t^\alpha \mu(x/t) \le \mu(x), \tag{2.5}$$

for all $0 < t \le 1$ and $x \in \mathbf{R}^n$. Then Ω is starshaped with respect to the origin, and moreover

$$t^\beta u(x/t) \le u(x) \tag{2.6}$$

for $\beta = (p + \alpha)/(p - 1)$ and all $0 < t \le 1$ and $x \in \Omega$.

3. Auxiliary lemmas

This section is devoted to some technical lemmas.

Lemma 1 (Hopf's comparison principle). *Let $D \subset \mathbf{R}^n$ be bounded, and $v_1, v_2 \in C^1(\overline{D})$, with*

$$\Delta_p v_1 \le \Delta_p v_2.$$

Then the following hold.

(a) *If $v_1 \ge v_2$ on ∂D, then $v_1 \ge v_2$ in D.*

(b) *Suppose $v_1 > v_2$ in D, $v_1(x^0) = v_2(x^0)$ for some $x^0 \in \partial D$, $|\nabla v_2| \ge \gamma > 0$ in D (for some $\gamma > 0$), and D satisfies the interior sphere condition. Then $\partial v_1/\partial\nu(x^0) < \partial v_2/\partial\nu(x^0)$, where ν is the unit outward normal vector on ∂D, at x^0.*

(c) *If $v_1 \ge v_2$ and $v_1 \not\equiv v_2$ in D, $|\nabla v_2| \ge \gamma > 0$ (for some $\gamma > 0$), then $v_1 > v_2$ in D.*

Remark. Throughout this paper we will, repeatedly, use Lemma 1. It is thus crucial that the relation $|\nabla v_2| > \gamma$, for some $\gamma > 0$, holds for the function v_2 in part (b) and (c) of the lemma. This, however, is not needed for part (a) of the lemma. Now in our applications of this lemma we will always consider part (a) in the entire domain in consideration and part (b) and (c) in a small subdomain with x^0 on its boundary. Since the magnitude of the gradient of any solution to (0.4) approaches one, continuously, the required condition in the lemma is fulfilled, near the boundary for solutions to (0.4). In the sequel we will omit mentioning this argument.

Lemma 2 (Serrin's boundary point lemma). *Let D be a domain with C^2 boundary and let T be a plane containing the normal to ∂D at some point $x^0 \in \partial D$. Denote by D' the portion of D that lies on some particular side of T. Suppose that w_1 and w_2 are of class $C^2(\overline{D})$, $w_1 \not\equiv w_2$ in D, $w_1(x^0) = w_2(x^0)$ and satisfy*

$$\Delta_p w_1 \leq \Delta_p w_2, \qquad w_1 \geq w_2 \qquad \text{in } D'.$$

Then either

$$\frac{\partial w_1}{\partial \tau} > \frac{\partial w_2}{\partial \tau} \qquad \text{or} \qquad \frac{\partial^2 w_1}{\partial \tau^2} > \frac{\partial^2 w_2}{\partial \tau^2} \qquad \text{at } x^0,$$

where τ is any direction entering D' non-tangentially.

Lemma 3. *Let (u, Ω) solve (0.4). Let T_λ be a plane containing the normal to $\partial\Omega$ at some point $x^0 \in \partial\Omega$. Suppose, moreover, $\widetilde{\Omega}_\lambda \subset \Omega$. Then the first and second derivatives of u and $\tilde{u}$ coincide at x^0.*

The proofs of Lemmas 1, 2 and 3 are much the same as that of the case $p = 2$. Indeed, since $|\nabla u| > 0$ near the boundary, in all three lemmas, we may rewrite the p-Laplacian as

$$h_1(|\nabla u|)\Delta u + h_2(|\nabla u|) \sum_{i,j} u_i u_j u_{ij},$$

where h_1 and h_2 are positive C^1 functions in their argument. Hence we will have a uniformly elliptic operator, which is translation and rotation invariant. However, for the reader's convenience we give exact references for the proofs of the lemmas. Lemma 1 is proven in [T, Lemma 3.2, Propositions 3.4.1, 3.4.2]. A proof for Lemma 2 can be found in ([Se, Lemmas 1 and 2]). The proof of Lemma 3 is also the same as in ([Se, p. 307]).

Lemma 4. *Let (u, Ω) be a solution to (0.4). Consider a hyperplane T, which cuts off Ω a cap Ω', and such that $\mathrm{supp}\,(\mu) \cap \overline{\Omega'} = \emptyset$. Then*

$$d := \sup_{x \in \partial\Omega'} \mathrm{dist}\,(x, T) < \sup_T u. \tag{3.1}$$

Moreover if $x^0 \in T$ *is such that* $u(x^0) = \sup_T u$, *then*

$$-\frac{\partial u}{\partial a}(x^0) > 1, \tag{3.2}$$

where a *is the unit normal vector to* T *pointing inward* Ω'.

Proof. Since the problem is rotation and translation invariant we may assume that $T = \{x_1 = 0\}$, and $\Omega' = \{x_1 > 0\} \cap \Omega$. Now let $y \in \partial\Omega'$, be such that $d = \operatorname{dist}(y, T)$ and observe that $(\partial u/\partial x_1)(y) = -1$. Then define

$$h(x) = s(d - x_1),$$

where

$$s := \frac{\sup_T u}{d}.$$

Now observing that

$$\Delta_p h = \Delta_p u = 0 \quad \text{in } \Omega', \qquad h \geq u \quad \text{on } \partial\Omega', \qquad u(y) = h(y) = 0 \tag{3.3}$$

we may invoke Lemma 1 (a)–(c) to deduce

$$-s = \frac{\partial h}{\partial x_1}(y) < \frac{\partial u}{\partial x_1}(y) = -1,$$

i.e.

$$\frac{\sup_T u}{d} = s > 1, \tag{3.4}$$

which proves (3.1). Next using (3.3)–(3.4) and the fact that $u(x^0) = h(x^0)$ we obtain

$$1 < s = -\frac{\partial h}{\partial x_1}(x^0) \leq -\frac{\partial u}{\partial x_1}(x^0),$$

i.e. (3.2) holds. □

Lemma 5 (Pasting lemma, [HKM, 7.9]). *Let* u *be any solution to* (0.4) *and extend it to the entire* $\mathbf{R}^n$ *by defining it to be zero in the exterior of* Ω. *Then*

$$\Delta_p(u - c) \geq -\mu,$$

for any constant c.

The proof of this lemma follows easily from the pasting lemma [HKM, 7.9]. However, for the reader's convenience we give the proof.

Proof. Take a small neighborhood N of $\partial\Omega$ such that $\operatorname{supp}(\mu) \cap N = \emptyset$, and define

$$v = \begin{cases} \max(u, 0) & \text{in } N \cap \Omega, \\ 0 & \text{in } \mathbf{R}^n \setminus \Omega. \end{cases}$$

Then, by [HKM, 7.9], $-v$ is p-superharmonic in N. Hence $v - c$, is p-subharmonic in N which is the desired result. □

4. Proofs of the main theorems

Proof of Theorem 1. Since the proofs of (1) and (2) are similar, we only prove (1); (3) follows from (1) or (2). Suppose $\Omega_2 \setminus \Omega_1 \neq \emptyset$, then we reach a contradiction. Extend u_j by zero to $\mathbf{R}^n \setminus \Omega_j$, for $j = 1, 2$, and let $x^0 \in \partial\Omega_1$ be such that $u_2(x^0) = \sup_{\partial\Omega_1} u_2$. Define now $w(x) = u_2 - u_2(x^0)$ in Ω_1. Then by Lemma 5 $\Delta_p w \geq -\mu$. Hence

$$\Delta_p w \geq \Delta_p u_1 \quad \text{in } \Omega_1, \qquad w \leq u_1 \quad \text{on } \partial\Omega_1, \qquad w(x^0) = u_1(x^0) = 0.$$

We may thus apply Lemma 1 to deduce that

$$\partial u_2/\partial\nu(x^0) = \partial w/\partial\nu(x^0) > \partial u_1/\partial\nu(x^0) = -1$$

i.e.

$$-\partial u_2/\partial\nu(x^0) < 1. \tag{4.1}$$

Now, using the convexity of $\partial\Omega_1$ and that $\operatorname{supp}(\mu) \subset \Omega_1 \cap \Omega_2$, we can take a supporting plane T at x^0 such that the assumptions of Lemma 4 are fulfilled. But then, (4.1) contradicts (3.2). This completes the proof of the theorem. □

Proof of Theorem 2. By scaling, we may assume $\operatorname{supp}(\mu) \subset B(0,1)$ and $\mu \leq 1$. We also consider the case $1 < p < n$; since for $p = n$ we need only to replace the fundamental solution $|x|^{(p-n)/(p-1)}$, in the definition below, with the logarithmic kernel. Define

$$v = \begin{cases} a|x|^{p/(p-1)} + b, & |x| \leq 1, \\ c|x|^{(p-n)/(p-1)}, & |x| \geq 1, \end{cases}$$

where

$$a = -n^{1/(1-p)}\left(\frac{p-1}{p}\right), \qquad c = \left(\frac{p-1}{n-p}\right)n^{1/(1-p)}, \qquad b = c - a.$$

Then v satisfies

$$\Delta_p v = -\chi_{B(0,1)} \leq -\mu \leq \Delta_p u \quad \text{in } \Omega, \qquad v > 0 \quad \text{in } \mathbf{R}^n.$$

Hence Lemma 1 can be applied to obtain $v \geq u$. Define now

$$d := \sup_{x \in \partial\Omega} \operatorname{dist}\big(x, B(0,1)\big).$$

Then by Lemma 4, and the above

$$d \leq \sup_{\partial B(0,1)} u \leq \sup_{\partial B(0,1)} v \leq c.$$

This proves the theorem. □

Proof of Theorem 3. Obviously $\Omega_\lambda = \emptyset$, for large values of λ. We start moving T_λ, by decreasing λ, towards Ω and until it intersects Ω and produces the cap Ω_λ. Then, at the beginning of this intersection the reflected cap $\widetilde{\Omega}_\lambda$ remains inside Ω, since $\partial\Omega$ is C^2. Now if the theorem fails, then for some $\lambda > \lambda_0$ one of the following can occur.

(1) $\widetilde{\Omega}_\lambda$ becomes internally tangent to $\partial\Omega$ at some point not on T_λ.

(2) T_λ reaches a position where it is orthogonal to $\partial\Omega$ at some point x^0, on $\partial\Omega$.

Suppose situation (1) occurs. Then u^λ, the reflection of u in T_λ, is p-harmonic in $\widetilde{\Omega}_\lambda$, and satisfies $u^\lambda \le u$ on $\partial\widetilde{\Omega}_\lambda$. Therefore we can apply part (a) of Lemma 1 to conclude that $u^\lambda \le u$. Next we observe that near x^0, the touching point of $\partial\widetilde{\Omega}_\lambda$ and $\partial\Omega$, both $|\nabla u^\lambda|$ and $|\nabla u|$ are > 0. Hence part (b) and (c) of Lemma 1 can be applied to deduce

$$-1 = \partial u^\lambda/\partial\nu(x^0) > \partial u/\partial\nu(x^0) = -1,$$

which is a contradiction, and implies that situation (1) above cannot occur.

Now if situation (2) arises, then one may use Lemmas 2 and 3 to reach a contradiction. This completes the proof of (2.1).

Next (2.3) follows from (2.2). Now by (2.1) $u^\lambda \ge u = 0$ on $\partial\Omega \cap T_\lambda^+$, and $u^\lambda = u$ on T_λ. Since $\Delta_p u^\lambda \le \Delta_p u$ in Ω_λ we can apply Lemma 1 (a) to deduce (2.2). ◻

Proof of Theorem 4. For $\beta = (p+\alpha)/(p-1)$, define

$$u_t(x) = t^\beta u(x/t), \qquad \Omega_t = \{x : x/t \in \Omega\},$$

and observe that by (2.5), the origin is in the interior of the support of μ and thus inside Ω. Hence for small values of t, $\Omega_t \subset \Omega$. Now define t_0 to be the largest number such that $\Omega_t \subset \Omega$ for all $0 < t < t_0$, and let $y \in \partial\Omega \cap \partial\Omega_{t_0}$. Since μ is of relation (2.5), we have

$$\Delta_p u_{t_0} \ge \Delta_p u, \quad \text{in } \Omega_{t_0}, \qquad 0 = u_{t_0} \le u \quad \text{on } \partial\Omega_{t_0}, \qquad u_{t_0}(y) = u(y) = 0.$$

If the conclusion in the theorem fails, then

$$t_0 < 1, \tag{4.2}$$

and $u_{t_0} \not\equiv u$. We may thus invoke Lemma 1 (a)–(c) to conclude

$$-t_0^{\beta-1} = \frac{\partial u_{t_0}}{\partial\nu}(y) > \frac{\partial u}{\partial\nu}(y) = -1. \tag{4.3}$$

The assumption $\alpha \le -1$ implies $\beta - 1 \le 0$, which in turn gives that (4.3) is in direct contradiction with (4.2). We thus conclude that $t_0 = 1$ and Ω is starshaped. To show (2.6), we consider once again $u_t(x)$ as above and observe, by the starshapeness, $\Omega_t \subset \Omega$ (for all $0 < t \le 1$). Hence by (2.5) $\Delta_p u_t \ge \Delta_p u$ in Ω_t, and $u_t \le u$ on $\partial\Omega_t$. Now Lemma 1 (a) gives the desired result. ◻

5. Concluding remarks

Most of the results of this paper can easily be generalized to hold for the following situations:

(1) Degenerate operators of the form $\operatorname{div}\big(\varphi_1(u,|\nabla u|)\nabla u\big)+\varphi_2(u,|\nabla u|)$, with regular ingredients;

(2) μ a positive measure;

(3) $\partial\Omega$ has finite perimeter with no a priori smoothness assumptions;

(4) relaxation of $\partial u/\partial\nu=-1$ on $\partial\Omega$ to $\overline{\lim}_{\Omega\ni x\to\partial\Omega}|\nabla u|=1$.

The reason that we work with L^∞-functions rather than measures is the belonging of the solution to the space $W^{1,p}(\mathbf{R}^n)$. However, outside the support of μ, the solution is as good as Lipschitz, if e.g. we have the boundary behavior for the gradient as in (0.4). Therefore if we assume the solution is in the space $W^{1,p}(\mathbf{R}^n)$, then we can repeat the above arguments, without any changes, to obtain Theorems 1–4, for arbitrary nonnegative measures.

The proof of Theorem 2, given here, does not apply when $p>n$. The reason is that in this case, we cannot find nonnegative supersolutions in $\mathbf{R}^n$, merely because of the behavior of the function $|x|^{(p-n)/(p-1)}$, near the infinity. We believe, however, that the theorem should hold for all $1<p<\infty$.

As mentioned earlier, most of the existing literature is concerned with the case $p=2$. It is known that measures, which are enough concentrated on their support, do admit solutions; e.g., for finite positive combinations of Dirac masses there always exists a solution for (0.4), when $p=2$. For other values of p, this is an open question. A natural way of finding solutions is through minimization of the functional

$$\int_{\mathbf{R}^n}\frac{1}{p}|\nabla v|^p-\mu v+\left(\frac{p-1}{p}\right)\chi_{\{v>0\}},$$

over the set

$$\{v: v\in W_0^{1,p}(\mathbf{R}^n),\ v\geq 0\}.$$

A minimizer, or stationary point, u of this functional expects to satisfy

$$\Delta_p u=-\mu+\mathscr{H}^{n-1}\lfloor\partial\Omega, \tag{5.1}$$

where $\Omega=\{u>0\}$, and $\mathscr{H}^{n-1}\lfloor\partial\Omega$ is the n-dimensional Hausdorff measure supported on $\partial\Omega$. Of course if $\partial\Omega$ is already known to be "nice" then it is not hard to prove that (5.1) holds. Also, for smooth boundaries, (5.1) expresses that $|\nabla u|=1$ on $\partial\Omega$. Thus the main problem, in using minimization, consists of showing the smoothness of the free boundary. This question (for $p=2$) has been treated by H.W. Alt and L.A. Caffarelli in their pioneering paper [AC]. It is most likely that the same technique works for the p-Laplacian.

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