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A FIXED POINT APPROACH TO THE STABILITY OF φ-MORPHISMS ON HILBERT C^*-MODULES

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ABSTRACT. Let E, F be two Hilbert C^*-modules over C^*-algebras A and B respectively. In this paper, by the alternative fixed point theorem, we give the Hyers-Ulam-Rassias stability of the equation

$$\langle U(x), U(y)\rangle = \varphi(\langle x, y\rangle) \qquad (x, y \in E),$$

where $U : E \to F$ is a mapping and $\varphi : A \to B$ is an additive map.

1. Introduction and Preliminaries

A *pre-Hilbert A-module* is a right module E over C^*-algebra A, with a map $\langle ., .\rangle : E \times E \to A$ which is conjugate linear in the first, linear in its second argument and satisfies

(i) $\langle x, ya\rangle = \langle x, y\rangle a \quad (x, y \in E, a \in A)$,
(ii) $\langle x, y\rangle^* = \langle y, x\rangle \quad (x, y \in E)$,
(iii) $\langle x, x\rangle \geq 0 \quad (x \in E)$,
(iv) $\langle x, x\rangle = 0 \Rightarrow x = 0$.

A *Hilbert A-module* (briefly *Hilbert module*) is a pre-Hilbert A-module that is complete in the norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. For more details about Hilbert modules see [12].

Let E, F be two Hilbert modules over C^*-algebras A and B respectively and $\varphi : A \to B$ be a map. A mapping $U : E \to F$ is called a φ-morphism if

$$\langle U(x), U(y)\rangle = \varphi(\langle x, y\rangle) \qquad (x, y \in E).$$

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This kind of mappings were introduced by Bakić and Guljaš [3]. The first author together with Moslehian and Niknam [1] used this kind of mappings to introduce dynamical systems on Hilbert modules. Also Abbaspour and Skeide in [2] investigated the relation between φ-morphisms, where they called them generalized module mappings, and ternary homomorphisms.

The stability problem of functional equations had been first raised by Ulam [18] by the following question: For what metric groups G is it true that an ϵ-automorphism of G is necessarily near to a strict automorphism? A partial answer to the above question has been given as follows. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \to E_2$ is a mapping. If there exist $\delta \geq 0$ and $p \geq 0, \ p \neq 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\delta\|x\|^p}{|2-2^p|} \qquad (x \in E_1).$$

This result is called the Hyers-Ulam-Rassias stability of the additive Cauchy equation. Indeed Hyers [10] obtained the above result for $p = 0$. Then Rassias [17] generalized the result of Hyers to the case where $0 \leq p < 1$. Gajda [9] solved the problem for $p > 1$ and gave an example that a similar result does not hold for $p = 1$. For the case $p < 0$, recently Lee [13] has shown that f should be an additive map. Thus the Hyers-Ulam-Rassias stability of the additive Cauchy equation holds for $p \in \mathbb{R} \setminus \{1\}$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq d(x, z) + d(z, x)$.

Generalized metric space (X, d) is called complete if each Cauchy sequence converges in X.

In 2003, Radu [16] employed the following theorem to prove the stability of a Cauchy functional equation. Later many authors, [7, 11, 14, 15] used this strategy to give the stability of functional equations. Before stating the theorem we recall that a mapping $J : X \to X$ is called a strictly contractive operator with the Lipschitz constant L, if

$$d(J(x), J(y)) < Ld(x, y) \quad (x, y \in X).$$

Theorem 1.1. ([8]) *Let (X, d) be a generalized complete metric space and $J : X \to X$ be a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(J^{k+1}x, J^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) *The sequence $\{J^n x\}$ converges to a fixed point x^* of J,*
(b) *x^* is the unique fixed point of J in*

$$X^* = \{y \in X \ | \ d(J^k x, y) < \infty\},$$

(c) *if* $y \in X^*$, *then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Jy, y).$$

In [4], Badora and Chmieliński, investigated the stability and superstability of inner product preserving mappings on Hilbert spaces. After then Chmieliński and Moslehian [6] investigated this problem in the framework of Hilbert C^*-modules; see also [5]. We mention that each φ-morphism is in fact a mapping preserving inner product modulo φ. In this paper, by using the alternative fixed point theorem for generalized metric spaces, the stability of φ-morphisms on Hilbert C^*-modules is considered. Throughout the paper we assume that E and F are two Hilbert C^*-modules over C^*-algebras A and B respectively and $\varphi : A \to B$ is an additive map.

2. MAIN RESULTS

Definition 2.1. A mapping $U : E \to F$ is called an approximate φ-morphism if there exists a control function $\tau : E^2 \to \mathbb{R}$ such that

$$\|\langle U(x), U(y)\rangle - \varphi(\langle x, y\rangle)\| \leq \tau(x, y)$$

holds for each $x, y \in E$.
As a consequence of Theorem 2.4 we will show that under some conditions on control function τ each approximate φ-morphism is near to a φ-morphism.

Example 2.2. We know that each C^*-algebra A is a Hilbert C^*-module over itself with the inner product defined by $\langle a, b\rangle = a^*b$. Let A be a unital C*-algebra , $a \in A$, $\epsilon = \|a^*a - 1\|$ and $\varphi : A \to A$ be a $*$-homomorphism. If we define $U(x) = a\varphi(x)$ then we have

$$\begin{aligned} \|\langle U(x), U(y)\rangle - \varphi\langle x, y\rangle\| &= \|\varphi(x^*)a^*a\varphi(y) - \varphi(x^*)\varphi(y)\| \\ &= \|\varphi(x^*)(a^*a - 1)\varphi(y)\| \\ &\leq \epsilon\|x\|\|y\| \\ &\leq \frac{\epsilon}{2}(\|x\|^2 + \|y\|^2) \end{aligned}$$

If a is an unitary element then U is a φ-morphism, otherwise U is an approximate φ-morphism with control function $\tau(x, y) = \frac{\epsilon}{2}(\|x\|^2 + \|y\|^2)$.

Lemma 2.3. *If* $U : E \to F$ *is a mapping such that* $\|U(x + y) - U(x) - U(y)\| \leq \tau(x, y)$ *for some control function* $\tau : E^2 \to \mathbb{R}$ *and there is* $0 < L < 1$ *with* $\tau(2x, 2y) \leq 2L\tau(x, y)$, *then there exists a unique additive map* $\psi : E \to F$ *such that* $\|U(x) - \psi(x)\| \leq \dfrac{1}{2-2L}\tau(x, x)$.

Proof. Let $X = \{g : E \to F \ : g \text{ is a mapping}\}$ and define

$$d(g, h) = \inf\{c \geq 0 \ : \|g(x) - h(x)\| \leq c\tau(x, x) \quad \forall x \in E\},$$

for $g, h \in X$. Then (X, d) is a complete generalized metric space. Now we consider the mapping $J : X \to X$ by $J(g)(x) = \frac{1}{2}g(2x)$. We can write for any $g, h \in X$,

$$\|g(x) - h(x)\| \leq d(g, h)\tau(x, x) \qquad (x \in E),$$

therefore for $x \in E$,

$$\|J(g)(x) - J(h)(x)\| = \|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\| \leq \frac{1}{2}d(g,h)\tau(2x,2x) \leq Ld(g,h)\tau(x,x).$$

Hence $d(J(g), J(h)) \leq Ld(g,h)$. Since $d(J(U), U) \leq \frac{1}{2} < \infty$, Theorem 1.1 implies that

(i) J has a unique fixed point $\psi : E \to F$ in the set $X^* = \{g \in X \ : d(g,U) < \infty\}$.
(ii) $d(J^n(U), \psi) \to 0$ as $n \to \infty$. This implies that $\lim_{n\to\infty} \frac{U(2^n x)}{2^n} = \psi(x)$ for all $x \in E$.
(iii) $d(U, \psi) \leq \frac{d(U,J(U))}{1-L} \leq \frac{1}{2-2L}$. That is, $\|U(x) - \psi(x)\| \leq \frac{1}{2-2L}\tau(x,x)$ for all $x \in E$.

Moreover, for each $x, y \in E$ we have,

$$\begin{aligned}
\|\psi(x+y) - \psi(x) - \psi(y)\| &= \lim_{n\to\infty} \|\frac{U(2^n(x+y))}{2^n} - \frac{U(2^n x)}{2^n} - \frac{U(2^n y)}{2^n}\| \\
&\leq \lim_{n\to\infty} \frac{1}{2^n}\tau(2^n x, 2^n y) \\
&\leq \lim_{n\to\infty} L^n \tau(x,y) \\
&= 0.
\end{aligned}$$

Hence ψ is an additive map. Now let $\psi' : E \to F$ be another additive map such that

$$\|U(x) - \psi'(x)\| \leq \frac{1}{2-2L}\tau(x,x) \qquad (x \in E),$$

so $J(\psi') = \psi'$ and $d(U, \psi') \leq \frac{1}{2-2L}$. In other words ψ' is a fixed point of J in X^*. Thus $\psi' = \psi$. □

Theorem 2.4. *Let $U : E \to F$ be a mapping and $\varphi : A \to B$ be an additive map such that for some control function $\rho : E^2 \to \mathbb{R}$, $\|\langle Ux, Uy\rangle - \varphi(\langle x, y\rangle)\| \leq \rho(x,y)$ for all $x, y \in E$. Let*

$$\tau(x,y) = \Big(\rho(x+y, x+y) + \rho(x+y, x) + \rho(x, x+y) + \rho(x+y, y) + \rho(y, x+y) + \rho(x,x) + \rho(y,y) + \rho(x,y) + \rho(y,x)\Big)^{\frac{1}{2}}$$

and suppose there is $0 < L < 1$ such that $\tau(2x, 2y) \leq 2L\tau(x,y)$. Then there exists a unique φ-morphism $T : E \to F$ such that $\|U(x) - T(x)\| \leq \dfrac{1}{2-2L}\tau(x,x)$ for all $x \in X$.

Proof. For all $x, y, z \in E$ we have

$$\begin{aligned}
&\|\langle U(x+y)-U(x)-U(y), U(z)\rangle\| \\
&\qquad = \|\langle U(x+y)-U(x)-U(y), U(z)\rangle - \varphi(\langle x+y, z\rangle) + \varphi(\langle x, z\rangle) \\
&\qquad\quad + \varphi(\langle y, z\rangle)\| \\
&\qquad \leq \|\langle U(x+y), U(z)\rangle - \varphi(\langle x+y, z\rangle)\| + \|\langle U(x), U(z)\rangle - \varphi(\langle x, z\rangle)\| \\
&\qquad\quad + \|\langle U(y), U(z)\rangle - \varphi(\langle y, z\rangle)\| \\
&\qquad \leq \rho(x+y, z) + \rho(x, z) + \rho(y, z)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|U(x+y)-U(x)-U(y)\|^2 \\
&\qquad = \|\langle U(x+y)-U(x)-U(y), U(x+y)-U(x)-U(y)\rangle\| \\
&\qquad \leq \|\langle U(x+y)-U(x)-U(y), U(x+y)\rangle\| \\
&\qquad\quad + \|\langle U(x+y)-U(x)-U(y), U(x)\rangle\| \\
&\qquad\quad + \|\langle U(x+y)-U(x)-U(y), U(y)\rangle\| \\
&\qquad \leq \rho(x+y, x+y) + \rho(x, x+y) + \rho(y, x+y) + \rho(x+y, x) + \rho(x, x) \\
&\qquad + \rho(y, x) + \rho(x+y, y) + \rho(x, y) + \rho(y, y).
\end{aligned}$$

It follows that

$$\|U(x+y)-U(x)-U(y)\| \leq \tau(x, y).$$

By Lemma 2.3, there is a unique additive map $T : E \to F$ such that

$$\|U(x) - T(x)\| \leq \frac{1}{2-2L}\tau(x, x) \qquad (x \in E).$$

Then

$$T(x) = \lim_{n\to\infty} \frac{U(2^n x)}{2^n}.$$

Now for each $x, y \in E$ we have

$$\begin{aligned}
\|\langle Tx, Ty\rangle - \varphi(\langle x, y\rangle)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|\langle U(2^n x), U(2^n y)\rangle - \varphi(\langle 2^n x, 2^n y\rangle)\| \\
&\leq \lim_{n\to\infty} \frac{1}{4^n}\rho(2^n x, 2^n y) \leq \lim_{n\to\infty} \frac{1}{4^n}\tau(2^n x, 2^n y)^2 \\
&= \lim_{n\to\infty} \left(\frac{1}{2^n}\tau(2^n x, 2^n y)\right)^2 \leq \left(\lim_{n\to\infty} L^n \tau(x, y)\right)^2 \\
&= 0.
\end{aligned}$$

This shows that T is a φ-morphism. Since each φ-morphism is an additive map Lemma 2.3 implies that T is the unique φ-morphism as desired. □

One can replace the condition $\tau(2x, 2y) \leq 2L\tau(x, y)$ on the control function τ by

$$\tau(x, y) \leq \frac{1}{2}L\tau(2x, 2y)$$

and obtain the following results.

Lemma 2.5. *If $U : E \to F$ is a mapping such that $\|U(x+y) - U(x) - U(y)\| \leq \tau(x,y)$ for some control function $\tau : E^2 \to \mathbb{R}$ and there is $0 < L < 1$ with $\tau(x,y) \leq \frac{1}{2}L\tau(2x,2y)$, then there exists a unique additive map $\psi : E \to F$ such that $\|U(x) - \psi(x)\| \leq \dfrac{L}{2-2L}\tau(x,x)$.*

Theorem 2.6. *Let $U : E \to F$ be a mapping and $\varphi : A \to B$ be an additive map such that for some control function $\rho : E^2 \to \mathbb{R}$, $\|\langle Ux, Uy\rangle - \varphi(\langle x, y\rangle)\| \leq \rho(x,y)$ for all $x, y \in E$. Let*

$$\tau(x,y) = \Big(\rho(x+y,x+y) + \rho(x+y,x) + \rho(x,x+y) + \rho(x+y,y) + \rho(y,x+y) + \rho(x,x) + \rho(y,y) + \rho(x,y) + \rho(y,x)\Big)^{\frac{1}{2}}$$

and suppose there is $0 < L < 1$ such that $\tau(x,y) \leq \frac{1}{2}L\tau(2x,2y)$. Then there exists a unique φ-morphism $T : E \to F$ such that $\|U(x) - T(x)\| \leq \dfrac{L}{2-2L}\tau(x,x)$ for all $x \in X$.

For a real number p let E_p denote either the whole space E if $p \geq 0$ or $E \setminus \{0\}$ if $p < 0$.

Corollary 2.7. *Let $U : E \to F$ be a mapping and $\varphi : A \to B$ be an additive map such that for some $p \neq 2$,*

$$\|\langle Ux, Uy\rangle - \varphi(\langle x, y\rangle)\| \leq c(\|x\|^p + \|y\|^p) \qquad (x, y \in E_p).$$

Then there exists a unique φ-morphism $T : E \to F$ such that

$$\|U(x) - T(x)\| \leq \frac{\sqrt{6c(2^p+2)}}{|2 - 2^{\frac{p}{2}}|}\|x\|^{\frac{p}{2}} \qquad (x \in E_p).$$

Proof. Define $\rho : E_p \times E_p \to \mathbb{R}$ by $\rho(x,y) = c(\|x\|^p + \|y\|^p)$, then apply Theorems 2.4 and 2.6 with

$$\tau(x,y) = \sqrt{6c(\|x+y\|^p + \|x\|^p + \|y\|^p)}$$

□

Remark 2.8. If E and F are two Hilbert C^*-modules over the same C^*-algebra A and $\varphi : A \to A$ is the identity map, then [6, Corollary 4.2] is a consequence of the above corollary.

Applying Theorem 2.4 and 2.6 with $\rho(x,y) = c\|x\|^p\|y\|^p$ we have the next result.

Corollary 2.9. *Let $U : E \to F$ be a mapping and $\varphi : A \to B$ be an additive map such that for some $p \neq 1$,*

$$\|\langle Ux, Uy\rangle - \varphi(\langle x, y\rangle)\| \leq c\|x\|^p\|y\|^p \qquad (x, y \in E_p).$$

Then there exists a unique φ-morphism $T : E \to F$ such that

$$\|U(x) - T(x)\| \leq \frac{\sqrt{c}(2^p+2)}{|2 - 2^p|}\|x\|^p \qquad (x \in E_p).$$

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