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REGULARITY AND FREE BOUNDARY REGULARITY FOR THE p LAPLACIAN IN LIPSCHITZ AND C^1 DOMAINS

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Abstract. In this paper we study regularity and free boundary regularity, below the continuous threshold, for the p Laplace equation in Lipschitz and C^1 domains. To formulate our results we let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p harmonic function in $\Omega \cap B(w, 4r)$, that u is continuous in $\bar{\Omega} \cap \bar{B}(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. We first prove, Theorem 1, that $\nabla u(y) \to \nabla u(x)$, for almost every $x \in \Delta(w, 4r)$, as $y \to x$ non tangentially in Ω. Moreover, $\| \log |\nabla u| \|_{BMO(\Delta(w,r))} \leq c(p, n, M)$. If, in addition, Ω is C^1 regular then we prove, Theorem 2, that $\log |\nabla u| \in VMO(\Delta(w, r))$. Finally we prove, Theorem 3, that there exists $\hat{M}$, independent of u, such that if $M \leq \hat{M}$ and if $\log |\nabla u| \in VMO(\Delta(w, r))$ then the outer unit normal to $\partial\Omega$, n, is in $VMO(\Delta(w, r/2))$.

1. Introduction

In this paper, which is the last paper in a sequence of three, we complete our study of the boundary behaviour of p harmonic functions in Lipschitz domains. In [LN] we established the boundary Harnack inequality for positive p harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$ and we carried out an in depth analysis of p capacitary functions in starlike Lipschitz ring domains. The study in [LN] was continued in [LN1] where we established Hölder continuity for ratios of positive p harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$. In [LN1] we also studied the Martin boundary problem for p harmonic functions in Lipschitz domains. In this paper we establish, in the setting of Lipschitz domains $\Omega \subset \mathbf{R}^n$, the analog for the p Laplace equation, $1 < p < \infty$, of the program carried out in the papers [D], [JK], [KT], [KT1] and [KT2] on regularity and free boundary regularity, below the continuous threshold, for the Poisson kernel associated to the Laplace operator when $p = 2$. Except for the work in [LN], where parts of this program

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were established for p capacitary functions in starlike Lipschitz ring domains, the results of this paper are, in analogy with the results in [LN] and [LN1], completely new in case $p \neq 2$, $1 < p < \infty$. We also refer to [LN2] for a survey of the results established in [LN], [LN1] and in this paper.

To put the contributions of this paper into perspective we consider the case of harmonic functions and we recall that in [D] B. Dahlberg showed for $p = 2$, that if Ω is a Lipschitz domain then the harmonic measure with respect to a fixed point, $d\omega$, and surface measure, $d\sigma$, are mutually absolutely continuous. In fact if $k = d\omega/d\sigma$, then Dahlberg showed that k is in a certain L^2 reverse Hölder class from which it follows that $\log k \in BMO(d\sigma)$, the functions of bounded mean oscillation with respect to the surface measure on $\partial\Omega$. Jerison and Kenig [JK] showed that if, in addition, Ω is a C^1 domain then $\log k \in VMO(d\sigma)$, the functions in $BMO(d\sigma)$ of vanishing mean oscillation. In [KT] this result was generalized to 'chord arc domains with vanishing constant'. Concerning reverse conclusions, Kenig and Toro [KT2] were able to prove that if $\Omega \subset \mathbf{R}^n$ is δ Reifenberg flat for some small enough $\delta > 0$, $\partial\Omega$ is Ahlfors regular and if $\log k \in VMO(d\sigma)$, then Ω is a chord arc domain with vanishing constant, i.e., the measure theoretical normal n is in $VMO(d\sigma)$.

The purpose of this paper is to prove for p harmonic functions, $1 < p < \infty$, and in the setting of Lipschitz domains, $\Omega \subset \mathbf{R}^n$, the results stated above for harmonic functions (i.e., $p = 2$). We also note that we intend to establish, in a subsequent paper, the full program in the setting of Reifenberg flat chord arc domains.

To state our results we need to introduce some notation. Points in Euclidean n space $\mathbf{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and we let $\bar{E}$, ∂E, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbf{R}^n$. We define $d(y, E)$ to equal the distance from $y \in \mathbf{R}^n$ to E and we let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbf{R}^n$. Moreover, $|x| = \langle x, x \rangle^{1/2}$ is the Euclidean norm of x, $B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$ is defined whenever $x \in \mathbf{R}^n$, $r > 0$, and dx denotes the Lebesgue n measure on $\mathbf{R}^n$. If $O \subset \mathbf{R}^n$ is open and $1 \leq q \leq \infty$ then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O. We let $\|f\|_{1,q} = \|f\|_q + \||\nabla f|\|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O, $C_0^\infty(O)$ denotes the class of infinitely differentiable functions with compact support in O and we let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Given a bounded domain G, i.e., a connected open set, and $1 < p < \infty$ we say that u is p harmonic in G provided $u \in W^{1,p}(G)$ and provided

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(G)$. Observe that, if u is smooth and $\nabla u \neq 0$ in G, then

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \text{ in } G \tag{1.2}$$

and u is a classical solution to the p Laplace partial differential equation in G. Here, as in the sequel, $\nabla\cdot$ is the divergence operator. We note that $\phi \colon E \to \mathbf{R}$ is said to be Lipschitz on E provided there exists b, $0 < b < \infty$, such that

$$|\phi(z) - \phi(w)| \leq b|z - w|, \text{ whenever } z, w \in E. \tag{1.3}$$

The infimum of all b such that (1.3) holds is called the Lipschitz norm of ϕ on E and is denoted $\|\phi\hat{\|}_E$. It is well known that if $E = \mathbf{R}^{n-1}$, then ϕ is differentiable almost everywhere on $\mathbf{R}^{n-1}$ and $\|\phi\hat{\|}_{\mathbf{R}^{n-1}} = \||\nabla\phi|\|_\infty$.

In the following we let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain, i.e., we assume that there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighbourhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned} \Omega \cap B(x_i, 4r_i) &= \{y = (y', y_n) \in \mathbf{R}^n \colon y_n > \phi_i(y')\} \cap B(x_i, 4r_i), \\ \partial\Omega \cap B(x_i, 4r_i) &= \{y = (y', y_n) \in \mathbf{R}^n \colon y_n = \phi_i(y')\} \cap B(x_i, 4r_i), \end{aligned} \tag{1.4}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i. The Lipschitz constant of Ω is defined to be $M = \max_i \||\nabla\phi_i|\|_\infty$. If the defining functions $\{\phi_i\}$ can be chosen to be C^1 regular then we say that Ω is a C^1 domain. If Ω is Lipschitz then there exists $r_0 > 0$ such that if $w \in \partial\Omega$, $0 < r < r_0$, then we can find points $a_r(w) \in \Omega \cap \partial B(w, r)$ with $d(a_r(w), \partial\Omega) \geq c^{-1}r$ for a constant $c = c(M)$. In the following we let $a_r(w)$ denote one such point. Furthermore, if $w \in \partial\Omega$, $0 < r < r_0$, then we let $\Delta(w, r) = \partial\Omega \cap B(w, r)$. Finally we let $e_i, 1 \leq i \leq n$, denote the point in $\mathbf{R}^n$ with one in the ith coordinate position and zeroes elsewhere and we let σ denote surface measure, i.e., the $(n-1)$-dimensional Hausdorff measure, on $\partial\Omega$.

Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and $w \in \partial\Omega$, $0 < r < r_0$. If $0 < b < 1$ and $x \in \Delta(w, 2r)$ then we let

$$\Gamma(x) = \Gamma_b(x) = \{y \in \Omega \colon d(y, \partial\Omega) > b|x - y|\} \cap B(w, 4r). \tag{1.5}$$

Given a measurable function k on $\bigcup_{x\in\Delta(w,2r)} \Gamma(x)$ we define the non tangential maximal function $N(k) \colon \Delta(w, 2r) \to \mathbf{R}$ for k as

$$N(k)(x) = \sup_{y\in\Gamma(x)} |k|(y) \text{ whenever } x \in \Delta(w, 2r). \tag{1.6}$$

We let $L^q(\Delta(w, 2r))$, $1 \leq q \leq \infty$, be the space of functions which are integrable, with respect to the surface measure, σ, to the power q on $\Delta(w, 2r)$. Furthermore, given a measurable function f on $\Delta(w, 2r)$ we say that f is of bounded mean oscillation on $\Delta(w, r)$, $f \in BMO(\Delta(w, r))$, if there exists A, $0 < A < \infty$, such that

$$\int_{\Delta(x,s)} |f - f_\Delta|^2 \, d\sigma \leq A^2 \sigma(\Delta(x, s)) \tag{1.7}$$

whenever $x \in \Delta(w, r)$ and $0 < s \leq r$. Here f_Δ denotes the average of f on $\Delta = \Delta(x, s)$ with respect to the surface measure σ. The least A for which (1.7) holds is denoted by $\|f\|_{BMO(\Delta(w,r))}$. If f is a vector valued function, $f = (f_1, \ldots, f_n)$, then $f_\Delta = (f_{1,\Delta}, \ldots, f_{n,\Delta})$ and the BMO-norm of f is defined as in (1.7) with $|f - f_\Delta|^2 = \langle f - f_\Delta, f - f_\Delta\rangle$. Finally we say that f is of vanishing mean oscillation on $\Delta(w, r)$, $f \in VMO(\Delta(w, r))$, provided for each $\varepsilon > 0$ there is a $\delta > 0$ such that

(1.7) holds with A replaced by ε whenever $0 < s < \min(\delta, r)$ and $x \in \Delta(w,r)$. For more on BMO we refer to [S1, chapter IV].

In this paper we first prove the following two theorems.

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p harmonic function in $\Omega \cap B(w,4r)$, u is continuous in $\bar\Omega \cap \bar B(w,4r)$ and $u = 0$ on $\Delta(w,4r)$. Then*

$$\lim_{y\in\Gamma(x),\, y\to x} \nabla u(y) = \nabla u(x)$$

for σ almost every $x \in \Delta(w,4r)$. Furthermore there exist $q > p$ and a constant c, $1 \le c < \infty$, which both only depend on p, n and M such that

(i) $N(|\nabla u|) \in L^q(\Delta(w,2r))$,

(ii) $$\int_{\Delta(w,2r)} |\nabla u|^q \, d\sigma \le c r^{(n-1)(\frac{p-1-q}{p-1})} \left(\int_{\Delta(w,2r)} |\nabla u|^{p-1} \, d\sigma \right)^{q/(p-1)},$$

(iii) $\log|\nabla u| \in BMO(\Delta(w,r))$, $\quad \|\log|\nabla u|\|_{BMO(\Delta(w,r))} \le c$.

Theorem 2. *Let Ω, M, p, w, r and u be as in the statement of Theorem 1. If, in addition, Ω is C^1 regular then*

$$\log|\nabla u| \in VMO(\Delta(w,r)).$$

Theorem 1 and Theorem 2 are proved in [LN] for p capacitary functions in starlike Lipschitz ring domains. Moreover, using Theorem 2 in [LN1] we can argue in a similar manner to obtain these theorems in general. Concerning converse results we in this paper prove the following theorem.

Theorem 3. *Let Ω, M, p, w, r and u be as in the statement of Theorem 1. Then there exists $\hat M$, independent of u, such that if $M \le \hat M$ and $\log|\nabla u| \in VMO(\Delta(w,r))$, then the outer unit normal to $\Delta(w,r)$ is in $VMO(\Delta(w,r/2))$.*

We let n denote the outer unit normal to $\partial\Omega$. To briefly discuss the proof of Theorem 3 we define

$$\eta = \lim_{\tilde r\to 0} \sup_{\tilde w\in\Delta(w,r/2)} \|n\|_{BMO(\Delta(\tilde w,\tilde r))}. \tag{1.8}$$

To prove Theorem 3 it is enough to prove that $\eta = 0$. To do this we argue by contradiction and assume that (1.8) holds for some $\eta > 0$. This assumption implies that there exist a sequence of points $\{w_j\}$, $w_j \in \Delta(w,r/2)$, and a sequence of scales $\{r_j\}$, $r_j \to 0$, such that $\|n\|_{BMO(\Delta(w_j,r_j))} \to \eta$ as $j \to \infty$. To establish a contradiction we then use a blow-up argument. In particular, let u be as in the statement of Theorem 3 and extend u to $B(w,4r)$ by putting $u = 0$ in $B(w,4r)\setminus\Omega$. For $\{w_j\}$, $\{r_j\}$ as above we define $\Omega_j = \{r_j^{-1}(x - w_j)\colon x \in \Omega\}$ and

$$u_j(z) = \lambda_j u(w_j + r_j z) \text{ whenever } z \in \Omega_j \tag{1.9}$$

where $\{\lambda_j\}$ is an appropriate sequence of real numbers defined in the bulk of the paper. We then show that subsequences of $\{\Omega_j\}$, $\{\partial\Omega_j\}$ converge to Ω_∞, $\partial\Omega_\infty$, in the Hausdorff distance sense, where Ω_∞ is an unbounded Lipschitz domain with Lipschitz constant bounded by M. Moreover, by our choice of the sequence $\{\lambda_j\}$ it follows that a subsequence of $\{u_j\}$ converges uniformly on compact subsets of $\mathbf{R}^n$ to u_∞, a positive p harmonic function in Ω_∞ vanishing continuously on $\partial\Omega_\infty$. Defining $d\mu_j = |\nabla u_j|^{p-1}\, d\sigma_j$, where σ_j is surface measure on $\partial\Omega_j$, it will also follow that a subsequence of $\{\mu_j\}$ converges weakly as Radon measures to μ_∞ and that

$$\int_{\mathbf{R}^n} |\nabla u_\infty|^{p-2}\langle \nabla u_\infty, \nabla\phi\rangle\, dx = -\int_{\partial\Omega_\infty} \phi\, d\mu_\infty \tag{1.10}$$

whenever $\phi \in C_0^\infty(\mathbf{R}^n)$. Moreover, we prove that the limiting measure, μ_∞, and the limiting function, u_∞, satisfy,

$$\mu_\infty = \sigma_\infty \text{ on } \partial\Omega_\infty, \quad c^{-1} \leq |\nabla u_\infty(z)| \leq 1 \text{ whenever } z \in \Omega_\infty. \tag{1.11}$$

In (1.11) σ_∞ is surface measure on $\partial\Omega_\infty$ and c is a constant, $1 \leq c < \infty$, depending only on p, n and M. Using (1.11) and results of Alt, Caffarelli and Friedman [ACF] we are then able to conclude that there exists $\hat{M}$, independent of u_∞, such that if $M \leq \hat{M}$ then (1.10) and (1.11) imply that Ω_∞ is a halfplane. In particular, this will contradict the assumption that η defined in (1.8) is positive. Hence $\eta = 0$ and we are able to complete the proof of Theorem 3. Thus a substantial part of the proof of Theorem 3 is devoted to appropriate limiting arguments in order to conclude (1.10) and (1.11).

Of paramount importance to arguments in this paper is a result in [LN1] (listed as Theorem 2.7 in section 2), stating that the ratio of two positive p harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$ is Hölder continuous up to the boundary. This result implies (see Theorem 2.8 in section 2), that if Ω, M, p, w, r and u are as in the statement of Theorem 1, then there exist c_3, $1 \leq c_3 < \infty$, $\hat{\lambda} > 0$, (both depending only on p, n, M) and $\xi \in \partial B(0,1)$, independent of u, such that if $x \in \Omega \cap B(w, r/c_3)$, then

$$\text{(i)}\ \hat{\lambda}^{-1}\frac{u(x)}{d(x,\partial\Omega)} \leq |\nabla u(x)| \leq \hat{\lambda}\frac{u(x)}{d(x,\partial\Omega)}, \quad \text{(ii)}\ \hat{\lambda}^{-1}\frac{u(x)}{d(x,\partial\Omega)} \leq \langle \nabla u(x), \xi\rangle. \tag{1.12}$$

If (1.12) (i) holds then we say that $|\nabla u|$ satisfies a uniform non-degeneracy condition in $\Omega \cap B(w, r/c_3)$ with constants depending only on p, n and M. Moreover, using this non-degeneracy property of $|\nabla u|$ it follows, by differentiation of (1.2), that if $\zeta = \langle \nabla u, \xi\rangle$, for some $\xi \in \mathbf{R}^n$, $|\xi| = 1$, then ζ satisfies, at x and in $\Omega \cap B(w, r/(2c_3))$, the partial differential equation $L\zeta = 0$, where

$$L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial}{\partial x_j}\right) \tag{1.13}$$

and

$$b_{ij}(x) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](x),\ 1 \leq i,j \leq n. \tag{1.14}$$

In (1.14) δ_{ij} denotes the Kronecker δ. Furthermore,

$$\left(\frac{u(x)}{c\,d(x,\partial\Omega)}\right)^{p-2}|\xi|^2 \le \sum_{i,j=1}^n b_{ij}(x)\xi_i\xi_j \le \left(\frac{c\,u(x)}{d(x,\partial\Omega)}\right)^{p-2}|\xi|^2. \tag{1.15}$$

To make the connection to the proof of Theorems 1–3 we first note that using (1.12)–(1.15) and we can use arguments from [LN] and apply classical theorems for elliptic PDE to get Theorems 1 and 2. The proof of Theorem 3 uses these results and the blow-up argument mentioned above and in the proof particular attention is paid to the proof of the refined upper bound for $|\nabla u_\infty|$ stated in (1.11).

The rest of the paper is organized as follows. In section 2 we state estimates for p harmonic functions in Lipschitz domains and we discuss elliptic measure defined with respect to the operator L defined in (1.13), (1.14). Most of this material is from [LN] and [LN1]. Section 3 is devoted to the proofs of Theorem 1 and Theorem 2. In section 4 we prove Theorem 3. In section 5 we discuss future work on free boundary problems beyond Lipschitz and C^1 domains.

Finally, we emphasize that this paper is not self-contained and that it relies heavily on work in [LN, LN1]. Thus the reader is advised to have these papers at hand[1].

2. Estimates for p harmonic functions in Lipschitz domains

In this section we consider p harmonic functions in a bounded Lipschitz domain $\Omega\subset\mathbf{R}^n$ having Lipschitz constant M. Recall that $\Delta(w,r)=\partial\Omega\cap B(w,r)$ whenever $w\in\partial\Omega$, $0<r$. Throughout the paper c will denote, unless otherwise stated, a positive constant ≥ 1, not necessarily the same at each occurrence, which only depends on p, n and M. In general, $c(a_1,\dots,a_n)$ denotes a positive constant ≥ 1, not necessarily the same at each occurrence, which depends on p, n, M and $a_1,\dots,a_n$. If $A\approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, only depend on p, n and M. Moreover, we let $\max\limits_{B(z,s)} u$, $\min\limits_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z,s)$ whenever $B(z,s)\subset\mathbf{R}^n$ and u is defined on $B(z,s)$.

2.1. Basic estimates. For proofs and for references to the proofs of Lemma 2.1–2.5 stated below we refer to [LN].

Lemma 2.1. *Given p, $1<p<\infty$, let u be a positive p harmonic function in $B(w,2r)$. Then*

$$\text{(i)}\quad r^{p-n}\int\limits_{B(w,r/2)}|\nabla u|^p\,dx \le c(\max_{B(w,r)} u)^p,$$

$$\text{(ii)}\quad \max_{B(w,r)} u \le c\min_{B(w,r)} u.$$

[1]For preprints we refer to www.ms.uky.edu/∼john and www.math.umu.se/personal/nystrom_kaj.

Furthermore, there exists $\alpha = \alpha(p,n,M) \in (0,1)$ such that if $x,y \in B(w,r)$ then

$$\text{(iii)} \quad |u(x)-u(y)| \le c\left(\tfrac{|x-y|}{r}\right)^{\alpha} \max_{B(w,2r)} u.$$

Lemma 2.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and suppose that p is given, $1<p<\infty$. Let $w \in \partial\Omega$, $0<r<r_0$ and suppose that u is a positive p harmonic function in $\Omega \cap B(w,2r)$, continuous in $\bar\Omega \cap B(w,2r)$ and that $u=0$ on $\Delta(w,2r)$. Then*

$$\text{(i)} \quad r^{p-n} \int\limits_{\Omega\cap B(w,r/2)} |\nabla u|^p\,dx \le c(\max_{\Omega\cap B(w,r)} u)^p.$$

Furthermore, there exists $\alpha = \alpha(p,n,M) \in (0,1)$ such that if $x,y \in \Omega \cap B(w,r)$ then

$$\text{(ii)} \quad |u(x)-u(y)| \le c\left(\tfrac{|x-y|}{r}\right)^{\alpha} \max_{\Omega\cap B(w,2r)} u.$$

Lemma 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and suppose that p is given, $1<p<\infty$. Let $w \in \partial\Omega$, $0<r<r_0$, and suppose that u is a positive p harmonic function in $\Omega \cap B(w,2r)$, continuous in $\bar\Omega \cap B(w,2r)$ and that $u=0$ on $\Delta(w,2r)$. There exists $c=c(p,n,M)\ge 1$ such that if $\bar r = r/c$, then*

$$\max_{\Omega\cap B(w,\bar r)} u \le c u(a_{\bar r}(w)).$$

Lemma 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and suppose that p is given, $1<p<\infty$. Let $w \in \partial\Omega$, $0<r<r_0$ and suppose that u is a positive p harmonic function in $\Omega \cap B(w,4r)$, continuous in $\bar\Omega \cap B(w,4r)$ and that $u=0$ on $\Delta(w,4r)$. Extend u to $B(w,4r)$ by defining $u \equiv 0$ on $B(w,4r)\setminus\Omega$. Then u has a representative in $W^{1,p}(B(w,4r))$ with Hölder continuous partial derivatives in $\Omega\cap B(w,4r)$. In particular, there exists $\sigma \in (0,1]$, depending only on p, n such that if $B(\tilde w,4\tilde r) \subset \Omega \cap B(w,4r)$ and $x,y \in B(\tilde w,\tilde r/2)$, then*

$$c^{-1}|\nabla u(x)-\nabla u(y)| \le (|x-y|/\tilde r)^{\sigma} \max_{B(\tilde w,\tilde r)} |\nabla u| \le c\tilde r^{-1}(|x-y|/\tilde r)^{\sigma} \max_{B(\tilde w,2\tilde r)} u.$$

Lemma 2.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Given p, $1<p<\infty$, $w\in\partial\Omega$, $0<r<r_0$, suppose that u is a positive p harmonic function in $\Omega\cap B(w,2r)$, continuous in $\bar\Omega\cap B(w,2r)$ with $u=0$ on $\Delta(w,2r)$. Extend u to $B(w,2r)$ by defining $u\equiv 0$ on $B(w,2r)\setminus\Omega$. Then there exists a unique finite positive Borel measure μ on $\mathbf{R}^n$, with support in $\Delta(w,2r)$, such that*

$$\text{(i)} \quad \int\limits_{\mathbf{R}^n} |\nabla u|^{p-2}\langle \nabla u, \nabla\phi\rangle\,dx = -\int\limits_{\mathbf{R}^n} \phi\,d\mu$$

whenever $\phi \in C_0^\infty(B(w,2r))$. Moreover, there exists $c=c(p,n,M)\ge 1$ such that if $\bar r = r/c$, then

$$\text{(ii)} \quad c^{-1}\bar r^{p-n}\mu(\Delta(w,\bar r)) \le (u(a_{\bar r}(w)))^{p-1} \le c\bar r^{p-n}\mu(\Delta(w,\bar r)).$$

2.2. Refined estimates. In the following we state a number of results and estimates proved in [LN] and [LN1]. In particular, for the proof of Theorems 2.6–2.8 stated below we refer to [LN] and [LN1] and we note that Theorem 2.8 is referred to as Lemma 4.28 in [LN1] while Theorem 2.6 and Theorem 2.7 are two of the main results established in [LN] and [LN1] respectively.

Theorem 2.6. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u and v are positive p harmonic functions in $\Omega \cap B(w, 2r)$. Assume also that u and v are continuous in $\bar{\Omega} \cap B(w, 2r)$, and $u = 0 = v$ on $\Delta(w, 2r)$. Under these assumptions there exists c_1, $1 \leq c_1 < \infty$, depending only on p, n and M, such that if $\tilde{r} = r/c_1$, $u(a_{\tilde{r}}(w)) = v(a_{\tilde{r}}(w)) = 1$, and $y \in \Omega \cap B(w, \tilde{r})$, then*

$$\frac{u(y)}{v(y)} \leq c_1.$$

Theorem 2.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u and v are positive p harmonic functions in $\Omega \cap B(w, 2r)$. Assume also that u and v are continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0 = v$ on $\Delta(w, 2r)$. Under these assumptions there exist c_2, $1 \leq c_2 < \infty$, and $\alpha \in (0, 1)$, both depending only on p, n and M, such that if $y_1, y_2 \in \Omega \cap B(w, r/c_2)$ then*

$$\left|\log\left(\frac{u(y_1)}{v(y_1)}\right) - \log\left(\frac{u(y_2)}{v(y_2)}\right)\right| \leq c_2\left(\frac{|y_1 - y_2|}{r}\right)^{\alpha}.$$

Theorem 2.8. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that (1.4) holds with x_i, r_i, ϕ_i replaced by w, r, ϕ. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p harmonic function in $\Omega \cap B(w, 2r)$. Assume also that u is continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Then there exist c_3, $1 \leq c_3 < \infty$ and $\hat{\lambda} > 0$, depending only on p, n and M, such that if $y \in \Omega \cap B(w, r/c_3)$ then*

$$\hat{\lambda}^{-1}\frac{u(y)}{d(y, \partial\Omega)} \leq \langle \nabla u(y), e_n \rangle \leq |\nabla u(y)| \leq \hat{\lambda}\frac{u(y)}{d(y, \partial\Omega)}.$$

We note that Lemmas 2.9–2.12 below are stated and proved, for p capacitary functions in starlike Lipschitz ring domains, as Lemma 2.5 (iii), Lemma 2.39, Lemma 2.45 and Lemma 2.54 in [LN]. However armed with Theorem 2.8 the proofs of these lemmas can be extended to the more general situation of positive p harmonic functions vanishing on a portion of the boundary of a Lipschitz domain. Lemma 2.9 is only stated as it is used in the proof of Lemmas 2.10–2.12 as outlined in [LN], while Lemmas 2.10-2.12 are used in the proof of Theorems 1–3. We refer to [LN] for details (see also the discussion after Lemma 2.8 in [LN1]).

Lemma 2.9. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p harmonic* *function in* $\Omega \cap B(w, 2r)$ *and that* u *is continuous in* $\bar{\Omega} \cap B(w, 2r)$ *with* $u = 0$ *on* $\Delta(w, 2r)$. *Then there there exists a constant* $c = c(p, n, M)$, $1 \le c < \infty$, *such that*

$$\max_{B(x,\frac{s}{2})} \sum_{i,j=1}^{n} |u_{y_i y_j}| \le c \Bigg(s^{-n} \int\limits_{B(x,3s/4)} \sum_{i,j=1}^{n} |u_{y_i y_j}|^2 \, dy \Bigg)^{1/2} \le c^2 u(x)/d(x, \partial\Omega)^2$$

whenever $x \in \Omega \cap B(w, r/c)$ *and* $0 < s \le d(x, \partial\Omega)$.

Lemma 2.10. *Let* Ω, M, p, w, r *and* u *be as in the statement of Lemma 2.9. Let* μ *be as in Lemma 2.5. Then there exists a constant* $c = c(p, n, M)$, $1 \le c < \infty$, *such that* $d\mu/d\sigma = k^{p-1}$ *on* $\Delta(w, 2r/c)$ *and*

$$\int\limits_{\Delta(w,r/c)} k^p \, d\sigma \le c r^{-\frac{n-1}{p-1}} \Bigg(\int\limits_{\Delta(w,r/c)} k^{p-1} \, d\sigma \Bigg)^{p/(p-1)}.$$

Recall that a bounded domain $\Omega \subset \mathbf{R}^n$ is said to be starlike Lipschitz, with respect to $\hat{x} \in \Omega$, provided $\partial\Omega = \{\hat{x} + R(\omega)\omega \colon \omega \in \partial B(0,1)\}$ where $\log R \colon \partial B(0,1) \to \mathbf{R}$ is Lipschitz on $\partial B(0,1)$. We refer to $\| \log R \hat{\|}_{\partial B(0,1)}$ as the Lipschitz constant for Ω and we observe that this constant is invariant under scalings about $\hat{x}$.

Lemma 2.11. *Let* Ω, M, p, w, r *and* u *be as in the statement of Lemma 2.9. Then there exist a constant* $c = c(p, n, M)$, $1 \le c < \infty$, *and a starlike Lipschitz domain* $\tilde{\Omega} \subset \Omega \cap B(w, 2r)$, *with center at a point* $\tilde{w} \in \Omega \cap B(w, r)$, $d(\tilde{w}, \partial\Omega) \ge c^{-1} r$, *and with Lipschitz constant bounded by* c, *such that*

$$c\sigma(\partial\tilde{\Omega} \cap \Delta(w, r)) \ge r^{n-1}.$$

Moreover, the following inequality is valid for all $x \in \tilde{\Omega}$,

$$c^{-1} r^{-1} u(\tilde{w}) \le |\nabla u(x)| \le c r^{-1} u(\tilde{w}).$$

Lemma 2.12. *Let* Ω, M, p, w, r *and* u *be as in the statement of Lemma 2.9. Let* $\tilde{\Omega}$ *be constructed as in Lemma 2.11. Define, for* $y \in \tilde{\Omega}$, *the measure*

$$d\tilde{\gamma}(y) = d(y, \partial\tilde{\Omega}) \max_{B(y, \frac{1}{2} d(y, \partial\tilde{\Omega}))} \{ |\nabla u|^{2p-6} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \} \, dy.$$

Then $\tilde{\gamma}$ *is a Carleson measure on* $\tilde{\Omega}$ *and there exists a constant* $c = c(p, n, M)$, $1 \le c < \infty$, *such that if* $z \in \partial\tilde{\Omega}$ *and* $0 < s < r$, *then*

$$\tilde{\gamma}(\tilde{\Omega} \cap B(z, s)) \le c s^{n-1} (u(\tilde{w})/r)^{2p-4}.$$

Let u, $\tilde{\Omega}$, be as in Lemma 2.12. We end this section by considering the divergence form operator L defined as in (1.13), (1.14), relative to u, $\tilde{\Omega}$. In particular, we state a number of results for this operator which we will make use of in the following sections. Arguing as above (1.13) we first observe that

$$L(\langle \nabla u, \xi \rangle) = 0 \text{ weakly in } \tilde{\Omega} \tag{2.13}$$

whenever $\xi \in \partial B(0,1)$. Moreover, using Theorem 2.8, Lemma 2.11, and (1.15) we see that L is uniformly elliptic in $\tilde\Omega$. Using this fact it follows from [CFMS] that if $z \in \partial\tilde\Omega$, $0 < s < r$, and if v is a weak solution to L in $\tilde\Omega$ which vanishes continuously on $\partial\tilde\Omega \cap B(z,s)$, then there exist τ, $0 < \tau \le 1$, and $c \ge 1$, both depending only on p, n, M, such that

$$\max_{\tilde\Omega\cap B(z,t)} v \le c\,(t/s)^{\tau} \max_{\tilde\Omega\cap B(z,s)} v, \ \text{ whenever } 0 < t \le s. \tag{2.14}$$

Moreover, using Lemma 2.12 we observe that if

$$d\,\theta(y) = d(y,\partial\tilde\Omega) \max_{B(y,\frac12 d(y,\partial\tilde\Omega))} \{\sum_{i,j=1}^{n} |\nabla b_{ij}|^2\}\,dy,$$

where $\{b_{ij}\}$ is the matrix defining L in (1.14), then θ is a Carleson measure on $\tilde\Omega$ and

$$\theta(\tilde\Omega \cap B(z,s)) \le cs^{n-1}(u(\tilde w)/r)^{2p-4}$$

whenever $z \in \partial\tilde\Omega$ and $0 < s < r$. Let $\tilde\omega(\cdot,\tilde w)$ be elliptic measure defined with respect to L, $\tilde\Omega$, and $\tilde w$ (see [CFMS] for the definition of elliptic measure). We note that the above observation and the main theorem in [KP] imply the following lemma.

Lemma 2.15. *Let u, $\tilde\Omega$, $\tilde w$ be as in Lemma 2.12 and let L be defined as in (1.13), (1.14), relative to u, $\tilde\Omega$. Then $\tilde\omega(\cdot,\tilde w)$ and the surface measure on $\partial\tilde\Omega$ (denoted $\tilde\sigma$) are mutually absolutely continuous. Moreover, $\tilde\omega(\cdot,\tilde w)$ is an A^∞ weight with respect to $\tilde\sigma$. Consequently, there exist $c \ge 1$ and γ, $0 < \gamma \le 1$, depending only on p, n, M, such that*

$$\frac{\tilde\omega(E,\tilde w)}{\tilde\omega(\partial\tilde\Omega \cap B(z,s),\tilde w)} \le c\left(\frac{\tilde\sigma(E)}{\tilde\sigma(\partial\tilde\Omega\cap B(z,s))}\right)^{\gamma}$$

whenever $z \in \partial\tilde\Omega$, $0 < s < r$, and $E \subset \partial\tilde\Omega \cap B(z,s)$ is a Borel set.

For several other equivalent definitions of A^∞ weights we refer to [CF] or [GR].

3. Proof of Theorem 1 and Theorem 2

In this section we prove Theorem 1 and Theorem 2. Hence we let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M and for given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$ we suppose that u is a positive p harmonic function in $\Omega \cap B(w,4r)$, continuous in $\bar\Omega \cap \bar B(w,4r)$ with $u = 0$ on $\Delta(w,4r)$.

3.1. Proof of Theorem 1. We first note that we can assume, without loss of generality, that

$$\max_{\Omega\cap B(w,4r)} u = 1. \tag{3.1}$$

We extend u to $B(w,4r)$ by defining $u \equiv 0$ on $B(w,4r) \setminus \Omega$ and we let μ be the measure associated to u as in the statement of Lemma 2.5. Using Lemma 2.10, Lemma 2.5 (ii) and the Harnack inequality for p harmonic functions we see that if $y \in \partial\Omega$, $s > 0$ and $B(y, 2cs) \subset B(w, 4r)$, then $d\mu/d\sigma = k^{p-1}$ on $\Delta(y, 2s)$ and

$$\int_{\Delta(y,s)} k^p \, d\sigma \le c s^{-\frac{n-1}{p-1}} \Big(\int_{\Delta(y,s/2)} k^{p-1} \, d\sigma \Big)^{p/(p-1)}. \tag{3.2}$$

(3.2) and Lemma 2.5 (ii) imply (see [G], [CF]) that for some $q' > p$, depending only on p, n and M, we have

$$\int_{\Delta(w,3r)} k^{q'} \, d\sigma \le c r^{-\frac{(n-1)(q'+1-p)}{p-1}} \Big(\int_{\Delta(w,3r)} k^{p-1} \, d\sigma \Big)^{q'/(p-1)}. \tag{3.3}$$

Let $y \in \Delta(w, 2r)$ and let $z \in \Gamma(y) \cap B(y, r/(4c_3))$, where c_3 is the constant appearing in the statement of Theorem 2.8 and $\Gamma(y)$, for $y \in \Delta(w, 2r)$, is defined in (1.5). Using Theorem 2.8, with w replaced by y, $s = |z - y|$ and Lemma 2.5 (ii) we obtain

$$\begin{aligned} |\nabla u(z)| &\le c \frac{u(z)}{s} \le c^2 s^{-1} \Big(s^{p-n} \mu(\Delta(y,s)) \Big)^{1/(p-1)} \\ &= c^2 \Big(s^{1-n} \int_{\Delta(y,s)} k^{p-1} d\sigma \Big)^{1/(p-1)} \le c^2 (M(k^{p-1})(y))^{1/(p-1)}. \end{aligned} \tag{3.4}$$

In (3.4),

$$M(f)(y) = \sup_{0<s<r/4} s^{1-n} \int_{\Delta(y,s)} f \, d\sigma$$

whenever f is an integrable function on $\Delta(w, 3r)$. Next we define

$$N_1(|\nabla u|)(y) = \sup_{\Gamma(y) \cap B(y, r/(4c_3))} |\nabla u| \text{ whenever } y \in \Delta(w, 2r).$$

Using (3.3), (3.4) and the Hardy–Littlewood maximal theorem we see that if $q = (q' + p)/2$ then

$$\begin{aligned} \int_{\Delta(w,2r)} N_1(|\nabla u|)^q d\sigma &\le c \int_{\Delta(w,2r)} M(k^{p-1})^{q/(p-1)} \, d\sigma \\ &\le c^2 r^{-\frac{(n-1)(q+1-p)}{p-1}} \Big(\int_{\Delta(w,2r)} k^{p-1} \, d\sigma \Big)^{q/(p-1)}. \end{aligned} \tag{3.5}$$

Using Lemma 2.4 and (3.1) we also see that $|\nabla u(x)| \le c r^{-1}$ whenever $x \in \Gamma(y) \setminus B(y, r/(4c_3))$ and $y \in \Delta(w, 2r)$. Thus $N(|\nabla u|) \le N_1(|\nabla u|) + c r^{-1}$ on $\Delta(w, 2r)$. Therefore, using (3.5) as well as Lemma 2.5 (ii) and (3.1) once again we can conclude that statement (i) of Theorem 1 is true.

Next we prove by a contradiction argument that ∇u has non tangential limits for σ almost every $y \in \Delta(w, 4r)$. To argue by contradiction we suppose

$$\begin{array}{l} \text{that there exists a set } F \subset \Delta(w,4r),\ \sigma(F) > 0, \text{ such that if } y \in F \\ \text{then the limit of } \nabla u(z), \text{ as } z \to y \text{ with } z \in \Gamma(y), \text{ does not exist.} \end{array} \tag{3.6}$$

Assuming (3.6) we let $y \in F$ be a point of density for F with respect to σ. Then

$$t^{1-n}\sigma(\Delta(y,t) \setminus F) \to 0 \text{ as } t \to 0,$$

so we can conclude that if $t > 0$ is small enough, then

$$c\sigma(\partial\tilde{\Omega} \cap \Delta(y,t) \cap F) \geq t^{n-1}$$

where $\tilde{\Omega} \subset \Omega$ is the starlike Lipschitz domain defined in Lemma 2.11 with w, $\tilde{w}$, r replaced by y, $\tilde{y}$, t. Using Lemma 2.11 we also see that $|\nabla u| \approx C$ in $\tilde{\Omega}$ for some constant C. Let L be defined as in (1.13), (1.14), relative to u, $\tilde{\Omega}$. Then, from (2.13), (1.15) and the fact $|\nabla u| \approx C$ in $\tilde{\Omega}$, we have that L is uniformly elliptic on $\tilde{\Omega}$ and $Lu_{x_k} = 0$ weakly in $\tilde{\Omega}$. Moreover, since u_{x_k} is bounded on $\tilde{\Omega}$ for $1 \leq k \leq n$, we can therefore conclude, by well known arguments, see [CFMS], that u_{x_k} has non tangential limits at almost every boundary point of $\tilde{\Omega}$ with respect to elliptic measure, $\tilde{\omega}(\cdot, \tilde{y})$, associated with the operator L, the domain $\tilde{\Omega}$, and the point $\tilde{y}$. Now from Lemma 2.15 we see that $\tilde{\omega}(\cdot, \tilde{y})$ and surface measure, $\tilde{\sigma}$, on $\partial\tilde{\Omega}$ are mutually absolutely continuous. Hence u_{x_k} has non tangential limits at $\tilde{\sigma}$ almost every boundary point. Since non tangential limits in $\tilde{\Omega}$ agree with those in Ω, for σ almost every point in F, we deduce that this latter statement contradicts the assumption made in (3.6) that $\sigma(F) > 0$. Hence ∇u has non tangential limits for σ almost every $y \in \Delta(w, 4r)$.

In the following we let $\nabla u(y), y \in \Delta(w, 2r)$, denote the non tangential limit of ∇u whenever this limit exists. To prove statement (ii) of Theorem 1 we argue as follows. Let $y \in \Delta(w, 2r)$ and put $\tilde{r} = r/(4c_3)$ where c_3 is the constant appearing in the statement of Theorem 2.8. Using Theorem 2.8 we note, to start with, that $B(y, 2\tilde{r}) \cap \{u = t\}$, for $0 < t$ sufficiently small, can be represented as the graph of a Lipschitz function with Lipschitz constant bounded by $c = c(p, n, M)$, $1 \leq c < \infty$. In particular, c can be chosen independently of t. In fact we can conclude, see [LN, Lemma 2.4] for the proof, that u is infinitely differentiable and hence that $B(y, 2\tilde{r}) \cap \{u = t\}$ is a C^∞ surface. Let $d\mu_t = |\nabla u|^{p-1}\, d\sigma_t$ where σ_t is surface measure on $B(y, 2\tilde{r}) \cap \{u = t\}$. Using the definition of μ it is easily seen that μ_t converges weakly to μ as defined in Lemma 2.5 on $B(y, 2\tilde{r}) \cap \Omega$. Using the implicit function theorem, we can express $d\sigma_t$ and also $d\mu_t$ locally as measures on $\mathbf{R}^{n-1}$. Doing this, using non tangential convergence of ∇u, Theorem 1 (i), and dominated convergence we see first that

$$k(y) = |\nabla u|(y) \text{ and } d\mu = |\nabla u|^{p-1}\, d\sigma. \tag{3.7}$$

Then, using (3.7), (3.3), Lemma 2.5 (ii) and the Harnack inequality for p harmonic functions it follows that Theorem 1 (ii) holds. Finally, Theorem 1 (iii) follows from

Theorem 1 (ii) by standard arguments, see [CF]. The proof of Theorem 1 is therefore complete. $\square$

3.2. Proof of Theorem 2. Let Ω, M, p, w, r and u be as in the statement of Theorem 1. We prove that there exist $0 < \varepsilon_0$ and $\tilde r = \tilde r(\varepsilon)$, for $\varepsilon \in (0, \varepsilon_0)$, such that whenever $y \in \Delta(w, r)$ and $0 < s < \tilde r(\varepsilon)$ then

$$\fint_{\Delta(y,s)} |\nabla u|^p \, d\sigma \le (1+\varepsilon)\Big(\fint_{\Delta(y,s)} |\nabla u|^{p-1} \, d\sigma\Big)^{p/(p-1)}. \tag{3.8}$$

Here

$$\fint_E f \, d\sigma = (\sigma(E))^{-1} \int_E f \, d\sigma$$

whenever $E \subset \partial\Omega$ is Borel measurable with finite positive σ measure and f is a σ integrable function on E. Theorem 2 then follows, once (3.8) is established, from a lemma of Sarason, see [KT]. To prove (3.8) we argue by contradiction. Indeed, if (3.8) is false then

$$\begin{aligned}&\text{there exist two sequences } \{y_m\}_1^\infty,\ \{s_m\}_1^\infty \text{ satisfying } y_m \in \Delta(w, r)\\ &\text{and } s_m \to 0 \text{ as } m \to \infty \text{ such that (3.8) is false with}\\ &y,\ s \text{ replaced by } y_m,\ s_m \text{ for } m \in \mathbf{Z}_+ = \{1, 2, \dots\}.\end{aligned} \tag{3.9}$$

To continue we first note that using the assumption that Ω is C^1 regular it follows that $\Delta(w, 2r)$ is Reifenberg flat with vanishing constant. That is, for given $\hat\varepsilon > 0$, small, there exists a $\hat r = \hat r(\hat\varepsilon) < 10^{-6} r$, such that whenever $y \in \Delta(w, 2r)$ and $0 < s \le \hat r$, then

$$\begin{aligned}&\{z + tn \in B(y, s),\ z \in P,\ t > \hat\varepsilon s\} \subset \Omega,\\ &\{z - tn \in B(y, s),\ z \in P,\ t > \hat\varepsilon s\} \subset \mathbf{R}^n \setminus \bar\Omega.\end{aligned} \tag{3.10}$$

In (3.10) $P = P(y, s)$ is the tangent plane to $\Delta(w, 2r)$ relative to y, s, and $n = n(y)$ is the inner unit normal to $\partial\Omega$ at $y \in \Delta(w, 2r)$. We let, for each $m \in \mathbf{Z}_+$, $P(y_m) = P(y_m, s_m)$ denote the tangent plane to $\Delta(w, 2r)$ relative to y_m, s_m where y_m, s_m are as in (3.9).

In the following we let $A = e^{1/\varepsilon}$ and note that if we choose ε_0, and hence ε, sufficiently small then A is large. Moreover, for fixed $A > 10^6$ we choose $\hat\varepsilon = \hat\varepsilon(A) > 0$ in (3.10) so small that if $y_m' = y_m + A s_m n(y_m)$, then the domain $\Omega(y_m')$, obtained by drawing all line segments from points in $B(y_m', A s_m/4)$ to points in $\Delta(y_m, A s_m)$, is starlike Lipschitz with respect to y_m'. We assume, as we may, that $s_m \le \hat r(\hat\varepsilon)$ for $m \in \mathbf{Z}_+$ and we put $D_m = \Omega(y_m') \setminus \bar B(y_m', A s_m/8)$. From C^1 regularity of Ω we also see that D_m, for $m \in \mathbf{Z}_+$, has Lipschitz constant $\le c$ where c is an absolute constant. To continue we let u_m be the p capacitary function for D_m and we put $u_m \equiv 0$ on $\mathbf{R}^n \setminus \bar\Omega(y_m')$. From Theorem 2.7 with w, r, u_1, u_2 replaced by y_m, $A s_m/100$, u, u_m

we deduce that if $w_1, w_2 \in \Omega \cap B(y_m, 2s_m)$, then

$$\left|\log\left(\frac{u_m(w_1)}{u(w_1)}\right) - \log\left(\frac{u_m(w_2)}{u(w_2)}\right)\right| \leq cA^{-\alpha} \tag{3.11}$$

whenever m is large enough. The constants c, α in (3.11) are the constants in Theorem 2.7 and these constants are independent of m. If we let $w_1, w_2 \to z_1, z_2 \in \Delta(y_m, 2s_m)$ in (3.11) and use Theorem 1, we get, for σ almost all z_1, $z_2 \in \Delta(y_m, 2s_m)$, that

$$\left|\log\left(\frac{|\nabla u_m(z_1)|}{|\nabla u(z_1)|}\right) - \log\left(\frac{|\nabla u_m(z_2)|}{|\nabla u(z_2)|}\right)\right| \leq cA^{-\alpha}. \tag{3.12}$$

Therefore, taking exponentials in the inequality in (3.12) we see that, for A large enough,

$$(1-\tilde{c}A^{-\alpha})\frac{|\nabla u_m(z_1)|}{|\nabla u_m(z_2)|} \leq \frac{|\nabla u(z_1)|}{|\nabla u(z_2)|} \leq (1+\tilde{c}A^{-\alpha})\frac{|\nabla u_m(z_1)|}{|\nabla u_m(z_2)|}, \tag{3.13}$$

whenever $z_1, z_2 \in \Delta(y_m, 2s_m)$ and where $\tilde{c}$ depends only on p, n, and the Lipschitz constant for Ω. Using (3.13) we first obtain that

$$\frac{\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^p \, d\sigma}{\left(\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^{p-1} \, d\sigma\right)^{p/(p-1)}} \geq (1-cA^{-\alpha})\frac{\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u|^p \, d\sigma}{\left(\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u|^{p-1} \, d\sigma\right)^{p/(p-1)}}. \tag{3.14}$$

Secondly, using the assumption that (3.8) is false and (3.9), we from (3.14) obtain that

$$\frac{\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^p \, d\sigma}{\left(\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^{p-1} \, d\sigma\right)^{p/(p-1)}} \geq (1-cA^{-\alpha})(1+\varepsilon). \tag{3.15}$$

Next for $m \in \mathbf{Z}_+$, let T_m be a conformal affine mapping of $\mathbf{R}^n$ which maps the origin and e_n onto y_m and y'_m respectively and which maps $W = \{x \in \mathbf{R}^n \colon x_n = 0\}$ onto $P(y_m)$. T_m is the composition of a translation, rotation, dilation. Let D'_m, u'_m be such that $T_m(D'_m) = D_m$ and $u_m(T_m x) = u'_m(x)$ whenever $x \in D'_m$. Since the p Laplace equation is invariant under translations, rotations, and dilations, we see that u'_m is the p capacitary function for D'_m. Also, as

$$\frac{\displaystyle\fint_{\partial D'_m\cap B(0,1/A)} |\nabla u'_m|^p \, d\sigma'_m}{\left(\displaystyle\fint_{\partial D'_m\cap B(0,1/A)} |\nabla u'_m|^{p-1} \, d\sigma'_m\right)^{p/(p-1)}} = \frac{\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^p \, d\sigma}{\left(\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^{p-1} \, d\sigma\right)^{p/(p-1)}},$$

where σ'_m is the surface measure on $\partial D'_m$, we see, using (3.15), that

$$\frac{\displaystyle\fint_{\partial D'_m\cap B(0,1/A)} |\nabla u'_m|^p \, d\sigma'_m}{\left(\displaystyle\fint_{\partial D'_m\cap B(0,1/A)} |\nabla u'_m|^{p-1} \, d\sigma'_m\right)^{p/(p-1)}} \geq (1 - cA^{-\alpha})(1+\varepsilon). \tag{3.16}$$

Letting $m \to \infty$ we see from Lemmas 2.1, 2.2 and 2.3 that u'_m converges uniformly on $\mathbf{R}^n$ to u' where u' is the p capacitary function for the starlike Lipschitz ring domain, $D' = \Omega' \setminus B(e_n, 1/8)$. Also Ω' is obtained by drawing all line segments connecting points in $B(0,1) \cap W$ to points in $B(e_n, 1/4)$. We can now repeat, essentially verbatim, the argument in [LN, Lemma 5.28, (5.29)–(5.41)], to conclude that

$$\limsup_{m\to\infty} \frac{\displaystyle\fint_{\partial D'_m\cap B(0,1/A)} |\nabla u'_m|^p \, d\sigma'_m}{\left(\displaystyle\fint_{\partial D'_m\cap B(0,1/A)} |\nabla u'_m|^{p-1} \, d\sigma'_m\right)^{p/(p-1)}} \leq \frac{\displaystyle\fint_{W\cap B(0,1/A)} |\nabla u'|^p \, dx'}{\left(\displaystyle\fint_{W\cap B(0,1/A)} |\nabla u'|^{p-1} \, dx'\right)^{p/(p-1)}}. \tag{3.17}$$

Here dx' denotes surface measure on W. To complete the argument we show that (3.17) leads to a contradiction to our original assumption. Note that it follows from Schwarz reflection that u' has a p harmonic extension to $B(0,1/8)$ with $u' \equiv 0$ on $W \cap B(0,1/8)$. From barrier estimates we have $c^{-1} \leq |\nabla u'| \leq c$ on $B(0,1/16)$ where c depends only on p, n, and from Lemma 2.4 we find that $|\nabla u'|$ is Hölder continuous with exponent $\theta = \theta(p,n)$ on $W \cap \bar{B}(0,1/16)$. In fact in this case we could take $\theta = 1$. Therefore, using these facts we first conclude that, for some c,

$$(1 - cA^{-\theta})|\nabla u'(0)| \leq |\nabla u'(z)| \leq (1 + cA^{-\theta})|\nabla u'(0)|$$

whenever $z \in B(0,1/A)$ and then from (3.16), (3.17) that

$$(1 + cA^{-\theta}) \geq \frac{\displaystyle\fint_{W\cap B(0,1/A)} |\nabla u'|^p \, dx'}{\left(\displaystyle\fint_{W\cap B(0,1/A)} |\nabla u'|^{p-1} \, dx'\right)^{p/(p-1)}} \geq (1 - cA^{-\alpha})(1+\varepsilon).$$

As $A = e^{1/\varepsilon}$ the last inequality clearly can not hold if we choose ε_0, and hence ε, sufficiently small. From this contradiction we conclude that our original assumption was false, i.e., (3.9) can not hold. Hence (3.8) holds. This completes the proof of Theorem 2. □

4. Proof of Theorem 3

In this section we prove Theorem 3. Our argument is similar to the argument in [KT2], in that we argue by way of contradiction to get a sequence of blow-ups as in (1.8)–(1.10). We then use a theorem of [ACF] to show that a subsequence of this sequence converges to a linear function which turns out to be a contradiction. However, our argument is less voluminous and seems simpler to us than the one in [KT2]. The following lemma plays a key role in our blow-up argument.

4.1. A refined version of Lemma 2.11.

Lemma 4.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p harmonic function in $\Omega \cap B(w, 2r)$, u is continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Suppose also that $\log|\nabla u| \in VMO(\Delta(w, r))$. Given $\varepsilon > 0$ there exist $\tilde{r} = \tilde{r}(\varepsilon)$, $0 < \tilde{r} < r$, and $c = c(p, n, M)$, $1 \le c < \infty$, such that the following is true whenever $0 < r' \le \tilde{r}$. There exists a starlike Lipschitz domain $\tilde{\Omega} \subset \Omega\cap B(w, cr') \subset \Omega\cap B(w, r)$, with center at a point $\hat{w} \in \Omega\cap B(w, cr')$, $d(\hat{w}, \partial\Omega) \ge r'$, and with Lipschitz constant bounded by c, such that*

(a) $\dfrac{\sigma(\partial\tilde{\Omega} \cap \Delta(w, r'))}{\sigma(\Delta(w, r'))} \ge 1 - \varepsilon,$

(b) $(1 - \varepsilon)b^{p-1} \le \dfrac{\mu(\Delta(y, s))}{\sigma(\Delta(y, s))} \le (1 + \varepsilon)b^{p-1}$ *whenever* $0 < s < r',\ y \in \partial\tilde{\Omega} \cap \Delta(w, r')$.

Here μ is the measure associated with u as in Lemma 2.5 and $\log b$ is the average of $\log|\nabla u|$ on $\Delta(w, 4r')$. Moreover, for all $x \in \tilde{\Omega}$

$$c^{-1}\frac{u(\hat{w})}{r'} \,\le\, |\nabla u(x)| \,\le\, c\frac{u(\hat{w})}{r'}.$$

Proof. In the following we let $\tilde{\varepsilon} > 0$ and $r^*(\tilde{\varepsilon}) \ll r$ be small positive numbers. For the moment we allow $\tilde{\varepsilon}$ and r^* to vary but we shall later fix these numbers to satisfy several conditions depending on ε. Using the assumption that $\log|\nabla u| \in VMO(\Delta(w, r))$ we see there exists $\hat{r}$, $0 < \hat{r} \le r^*$, such that $\log|\nabla u| \in BMO(\Delta(w, 8\hat{r}))$ with BMO norm less than or equal to $\tilde{\varepsilon}^3$. Let A denote the average of $f = \log|\nabla u|$ with respect to surface measure over $\Delta(w, 4\hat{r})$. Using the definition of BMO, see (1.7), we have

$$(4.2)\quad \frac{\tilde{\varepsilon}\,\sigma(\{x \in \Delta(w, 4\hat{r})\colon |f(x) - A| > \tilde{\varepsilon}\})}{\sigma(\Delta(w, 4\hat{r}))} \le (\sigma(\Delta(w, 4\hat{r}))^{-1} \int\limits_{\Delta(w,4\hat{r})} |f - A|\, d\sigma \le c\tilde{\varepsilon}^3.$$

If $b = e^A$, then from (4.2) we see

$$\text{that there exists a set } E \subset \Delta(w, 4\hat{r}) \text{ such that } (1 - c\tilde{\varepsilon})b \leq |\nabla u| \leq (1 + c\tilde{\varepsilon})b \text{ on } E \text{ and if } F = \Delta(w, 4\hat{r}) \setminus E \text{ then } \sigma(F) \leq c\tilde{\varepsilon}^2\, \sigma(\Delta(w, 4\hat{r})). \tag{4.3}$$

In (4.3), c is a universal constant. We introduce, for σ integrable functions h defined on $\Delta(w, 5\hat{r})$ and for $x \in \Delta(w, 4\hat{r})$, the maximal function

$$M(h)(x) = \sup_{0<s<\hat{r}} \frac{1}{\sigma(\Delta(x,s))} \int_{\Delta(x,s)} h\, d\sigma.$$

Let $G = \{x \in \Delta(w, 4\hat{r})\colon M(\chi_F)(x) \leq \tilde{\varepsilon}\}$ where χ_F is the indicator functions for the set F introduced in (4.3) and define $K = \Delta(w, 4\hat{r}) \setminus G$. Using weak type estimates for the maximal function, see [S], it then follows that

$$\sigma(K) \leq c\tilde{\varepsilon}\sigma(\Delta(w, 4\hat{r})). \tag{4.4}$$

Let $y \in G \cap \Delta(w, \hat{r})$, $0 < s \leq \hat{r}$. Then from Lemma 2.10, Theorem 1 and (3.7) we deduce

$$\begin{aligned} \mu(\Delta(y,s)) &= \int_{\Delta(y,s)} |\nabla u|^{p-1}\, d\sigma = \int_{E\cap\Delta(y,s)} |\nabla u|^{p-1}\, d\sigma + \int_{F\cap\Delta(y,s)} |\nabla u|^{p-1}\, d\sigma \\ &= T_1 + T_2. \end{aligned} \tag{4.5}$$

From the definitions of the sets E, F, G, we see that

$$(1 - c\tilde{\varepsilon})\, b^{p-1}\sigma(\Delta(y,s)) \leq T_1 \leq (1 + c\tilde{\varepsilon})\, b^{p-1}\sigma(\Delta(y,s)), \tag{4.6}$$

for some $c = c(p, n, M)$, provided $\tilde{\varepsilon}$ is sufficiently small. Also from Hölder's inequality,

$$(\sigma(\Delta(y,s)))^{-1}T_2 \leq \left(\frac{1}{\sigma(\Delta(y,s))} \int_{\Delta(y,s)} |\nabla u|^p\, d\sigma\right)^{(p-1)/p} \left(\frac{\sigma(F\cap\Delta(y,s))}{\sigma(\Delta(y,s))}\right)^{1/p}. \tag{4.7}$$

Using $y \in G$ and the reverse Hölder inequality for $|\nabla u|$ in Theorem 1 we get from (4.7) that

$$T_2 \leq c\tilde{\varepsilon}^{1/p}\mu(\Delta(y,s)). \tag{4.8}$$

Using (4.6) and (4.8) in (4.5), we obtain that

$$(1 - c\,\tilde{\varepsilon}^{1/p})b^{p-1} \leq \frac{\mu(\Delta(y,s))}{\sigma(\Delta(y,s))} \leq (1 + c\tilde{\varepsilon}^{1/p})b^{p-1}. \tag{4.9}$$

To construct $\tilde{\Omega}$ we assume, as we may, that

$$\begin{aligned} \Omega \cap B(w, 4r) &= \{(x', x_n)\colon x_n > \phi(x')\} \cap B(w, 4r), \\ \partial\Omega \cap B(w, 4r) &= \{(x', x_n)\colon x_n = \phi(x')\} \cap B(w, 4r), \end{aligned} \tag{4.10}$$

where $\phi\colon \mathbf{R}^{n-1} \to \mathbf{R}$ is Lipschitz with $\||\nabla\phi|\|_\infty \leq M$. Let $r' = \hat{r}/c$ and $\hat{w} = w + \frac{1}{4}\hat{r}\, e_n$. Let $\tilde{\Omega}$ be the domain obtained from drawing all open line segments from points in

$B(\hat{w}, r')$ to points in $\Delta(w, r') \cap G$. If c is large enough and $\tilde{r}$ small enough, it follows from Lipschitzness of Ω and elementary geometry that $\tilde{\Omega} \subset \Omega$ is a starlike Lipschitz domain with center at $\hat{w}$ and Lipschitz constant $\tilde{M} = \tilde{M}(M)$. Now from (4.9) we see that if $\tilde{\varepsilon} = (\varepsilon/c)^p$ and $\tilde{r}(\varepsilon) = r^*(\tilde{\varepsilon})$, then (b) of Lemma 4.1 is valid. Also, (a) is an obvious consequence of (4.4) as $r' = \hat{r}/c$.

To prove the last display in Lemma 4.1 we first note from Theorem 2.8 that

$$c^{-1}\frac{u(x)}{d(x,\partial\Omega)} \le |\nabla u(x)| \le c\frac{u(x)}{d(x,\partial\Omega)} \tag{4.11}$$

whenever $x \in \Omega \cap B(w, r/c)$. Second we note that if $x \in \tilde{\Omega}$, there exists $y \in G$ with $d(x, \partial\Omega) \approx |x-y|$. If $s = |x-y|$, then from (4.11), the definition of the set G, Lemma 4.1 (b), Harnack's inequality, and Lemma 2.5 we find that

$$b^{p-1} \approx \frac{\mu(\Delta(y,s))}{\sigma(\Delta(y,s))} \approx \left(\frac{u(x)}{d(x,\partial\Omega)}\right)^{p-1} \approx |\nabla u(x)|^{p-1}. \tag{4.12}$$

From (4.12) and the fact that $\hat{w} \in \tilde{\Omega}$ we obtain the last display in Lemma 4.1. The proof of Lemma 4.1 is now complete. □

4.2. The blow-up argument. To begin the blow-up argument in the proof of Theorem 3 we first let

$$D(F_1, F_2) = \max\left(\sup\{d(x, F_2)\colon x \in F_1\}, \sup\{d(y, F_1)\colon y \in F_2\}\right)$$

be the Hausdorff distance between the sets $F_1, F_2 \subset \mathbf{R}^n$. Second, recall from section 1 that to prove Theorem 3 it suffices to obtain a contradiction to the assumption that

$$\eta = \lim_{\tilde{r}\to 0} \sup_{\tilde{w}\in\Delta(w,r/2)} \|n\|_{BMO(\Delta(\tilde{w},\tilde{r}))} \neq 0 \tag{4.13}$$

where n is the outer unit normal to Ω. Moreover if (4.13) is false then there exist sequences, see the discussion after (1.8), $\{w_j\}$, $w_j \in \Delta(w, r/2)$, and $\{r_j\}$, $r_j \to 0$, such that

$$\eta = \lim_{j\to\infty}\left(\frac{1}{\sigma(\Delta(w_j, r_j))}\int_{\Delta(w_j,r_j)} |n - n_{\Delta(w_j,r_j)}|^2\, d\sigma\right)^{1/2} \tag{4.14}$$

where $n_{\Delta(w_j,r_j)}$ denotes the average of n on $\Delta(w_j, r_j)$ with respect to σ. Let $\Omega \cap B(w, 4r)$ be as in (4.10) and let u be as in Theorem 3. Extend u to $B(w, 4r)$ by putting $u = 0$ in $B(w, 4r) \setminus \Omega$. Let $T_j(z) = w_j + r_j z$ and as in (1.9) we put, for $j = 1, 2, \ldots$,

$$\begin{aligned} \Omega_j &= T_j^{-1}(\Omega \cap B(w, 4r)) = \{r_j^{-1}(x - w_j)\colon x \in \Omega \cap B(w, 4r)\},\\ u_j(z) &= \lambda_j\, u(T_j(z)) \text{ whenever } z \in T_j^{-1}(B(w, 4r)). \end{aligned} \tag{4.15}$$

The sequence $\{\lambda_j\}$ used in (4.15) will be defined in (4.21) below. From translation and dilation invariance of the p Laplace equation we see that u_j is p harmonic in Ω_j

and continuous in $T_j^{-1}(B(w,4r))$ with $u_j \equiv 0$ in $T_j^{-1}(B(w,4r)\setminus\Omega)$. Also we note, for $j = 1,2,\dots$, that

$$\begin{aligned} \Omega_j &= \{(y', y_n)\colon y_n > \psi_j(y')\} \cap T_j^{-1}(B(w,4r)),\\ \partial\Omega_j &= \{(y', y_n)\colon y_n = \psi_j(y')\} \cap T_j^{-1}(B(w,4r)), \end{aligned} \tag{4.16}$$

where if $w_j = (w_j', (w_j)_n)$, then

$$\psi_j(y') = r_j^{-1}[\phi(r_j y' + w_j') - (w_j)_n] \text{ whenever } y' \in \mathbf{R}^{n-1}. \tag{4.17}$$

Clearly, ψ_j is Lipschitz with

$$\psi_j(0) = 0 \text{ and } \||\nabla\psi_j|\|_\infty = \||\nabla\phi|\|_\infty \le M \text{ for } j = 1,2,\dots. \tag{4.18}$$

Let μ, μ_j be the measures associated with u, u_j as in Lemma 2.5 and let σ, σ_j be the surface measures on $\partial\Omega$ and $\partial\Omega_j$ respectively. From (4.16)–(4.18) and the definition of u_j, we see that if H_j is a Borel subset of $\partial\Omega_j$, then

$$\sigma_j(H_j) = r_j^{1-n}\,\sigma(T_j(H_j)),\ \ \mu_j(H_j) = \lambda_j^{p-1}\, r_j^{p-n}\,\mu(T_j(H_j)). \tag{4.19}$$

We assume as we may that $2^j r_j \to 0$ as $j \to \infty$. We now apply Lemma 4.1 to u with w, r' replaced by w_j, $2^j r_j$ and with $\varepsilon = 2^{-2j^2}$. Then for j large enough there exists a starlike Lipschitz domain $\tilde\Omega = \tilde\Omega(j) \subset \Omega \cap B(w_j, c2^j r_j)$, with Lipschitz constant $\tilde M = \tilde M(M)$ and center at $\hat w_j$, such that $d(\hat w_j, \partial\Omega) \approx 2^j r_j$ and such that

$$\begin{aligned} &\text{(a')}\ \frac{\sigma(\partial\tilde\Omega \cap \Delta(w_j, 2^j r_j))}{\sigma(\Delta(w_j, 2^j r_j))} \ge 1 - 2^{-2j^2},\\ &\text{(b')}\ (1 - 2^{-2j^2}) b_j^{p-1} \le \frac{\mu(\Delta(y,s))}{\sigma(\Delta(y,s))} \le (1 + 2^{-2j^2}) b_j^{p-1} \text{ whenever } 0 < s < 2^j r_j\\ &\qquad \text{and } y \in \partial\tilde\Omega \cap \Delta(w, 2^j r_j),\\ &\text{(c')}\ c^{-1}\frac{u(\hat w_j)}{2^j r_j} \le |\nabla u(x)| \le c\frac{u(\hat w_j)}{2^j r_j} \text{ whenever } x \in \tilde\Omega. \end{aligned} \tag{4.20}$$

In (4.20) (b'), $\log b_j$ denotes the average of $\log|\nabla u|$ on $\Delta(w_j, 2^{j+2} r_j)$ with respect to σ. From (4.15), (4.19) and (4.20) we see that if

$$\lambda_j = (r_j b_j)^{-1},\ O_j = T_j^{-1}(\tilde\Omega(j)),\ \zeta_j = T_j^{-1}(\hat w_j), \tag{4.21}$$

then $O_j \subset \Omega_j \cap B(0, c2^j)$ is a starlike Lipschitz domain with center at ζ_j and Lipschitz constant $\tilde M = \tilde M(M)$. Moreover, $d(\zeta_j, \partial\Omega_j) \approx 2^j$ and

$$\begin{aligned} &(\alpha)\ \frac{\sigma_j(\partial O_j \cap \partial\Omega_j \cap B(0, 2^j))}{\sigma_j(\partial\Omega_j \cap B(0, 2^j))} \ge 1 - 2^{-2j^2},\\ &(\beta)\ (1 - 2^{-2j^2}) \le \frac{\mu_j(\partial\Omega_j \cap B(z,s))}{\sigma_j(\partial\Omega_j \cap B(z,s))} \le (1 + 2^{-2j^2}) \text{ whenever } 0 < s < 2^j\\ &\qquad \text{and } z \in \partial O_j \cap \partial\Omega_j,\\ &(\gamma)\ c^{-1} \le |\nabla u_j(x)| \le c \text{ whenever } x \in O_j. \end{aligned} \tag{4.22}$$

In fact, (4.22) $(\alpha),(\beta)$ are straightforward consequences of (4.20) (a'), (b') and (4.21). (4.22) (γ) follows from (4.20) (c'), (4.22) (β), and the fact that by Lemma 2.5,

$$\frac{\mu_j(\partial\Omega_j\cap B(0,2^j))}{\sigma_j(\partial\Omega_j\cap B(0,2^j))}\approx\left(\frac{u_j(\zeta_j)}{2^j}\right)^{p-1}.$$

Let $\hat\sigma_j$ denote the surface measure on ∂O_j. We next show that the following holds for j large enough,

$$\text{(4.23)}\qquad \begin{aligned}&(\hat\alpha)\ \hat\sigma_j\left((\partial O_j\setminus\partial\Omega_j)\cap B(0,2^{j/2})\right)\ \le c2^{-j^2},\\ &(\hat\beta)\ D(\partial\Omega_j\cap B(0,2^{j/2}),\partial O_j\cap B(0,2^{j/2}))\ \le c2^{-j^2/(n-1)}.\end{aligned}$$

To prove (4.23) we observe from (4.22) (α) that for large j,

$$\text{(4.24)}\qquad d(x,\partial O_j)\le 2^{-3j^2/(2(n-1))}\text{ whenever }x\in\partial\Omega_j\cap B(0,2^{j/2}).$$

In fact, if the statement in (4.24) is false then there exists $x\in\partial\Omega_j\cap B(0,2^{j/2})$ such that $B(x,2^{-3j^2/(2(n-1))})\cap\partial O_j=\emptyset$ and such that

$$\frac{\sigma_j(\partial O_j\cap\partial\Omega_j\cap B(0,2^j))}{\sigma_j(\partial\Omega_j\cap B(0,2^j))}\le\left(1-c2^{-(j(n-1)+3j^2/2)}\right).$$

As $1-c2^{-(j(n-1)+3j^2/2)}<1-2^{-2j^2}$ if j is large enough the statement in the last display contradicts (4.22) (α) and hence (4.24) must hold. Moreover, if $x\in(\partial O_j\setminus\partial\Omega_j)\cap B(0,2^{j/2})$, then we can project x onto $x^*\in\partial\Omega_j$ by way of radial projection from ζ_j. From the construction of O_j and (4.22) (α) we again see for large j that

$$d(x,\partial\Omega_j)\approx d(x^*,\partial O_j\cap\partial\Omega_j)\le 2^{-3j^2/(2(n-1))}.$$

Thus using the inequality in the last display and (4.24) we see that (4.23) $(\hat\beta)$ is true. (4.23) $(\hat\alpha)$ also follows from this inequality and a covering argument.

From (4.18) and a standard compactness argument we see there exists a subsequence $\{\psi_j'\}$ of $\{\psi_j\}$ with $\psi_j'\to\phi_\infty$ uniformly on compact subsets of $\mathbf{R}^{n-1}$ where ϕ_∞ is Lipschitz and

$$\text{(4.25)}\qquad \begin{aligned}&(*)\ \||\nabla\phi_\infty|\|_\infty\le M\text{ and }\phi_\infty(0)=0,\\ &(**)\int\limits_{\mathbf{R}^{n-1}}\frac{\partial\psi_j'}{\partial x_i}\,f\,dx'\to\int\limits_{\mathbf{R}^{n-1}}\frac{\partial\phi_\infty}{\partial x_i}\,f\,dx'\text{ as }j\to\infty\text{ for }1\le i\le n\\ &\qquad\text{and }f\in C_0^\infty(\mathbf{R}^{n-1}).\end{aligned}$$

Let $\Omega_j'=\{x\in\mathbf{R}^n\colon x_n>\psi_j'(x')\}$, $\Omega_\infty=\{x\in\mathbf{R}^n\colon x_n>\phi_\infty(x')\}$, and let n_j',σ_j' and n_∞,σ_∞ denote, respectively, the outer unit normal and the surface measure to $\partial\Omega_j'$ and $\partial\Omega_\infty$. From (4.25) we find that

$$\text{(4.26)}\qquad \begin{aligned}&(+)\ D(\partial\Omega_j'\cap B(0,R),\partial\Omega_\infty\cap B(0,R))\to 0\text{ as }j\to\infty\text{ for each }R>0,\\ &(++)\int\langle n_j,F\rangle\,d\sigma_j'\to\int\langle n,F\rangle\,d\sigma_\infty\text{ as }j\to\infty\text{ whenever }F=(F_1,\dots,F_n)\\ &\qquad\text{with }F_i\in C_0^\infty(\mathbf{R}^n)\text{ for }1\le i\le n.\end{aligned}$$

In the last inequality we have used the fact that if $y = (y', \psi'_j(y')) \in \partial\Omega'_j \cap B(0, 2^j)$, then

$$n'_j(y)\, d\sigma'_j(y) \;=\; (\nabla\psi_j(y'), -1).$$

(4.26) (++) and measure theoretic type arguments imply

$$\int\limits_{\partial\Omega_\infty} f\, d\sigma_\infty \le \liminf_{j\to\infty} \int\limits_{\partial\Omega'_j} f\, d\sigma'_j \text{ whenever } f \ge 0 \in C_0^\infty(\mathbf{R}^n). \tag{4.27}$$

Let $\{u'_j\}$, $\{\mu'_j\}$ be subsequences of $\{u_j\}$, $\{\mu_j\}$, corresponding to (Ω'_j). Then from Lemmas 2.1–2.5 applied to u'_j and (4.22) (β) we deduce that u'_j is bounded, Hölder continuous, and locally in $W^{1,p}$ on compact subsets of $\mathbf{R}^n$ with norms of all functions bounded above by constants which are independent of j. Also, if $B(x, 2\rho) \subset \Omega_\infty$, then for large j we see from (4.23)($\hat\beta$) and Lemma 2.4 that $\nabla u'_j$ is Hölder continuous and bounded on $B(x,\rho)$ with constants independent of j. Thus we assume, as we may, that $\{u'_j\}$ converges uniformly and weakly in $W^{1,p}$ on compact subsets of $\mathbf{R}^n$ to u_∞ and that $\{\nabla u'_j\}$ converges uniformly to ∇u_∞ on compact subsets of Ω_∞. Also, $u_\infty \ge 0$ is p harmonic in Ω_∞ and continuous on $\mathbf{R}^n$, with $u_\infty \equiv 0$ on $\mathbf{R}^n \setminus \Omega_\infty$. Furthermore, if μ_∞ denotes the measure associated with u_∞ as in Lemma 2.5 and $f \in C_0^\infty(\mathbf{R}^n)$, then

$$\begin{aligned} -\int\limits_{\mathbf{R}^n} f\, d\mu_\infty &= \int\limits_{\mathbf{R}^n} |\nabla u_\infty|^{p-2} \langle \nabla u_\infty, \nabla f\rangle\, dx \\ &= \lim_{j\to\infty} \int\limits_{\mathbf{R}^n} |\nabla u'_j|^{p-2} \langle \nabla u'_j, \nabla f\rangle\, dx = -\lim_{j\to\infty} \int\limits_{\mathbf{R}^n} f\, d\mu'_j. \end{aligned} \tag{4.28}$$

Thus $\{\mu'_j\}$ converges weakly to μ_∞.

Next we show that

$$\sigma_\infty \le \mu_\infty. \tag{4.29}$$

To do this we first observe from Theorem 1 and (3.7) that $d\mu'_j = |\nabla u'_j|^{p-1}\, d\sigma'_j$ on $\partial\Omega'_j$. Using this inequality, (4.22) (β), and differentiation theory we see that

$$1 - 2^{-2j^2} \le |\nabla u'_j| \le 1 + 2^{-2j^2} \tag{4.30}$$

σ'_j almost everywhere on $\partial\Omega'_j \cap \partial O'_j \cap B(0, 2^j)$, where $\{O'_j\}$ is the subsequence of $\{O_j\}$ corresponding to $\{\Omega'_j\}$. Let $f \in C_0^\infty(\mathbf{R}^n)$ and $f \ge 0$. From (4.28), (4.27), (4.30), and (4.22) (α) we find that

$$\begin{aligned} \int f\, d\mu_\infty &= \lim_{j\to\infty} \int\limits_{\partial\Omega'_j} f |\nabla u'_j|^{p-1}\, d\sigma'_j \ge \liminf_{j\to\infty} \int\limits_{\partial O'_j \cap \partial\Omega'_j} f\, |\nabla u'_j|^{p-1}\, d\sigma'_j \\ &\ge \liminf_{j\to\infty} (1 - 2^{-2j}) \int\limits_{\partial O'_j \cap \partial\Omega'_j} f\, d\sigma'_j = \liminf_{j\to\infty} \int\limits_{\partial O'_j \cap \partial\Omega'_j} f\, d\sigma'_j \ge \int f\, d\sigma_\infty. \end{aligned}$$

Thus (4.29) is true. We claim that

$$c^{-1} \le |\nabla u_\infty| \le 1 \text{ on } \Omega_\infty. \tag{4.31}$$

We note that once (4.31) is proved we get from Theorem 1 and (3.7) that

$$d\mu_\infty = |\nabla u_\infty|^{p-1}\, d\sigma_\infty \le d\sigma_\infty.$$

From this inequality and (4.29) we conclude

$$\sigma_\infty = \mu_\infty. \tag{4.32}$$

To prove (4.31) let $x \in \Omega_\infty$ and suppose that j is so large that $|x| \le 2^{j/4}$ and $d(x, \partial O'_j) \ge \frac{1}{2} d(x, \partial\Omega_\infty)$. The last assumption is permissible as we see from (4.23) and (4.26) (+). Let $\xi \in \partial B(0,1)$ and for fixed j we set $v = \langle \nabla u'_j, \xi\rangle$. Let $\omega'_j(\cdot, x)$ denote elliptic measure at $x \in O'_j$ with respect to the operator L in (1.13), where u in (1.14) is replaced by u'_j. From (1.15) and (4.22) (γ) we see that

$$|v| \le c \text{ and } Lv \equiv 0 \text{ weakly in } O'_j. \tag{4.33}$$

Let $\hat\sigma'_j$ be surface measure on $\partial O'_j$. Using Lemma 2.15 and Harnack's inequality for the operator L we see that $\hat\sigma'_j$ and $\omega'_j(x,\cdot)$ are mutually absolutely continuous. Hence, arguing as in [CFMS] we get that v has non-tangential limits $\hat\sigma'_j$ almost everywhere on $\partial O'_j$. Moreover, v can be interpreted as the 'Poisson integral' of its boundary values. Using these facts, (4.33), (4.22) (α) and the maximum principle for the operator L, we deduce that

$$|v(x)| \le (1 + 2^{-2j^2}) T_1(x) + c(T_2(x) + T_3(x)) \tag{4.34}$$

where

$$\begin{aligned} T_1(x) &= \omega'_j(\partial O'_j \cap \partial\Omega'_j \cap B(0, 2^{j/2}),\, x) \\ T_2(x) &= \omega'_j((\partial O'_j \setminus \partial\Omega'_j) \cap B(0, 2^{j/2}),\, x) \\ T_3(x) &= \omega'_j(\partial O'_j \setminus B(0, 2^{j/2}),\, x). \end{aligned}$$

Next we estimate $T_1(x)$, $T_2(x)$ and $T_3(x)$ for $|x| \le 2^{j/4}$. In particular, using (2.14) we see that if $|x| \le 2^{j/4}$ then

$$T_3(x) \le c 2^{-j\tau/4} \tag{4.35}$$

where $c \ge 1$, $0 < \tau \le 1$, depend only on p, n, M. Also from Lemma 2.15 and (4.23) $(\hat\alpha)$ we obtain

$$T_2(\zeta'_j) \le c \left(\frac{\sigma'_j((\partial O'_j \setminus \partial\Omega'_j) \cap B(0, 2^{j/2}))}{\sigma'_j(\partial O'_j \cap B(0, 2^{j/2}))} \right)^{\gamma} \le c 2^{-\gamma j^2/2} \tag{4.36}$$

for j large enough. Here ζ'_j is the center of O'_j. Moreover, using Harnack's inequality for the operator L and the fact that $d(\zeta'_j, \partial O'_j) \approx 2^j$ we see there exist $c \ge 1$ and $\kappa \ge 1$, depending only on p, n, M, such that

$$T_2(x) \le c T_2(\zeta'_j)\, (2^j / d(x, \partial\Omega_\infty))^{\kappa}$$

provided j is large enough. In view of this inequality and (4.36) we can conclude that

$$T_2(x) \le 2^{-\gamma j^2/4}\, d(x, \partial\Omega_\infty)^{-\kappa} \tag{4.37}$$

for large j. Using (4.34), the fact that $T_1 \le 1$, (4.37) as well as (4.35) we find, by taking limits, that

$$|\langle \nabla u_\infty, \xi\rangle|(x) = \lim_{j\to\infty} |\langle \nabla u_j', \xi\rangle|(x) \le 1.$$

Since $x \in \Omega_\infty$ and $\xi \in \partial B(0,1)$ are arbitrary, we conclude that the righthand inequality in (4.31) is true. The lefthand inequality in (4.31) follows from (4.22) (γ) and the fact that $\{\nabla u_j'\}$ converges to ∇u_∞ uniformly on compact subsets of Ω_∞.

4.3. The final proof. For those well versed in [ACF] we can now rapidly obtain a contradiction to (4.14) and thus prove Theorem 3. Indeed from (4.31), (4.32), (4.25) ($*$), and [ACF] it follows, for $\hat{M}$ small enough, that if $M \le \hat{M}$ then

$$u_\infty = \langle x, \nu\rangle \text{ and } \Omega_\infty = \{x \in \mathbf{R}^n : \langle x, \nu\rangle\} > 0 \text{ for some } \nu \in \partial B(0,1). \tag{4.38}$$

Using (4.38) and (4.26) (++) we see that

$$\lim_{j\to\infty} \int\limits_{\partial\Omega_j' \cap B(0,1)} \langle n_j', \nu\rangle\, d\sigma_j' = -\sigma_\infty(\partial\Omega_\infty \cap B(0,1)). \tag{4.39}$$

Also from (4.31), (4.22), and the fact that $d\sigma_\infty = d\mu_\infty$, see (4.32), we obtain for $f \ge 0$ and $f \in C_0^\infty(\mathbf{R}^n)$, as in the argument leading to (4.29),

$$\begin{aligned} \int f\, d\sigma_\infty &= \lim_{j\to\infty} \int\limits_{\partial\Omega_j'} f|\nabla u_j'|^{p-1}\, d\sigma_j' \ge \limsup_{j\to\infty} \int\limits_{\partial O_j' \cap \partial\Omega_j'} f\,|\nabla u_j'|^{p-1}\, d\sigma_j' \\ &\ge \limsup_{j\to\infty} (1 - 2^{-2j^2}) \int\limits_{\partial O_j' \cap \partial\Omega_j'} f\, d\sigma_j' = \limsup_{j\to\infty} \int\limits_{\partial\Omega_j'} f\, d\sigma_j'. \end{aligned} \tag{4.40}$$

Combining (4.40) and (4.27) we see that

$$\sigma_j' \to \sigma_\infty \text{ weakly as } j \to \infty. \tag{4.41}$$

Finally, let a_j' denote the average of n_j' on $\partial\Omega_j' \cap B(0,1)$ with respect to σ_j'. From (4.41) and (4.26) (++) we deduce that $a_j \to -\nu$ as $j \to \infty$. Using this fact, (4.41), (4.39), the fact that (4.14) is scale invariant, and the triangle inequality, we get

$$\begin{aligned} 0 < \eta &= \lim_{j\to\infty} \left(\frac{1}{\sigma_j'(\partial\Omega_j' \cap B(0,1))} \int\limits_{\partial\Omega_j' \cap B(0,1)} |n_j' - a_j'|^2 d\sigma_j' \right)^{1/2} \\ &\le \limsup_{j\to\infty} \left(\frac{1}{\sigma_j'(\partial\Omega_j' \cap B(0,1))} \int\limits_{\partial\Omega_j' \cap B(0,1)} |n_j' + \nu|^2 d\sigma_j' \right)^{1/2} + \lim_{j\to\infty} |a_j + \nu| \end{aligned} \tag{4.42}$$

$$= \limsup_{j\to\infty} \left(\frac{1}{\sigma'_j(\partial\Omega'_j \cap B(0,1))} \int\limits_{\partial\Omega'_j \cap B(0,1)} 2(1 + \langle n'_j, \nu \rangle)\, d\sigma'_j \right)^{1/2} = 0.$$

We have therefore reached a contradiction and thus Theorem 3 is true. □

For the reader not so well versed in [ACF] we outline the proof of (4.38). First we remark that from (4.31) it follows (see [LN, Lemma 2.4]) that u_∞ is infinitely differentiable in Ω_∞. Using this fact and (4.31) once again it is easily checked that the argument in sections 5 and 6 of [ACF] applies to u_∞. To briefly outline these sections in our situation we need a definition.

Definition 4.43. Let $0 \le \sigma_+, \sigma_- \le 1$, $\xi \in \partial B(0,1)$ and $\lambda \in (0,1]$. For fixed $p, 1 < p < \infty$, we say that u belongs to the class $F(\sigma_+, \sigma_-, R, \xi, \lambda)$, $0 < R$, if the following conditions are fulfilled,

(i) $u(x) \ge \langle x, \xi \rangle - \sigma_+ R$ whenever $x \in B(0,R)$ and $\langle x, \xi \rangle \ge \sigma_+ R$,

(ii) $u(x) = 0$ whenever $x \in B(0,R)$ and $\langle x, \xi \rangle \le \sigma_- R$,

(iii) $\lambda \le |\nabla u(x)| \le 1$ whenever $x \in \Omega_\infty \cap B(0,R)$,

(iv) $u \ge 0$ is p harmonic in $\{u > 0\} \cap B(0,R)$ and continuous in $B(0,R)$.

From (4.31), (4.32), one can deduce, as in the proof Theorem 5.1 and Lemma 5.6 in [ACF] (see also Lemma 7.2 and Lemma 7.9 in [AC]), that the following two lemmas hold.

Lemma 4.44. *There exist constants* $0 < \sigma_1$ *and* $0 < c_1$ *such that if* $0 < \sigma \le \sigma_1$ *and if* $u_\infty \in F(1, \sigma, R, \xi, \lambda)$ *then* $u_\infty \in F(c_1\sigma, 2\sigma, R/2, \xi, \lambda)$.

Lemma 4.45. *Given* $\theta \in (0,1)$ *there exist constants* $0 < \sigma_2 = \sigma_2(\theta)$ *and* $\beta = \beta(\theta) \in (0,1)$ *such that if* $0 < \sigma \le \sigma_2$ *and if* $u_\infty \in F(\sigma, \sigma, R, \xi, \lambda)$ *then* $u_\infty \in F(1, \theta\sigma, \beta R, \tilde\xi, \lambda)$ *for some* $\tilde\xi \in \partial B(0,1)$ *with* $|\xi - \tilde\xi| \le c\sigma$.

In the following we let $\tilde\theta \in (0, 1/2)$ be a constant to be chosen. Let $\delta = \sigma_2(\tilde\theta)$ where σ_2 is as in Lemma 4.45. Note from (4.25)(∗) and (4.31), that there exists $\hat M = \hat M(\delta)$ such that if $\xi_0 = e_n$, $M \le \hat M$, and $\lambda = c^{-1}$, c as in (4.31), then $u_\infty \in F(\delta, \delta, R, \xi_0, \lambda)$ for any $R > 0$. We can now apply Lemma 4.45 to conclude that $u_\infty \in F(1, \tilde\theta\delta, \beta(\tilde\theta)R, \xi_1, \lambda)$ where $|\xi_0 - \xi_1| \le c\delta$. Subsequently using Lemma 4.44 we also see that $u_\infty \in F(c_1\tilde\theta\delta, 2\tilde\theta\delta, \beta(\tilde\theta)R/2, \xi_1, \lambda)$. We let $\theta = \max\{c_1\tilde\theta, 2\tilde\theta\}$ and choose $\tilde\theta \in (0, 1/2)$ so small that $\theta < 1$. We also let $\beta = \beta(\tilde\theta)/2$. Based on this we can conclude that if $u_\infty \in F(\delta, \delta, R, \xi_0, \lambda)$ then $u_\infty \in F(\theta\delta, \theta\delta, \beta R, \xi_1, \lambda)$ and $|\xi_0 - \xi_1| \le c\delta$. By iteration we see that,

$$u_\infty \in F(\theta^m\delta, \theta^m\delta, \beta^m R, \xi_m, \lambda) \text{ and } |\xi_m - \xi_{m-1}| \le c\theta^m\delta \text{ for } m = 1, 2, \ldots. \tag{4.46}$$

If we let $R = m\beta^{-m}$ for a fixed positive integer m, then we note from (4.46) that if $x \in \partial\Omega_\infty \cap B(0,m)$, then

$$|\langle x, \xi_m \rangle| \le c\theta^m\delta. \tag{4.47}$$

Letting $m \to \infty$ in (4.47) we see that (4.38) is valid, where ν is the limit of a certain subsequence of $\{\xi_m\}$.

5. Closing remarks

As noted in section 1, in a future paper, we shall prove Theorems 1–3 in the setting of Reifenberg flat chord arc domains and thus carry out the full program in [KT], [KT1], [KT2] when $1 < p < \infty$, $p \neq 2$. We also plan to study and remove the smallness assumption in Theorem 3 on M by generalizing the results in [C] for harmonic functions (see also [C1], [C2], [J]) to p harmonic functions. We also note that one can state interesting codimension problems similar to Theorems 1–3 for certain values of p. For example if $\gamma \subset B(0,1/2) \subset \mathbf{R}^3$ is a curve and $p > 2$, then there exists a unique p harmonic function u in $B(0,1) \setminus \gamma$ which is continuous in $\bar{B}(0,1)$ with boundary values $u = 0$ on γ and $u = 1$ on $\partial B(0,1)$. Moreover, there exists a unique measure μ with support $\subset \gamma$. If γ is Lipschitz, is it true that μ is absolutely continuous with respect to Hausdorff one measure (H^1) on γ? If so, we next assume γ is C^1, and put $k = d\mu/d\sigma$. Is it true that $\log k \in VMO(\gamma)$, where integrals are taken with respect to H^1 measure? If $\mu = H^1$ measure on γ, is it true that γ is a line segment or a circular arc? That is, to what extent do the theorems of Caffarelli and coauthors generalize to the codimension > 1 case.

As for related problems, we note that in [LV], see also [LV1], Lewis and Vogel study over-determined boundary conditions for positive solutions to the p Laplace equation in a bounded domain Ω. They prove that conditions akin to (4.32) imply uniqueness in certain free boundary problems. In particular, in [LV] the following free boundary problem is considered. Given a compact convex set $F \subset \mathbf{R}^n$, $a > 0$, and $1 < p < \infty$, find a function u, defined in a domain $\Omega = \Omega(a,p) \subset \mathbf{R}^n$, such that $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ weakly in $\Omega \setminus F$, $u(x) \to 1$ whenever $x \to y \in F$, $u(x) \to 0$ whenever $x \to y \in \partial\Omega$ and such that $\mu = a^{p-1}H^{n-1}$ on $\partial\Omega$. Here H^{n-1} denotes $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$ and μ is the unique finite positive Borel measure associated with u as in Lemma 2.5. If in addition, μ is upper Ahlfors regular, then the above authors show that this over-determined boundary value problem has a unique solution. An important part of their argument is to show that $\limsup_{x\to\partial\Omega} |\nabla u(x)| \leq a$. If $\partial\Omega$ is Lipschitz we note that this inequality is an easy consequence of Theorem 1 and (3.7). However, in [LV] it is only assumed that Ω is bounded, so a different argument, based on finiteness of a certain square function, is used.

References

[AC] Alt, H. W., and L. Caffarelli: Existence and regularity for a minimum problem with free boundary. - J. Reine Angew. Math. 325, 1981, 105–144.

[ACF] Alt, H. W., L. Caffarelli, and A. Friedman: A free boundary problem for quasilinear elliptic equations. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 11:1, 1984, 1–44.

[C] Caffarelli, L. A.: A Harnack inequality approach to the regularity of free boundaries. Part 1: Lipschitz free boundaries are $C^{1,\alpha}$. - Rev. Mat. Iberoamericana 3, 1987, 139–162.

[C1] Caffarelli, L. A.: A Harnack inequality approach to the regularity of free boundaries. Part 2: Flat free boundaries are Lipschitz. - Comm. Pure Appl. Math. 42, 1989, 55–78.

[C2] Caffarelli, L. A.: A Harnack inequality approach to the regularity of free boundaries. Part 3: Existence theory, regularity, and dependence on X. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 15, 1988, 583–602.

[CFMS] Caffarelli, L., E. Fabes, S. Mortola, and S. Salsa: Boundary behavior of non-negative solutions of elliptic operators in divergence form. - Indiana J. Math. 30:4, 1981, 621–640.

[CF] Coifman, R., and C. Fefferman: Weighted norm inequalities for maximal functions and singular integrals. - Studia Math. 51, 1974, 241–250.

[D] Dahlberg, B.: On estimates of harmonic measure. - Arch. Ration. Mech. Anal. 65, 1977, 275–288.

[GR] Garcia-Cuerva, J., and J. L. Rubio de Francia: Weighted norm inequalities and related topics. - North-Holland Math. Stud. 116, North-Holland, Amsterdam, 1985.

[J] Jerison, D.: Regularity of the Poisson kernel and free boundary problems. - Colloq. Math. 60-61, 1990, 547–567.

[JK] Jerison, D., and C. Kenig: The logarithm of the Poisson kernel of a C^1 domain has vanishing mean oscillation. - Trans. Amer. Math. Soc. 273, 1982, 781–794.

[KT] Kenig, C., and T. Toro: Harmonic measure on locally flat domains. - Duke Math. J. 87, 1997, 501–551.

[KT1] Kenig, C., and T. Toro: Free boundary regularity for harmonic measure and Poisson kernels. - Ann. of Math. (2) 150, 1999, 369–454.

[KT2] Kenig, C., and T. Toro: Poisson kernel characterization of Reifenberg flat chord arc domains. - Ann. Sci. École Norm. Sup. (4) 36:3, 2003, 323–401.

[KP] Kenig, C., and J. Pipher: The Dirichlet problem for elliptic operators with drift term. - Publ. Mat. 45:1, 2001, 199–217.

[LN] Lewis, J., and K. Nyström: Boundary behaviour for p harmonic functions in Lipschitz and starlike Lipschitz ring domains. - Ann. Sci. École Norm. Sup. (4) 40:4, 2007, 765–813.

[LN1] Lewis, J., and K. Nyström: Boundary behaviour and the Martin Boundary problem for p harmonic functions in Lipschitz domains. - Submitted.

[LN2] Lewis, J., and K. Nyström: New results on p harmonic functions. - Pure Appl. Math. Quart. (to appear).

[LV] Lewis, J., and A. Vogel: Uniqueness in a free boundary problem. - Comm. Partial Differential Equations 31, 2006, 1591–1614.

[LV1] Lewis, J., and A. Vogel: Symmetry theorems and uniform rectifiability. - Bound. Value Probl. 2007, 2007, 1–59.

[S] Stein, E. M.: Singular integrals and differentiability properties of functions. - Princeton Univ. Press, Princeton, NJ, 1970.

[S1] Stein, E. M.: Harmonic analysis. - Princeton Univ. Press, Princeton, NJ, 1993.

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