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SPECTRAL MAPPING THEOREMS AND STABILITY THEORY IN LINEAR DYNAMICAL SYSTEMS

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ABSTRACT. The spectral properties of the operators inducing linear dynamical systems on Banach spaces determine the asymptotic properties of the systems. In this note an interaction between the spectral mapping properties of operators and stability properties of the corresponding dynamical systems is presented.

1. INTRODUCTION

As we all can visualise, Mathematics, in a broader sense is a study of measurements, changes, forms and patterns. Study of changes created a system known as dynamical system which has made contact with many areas of mathematics, physics and social sciences. We shall present a concise historical development of the evolution of the dynamical systems. It began in work of Newton in mid 1600's as a branch of physics. He gave the laws of motion and solved the 2-body problem involving Sun and Earth. He invented differential calculus for this purpose which became starting point of modern mathematics. Later mathematicians and physicists tried to solve 3-body problem involving Sun, Earth and Moon and it was realised that it was impossible to solve it in sense of explicit formulas for motion of the three bodies. At this juncture the situation became hopeless. A break-through appeared in late 1800's with the work of Poincare and Liapunov. Poincare considered qualitative rather than quantitative questions. They asked about the stability of solar system and developed a powerful geometric approach to analyse some questions. This became starting point of dynamical systems where differential equations & iterated maps played main roles in the study. Henri Poincare (1854-1912), Alexander Liapunov (1857-1918) and George

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Birkhoff (1884-1944) are regarded as founders of dynamical systems. Major work on stability was done by A. Liapunov in his paper of 1892 "General problem in stability of movements". Liapunov functions & Liapunov exponents are used in the study of some dynamical systems. Concept of modern dynamical systems as we know today was developed by Birkhoff in early part of the 20^{th} century. Around that time functional analysis was being developed in work of Hilbert, Banach, Lebesgue etc. Functional analysis entered in a big way in study of dynamical systems. Infinite dimensional challenges came in dynamical systems. Hilbert's work and Banach's book on the "theory of linear operators" became tools & base for infinite dimensional dynamical systems. In linear case the theory of semi-groups of operators was developed to study evolutionary equations. By 1930's basic theory of dynamical systems was well in place and by 1970 or so finite dimensional & infinite dimensional dynamical systems were united and since then it became study of motions (dynamics) of evolutionary equations [22]. By middle of the 20^{th} century it developed in many directions having rapport with differential equations, operator theory, transformation groups, differential geometry and having applications in engineering, biology, physics and social sciences. For further details we refer to [4, 14, 22, 23]. In this more or less expository article we demonstrate an interplay of operator theoretic properties and dynamical properties of linear dynamical systems induced by operators. The spectral mapping properties play significant role in this interplay.

2. Time dependent dynamical systems

A continuous dynamical system is a triple $(\pi, \mathbb{R}, X)$, where X is a topological space, $\mathbb{R}$ is the additive group of real numbers with usual topology, and $\pi : \mathbb{R} \times X \to X$ is a continuous mapping such that

(1) $\pi(0, x) = x$, for every $x \in X$.
(2) $\pi(s + t, x) = \pi(s, \pi(t, x))$ for every $s, t \in \mathbb{R}$ & $x \in X$.

X is called the phase space or state space and π is called flow, motion or action. If $t \in \mathbb{R}$, then $\pi^t : X \to X$ defined as $\pi^t(x) = \pi(t, x)$, is a homeomorphism with $(\pi^t)^{-1} = \pi^{-t}$. Let $H^1 = \{\pi^t : t \in \mathbb{R}\}$. Then H^1 is a subgroup of the group H of all homeomorphisms of X. Let $x \in X$ and let $\pi_x : \mathbb{R} \to X$ be defined as $\pi_x(t) = \pi(t, x)$ for every $t \in \mathbb{R}$. Then range $\pi_x = \{\pi(t, x) : t \in \mathbb{R}\}$ is called the orbit of x. If $\pi(t, x) = x$ for every $t \in \mathbb{R}$, then x is called a fixed point or an equilibrium point of X(or of π). By a periodic point we mean a non-fixed point $x \in X$ such that $\pi(t, x) = x$, for some non-zero $t \in \mathbb{R}$. If at place of $\mathbb{R}$, we take the discrete group $\mathbb{Z}$, then what we get is known as discrete dynamical system. Every homeomorphism ϕ of X gives rise to a discrete dynamical system given by

$$\pi(n, x) = \phi^n(x); \text{ where } \phi^n = \phi \circ \phi \circ \phi \circ \phi \text{...........} \circ \phi \text{ (iteration } n \text{ times).}$$

Actually every discrete dynamical system comes from a homeomorphism [8]. If at place of $\mathbb{R}$(or $\mathbb{Z}$) we take $\mathbb{R}^+$(or $\mathbb{Z}^+$), the semigroups of numbers, then we get semidynamical systems, whose evolution is future-dependent. In most of the work they are taken to be dynamical systems. In this case every discrete semidynamical system comes from a self continuous map of X.

Note. If at place of $\mathbb{R}$ we take any topological group G, then the triple $(\pi, G.X)$ is called a transformation group and π is an action of G on X. This is a very broad area and includes many algebraic and topological systems. To make contact with reality we shall further specialize by taking X as a complex Banach space and $(\pi, \mathbb{R}^+, X)$ or $(\pi, \mathbb{Z}^+, X)$ as linear semi dynamical system. The motion $\pi : \mathbb{R} \times X \to X$ is linear if

$$\pi(t, \alpha x + \beta y) = \alpha\pi(t, x) + \beta\pi(t, y) \text{ for } t \in \mathbb{R} \text{ and } x, y \in X,\ \alpha, \beta \in \mathbb{C}.$$

Definition 2.1. Let x_0 be a fixed point of the dynamical system $(\pi, \mathbb{R}, X)$ on a Banach space X. Then

(1) x_0 is called a stable fixed point if for every $\epsilon > 0$, there exits $\delta > 0$ such that $\|x_0 - x\| < \delta$ implies that $\|\pi(t, x) - x_0\| < \epsilon$ for every $t \in \mathbb{R}^+$.
(2) x_0 is called asymptotically stable if there exits $\delta > 0$ such that $\|x_0 - x\| < \delta$ implies that $\lim\limits_{t\to\infty} \|\pi(t, x) - x_0\| = 0$.
(3) x_0 is exponentially stable if there exits $\delta > 0, \alpha > 0$ & $M > 0$ such that $\|x - x_0\| < \delta$ implies that $\|\pi(t, x) - x_0\| \le Me^{-\alpha t}\|x - x_0\|$ for every $t \in \mathbb{R}^+$.

For details on stable motions we refer to [2, 3, 4].

By $B(X)$ we denote the Banach algebra of all bounded linear operators on X. We shall concentrate on dynamical systems induced by operators and present a relation between operator theoretic properties and dynamical concepts in the induced system. Recall that if W is a Banach algebra with identity e and $x \in W$, then spectrum of x, denoted by $\sigma(x)$ is defined by

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } W\}.$$

It is known that $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$. The proof follows from two important theorems, namely, Hahn–Banach theorem and Liouville's theorem. Let $\rho(x) = \mathbb{C}\backslash\sigma(x)$ and let $f : \rho(x) \to W$, be defined by

$$f(\lambda) = (x - \lambda e)^{-1}.$$

Then f is analytic $W-$valued and $f'(\lambda) = -(x - \lambda e)^{-2}$. If $\sigma(x)$ is empty, then $\phi \circ f : \mathbb{C} \to \mathbb{C}$ is a bounded entire function vanishing at infinity for every $\phi \in W^*$, the dual of W. Thus it is constant 0, which is a contradiction since $f'(\lambda) = -(x - \lambda e)^{-2} \neq 0$. Since $\sigma(x)$ is closed and bounded, it is compact [7]. Theory of Banach algebra and C^*-algebra was developed extensively by Gelfand and Naimark around middle of the last century. Among other things they proved spectral mapping theorem, which became a powerful tool in study of stability of dynamical systems induced by operators. Let $E(\mathbb{C})$ denote the algebra of all entire functions. Let $x \in W$, a Banach algebra, and define $\phi_x : E(\mathbb{C}) \to W$ as $\phi_x(f) = f(x)$. Then ϕ_x is an algebra homomorphism. By $H(\sigma(x))$ we denote the algebra of all holomorphic functions on open sets containing $\sigma(x)$. Thus $f \in H(\sigma(x))$ if there exists an open set G containing $\sigma(x)$ and $f : G \to \mathbb{C}$ is holomorphic. The spectral mapping theorem is given in the following three cases. Further details on spectral theory can be seen in [1, 6, 7].

Theorem 2.2. *(spectral mapping theorem): Let W be a complex Banach algebra with identity. Then*

(1) *For every $f \in E(\mathbb{C})$ and every $x \in W$,*

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) : \lambda \in \sigma(x)\}.$$

(2) *For every $f \in H(\sigma(x))$,*

$$\sigma(f(x)) = f(\sigma(x)) \text{ for } x \in W.$$

(3) *In case X is a C^*-algebra, and x is a normal element of W, we have*

$$\sigma(f(x)) = f(\sigma(x)) \text{ for every } f \in C(\sigma(x)),$$

where $C(\sigma(x))$ is the Banach algebra of all continuous functions on $\sigma(x)$.

Outline of the proof.

(2) Let $f \in H(\sigma(x))$ and let $\lambda \in \sigma(x)$. Then $f(z) - f(\lambda) = (z - \lambda)g(z)$ for some $g \in H(\sigma(x))$. In case $f(\lambda) \notin \sigma(f(x))$, $f(x) - f(\lambda)e$ is invertible, and hence $x - \lambda e$ is invertible. Thus $f(\lambda) \in \sigma(f(x))$. This shows that $f(\sigma(x)) \subset \sigma(f(x))$. If $\alpha \notin f(\sigma(x))$, then $(f(z) - \alpha)^{-1} \in H(\sigma(x))$. Hence $f(x) - \alpha e$ is invertible. This shows that $\sigma(f(x)) \subset f(\sigma(x))$. Thus $\sigma(f(x)) = f(\sigma(x))$. The proof of (1) follows from proof of (2).

(3). If X is a C^*-algebra and x is a normal element, then the C^*-algebra $C^*(x)$ generated by x and x^* is commutative and hence by Gelfand–Naimark theorem the mapping $\phi : C(\sigma(x)) \to C^*(x)$ defined by $\phi(f) = f(x)$ is $*-$isometric isomorphism. Hence $\sigma(f(x)) = \sigma(\phi(f)) = \sigma(f)$. Since $f \in C(\sigma(x))$ and $\sigma(f) = f(\sigma(x))$, we prove that $\sigma(f(x)) = f(\sigma(x))$. This proves (3).

3. Operator-induced dynamical systems on Banach spaces

Let X be a Banach space, and let $A \in B(X)$, the Banach algebra of all bounded linear operators on X. Let $t \in \mathbb{R}^+$. Let

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \in B(X).$$

Define $\pi_A^c : \mathbb{R}^+ \times X \to X$ by $\pi_A^c(t, x) = e^{tA}(x)$. Then $(\pi_A^c, \mathbb{R}^+, X)$ is a continuous semidynamical system on X. If we define $\pi_A^d : \mathbb{Z}^+ \times X \to X$ as $\pi_A^d(n, X) = A^n(x)$, then π_A^d is a discrete dynamical system on X. Thus every operator A on a Banach space gives rise to at least two linear dynamical systems. If $X = \mathbb{C}^n$, the Banach space of all n-tupples of complex numbers and A is an $n \times n$ matrix, then A gives rise to a continuous motion and a discrete motion on $\mathbb{C}^n$. The study of these motions has been done in classical dynamical systems at juncture of 19^{th} and 20^{th} centuries. A continuous function $T : \mathbb{R}^+ \to B(X)$ is called a functional homomorphism if

$$T(s + t) = T(s)T(t) \text{ and } T(0) = I \text{ for } s, t \in \mathbb{R}^+.$$

If T is continuous with respect to strong operator topology on $B(X)$, then it is called c_0-functional homomorphism [20]. It is well known that every norm continuous functional homomorphism is of type $T(t) = e^{tA}$ for some operator $A \in B(X)$. There are c_0-functional homomorphisms which are not of this type.

For details and examples we refer to [5] and [10]. A c_0-functional homomorphism T is also known as a c_0-semigroup of operators on X and $\{T(t) : t \in \mathbb{R}^+\}$ is a commutative semigroup of $B(X)$. Every c_0-functional homomorphism T on X induces a continuous dynamical system π_T on X defined by

$$\pi_T(t, x) = T(t)x \text{ for } t \in \mathbb{R}^+ \text{ and } x \in X.$$

Let $T : \mathbb{R}^+ \to B(X)$ be a c_0-functional homomorphism. Then infinitesimal generator A of T is defined as

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t},$$

wherever this limit exits. A is also a closed linear operator with dense domain contained in X. This operator is not necessarily bounded [10]. If $A \in B(X)$ and $T(t) = e^{tA}$, then infinitesimal generator of T is A itself. If $X = L^2(\mathbb{R})$ and $(T(t)f)(s) = f(s + t)$ for $s \in \mathbb{R}$, then infinitesimal generator of T is differentiation operator i.e $Af = f'$, which is not a bounded operator. Not every unbounded operator on X is a generator of a c_0-functional homomorphism. But under certain conditions an unbounded operator is an infinitesimal generator of a c_0-functional homomorphism. In 1947 Hille and Yosida characterized generators of c_0-functional homomorphisms. For details we refer to [11] and [13]. If A is an unbounded generator of c_0-functional homomorphism $T : \mathbb{R}^+ \to B(X)$, then we also write $T(t) = e^{tA}$. Associated with an unbounded generator A of a c_0-functional homomorphism T, we get a dynamical system $\pi_A : \mathbb{R}^+ \times X \to X$ defined as

$$\pi_T(t, x) = T(t)x = e^{tA}(x), t \in \mathbb{R}, x \in X.$$

Abstract Cauchy Problem (ACP) associated with A is the following boundary-value problem

$$\begin{matrix} \frac{dx}{dt} = Ax \\ x(0) = x_0 \end{matrix} .$$

If $x(t) = T(t)x_0$, then x is a solution of this Abstract Cauchy Problem. The orbit function associated with x_0 is a solution of ACP. There are several ways of creating c_0-functional homomorphism associated with generator A [10]. In order to extend some of the results of dynamical systems induced by unbounded linear operators the spectral mapping theorems for c_0-functional homomorphisms are useful. Before recording the results we shall need the following definitions

Definition 3.1. : Let A be a closed operator on X with domain $D(A)$ and let

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in B(X)\}.$$

Then the set $\rho(A)$ is called resolvent set and $(\lambda I - A)^{-1}$ is called resolvent operator with respect to $\lambda \in \rho(A)$. We denote $(\lambda I - A)^{-1}$ by $R(\lambda, A)$. The complement of $\rho(A)$ in $\mathbb{C}$ is called the spectrum of A, denoted by $\sigma(A)$ as usual. By $\sigma_p(A)$ we denote the point spectrum of A, i.e., the set of all eigenvalues of A. By $\sigma_c(A)$ we denote the continuous spectrum of A, which is defined as

$$\sigma_c(A) = \{\lambda \in \mathbb{C} : \lambda \notin \sigma_p(A) \text{ and range of } \lambda I - A \text{ is a non-closed dense subset of } X\}.$$

Let $\sigma_{ap}(A)=\{\lambda\in\mathbb{C}:\lambda\in\sigma_p(A)$ or range of $\lambda I-A$ is not closed$\}$. The set $\sigma_{ap}(A)$ is called approximate point spectrum of A. Unlike bounded operators $\sigma(A)$ of an unbounded operator A may be empty or unbounded subset of $\mathbb{C}$. For example and details we refer to [10]. The following theorem contains some results about spectral mapping theorems for c_0-functional homomorphisms.

Theorem 3.2. *(spectral mapping theorem for c_0-functional homomorphisms): Let $T:\mathbb{R}^+\to B(X)$ be a c_0-functional homomorphism generated by a linear operator A. Then*

(1) $e^{t\sigma(A)}\subset\sigma(T(t))$ *for every* $t\in\mathbb{R}^+$.
(2) $e^{t\sigma_p(A)}\subset\sigma_p(T(t))$.
(3) *If T is eventually norm continuous or uniformly continuous or eventually compact, then T has spectral mapping property i.e*

$$\sigma(T(t))\backslash\{0\}=e^{t\sigma(A)}.$$

(4) *If X is a Hilbert space, then* $\sigma(T(t))\backslash\{0\}=\{e^{\alpha t}:\frac{\alpha+2\pi ik}{t}\in\sigma(A)$ *for some* $k\in\mathbb{Z}$ *or sequence* $(\|R(\alpha_k,A)\|)$ *is unbounded , where* $\alpha_k=\frac{\alpha+2\pi ik}{t}\}$.

Outline of the proof.

(1) Let $x\in X$. Then by definition of the infinitesimal generator A of T,

$$\begin{aligned}
A\int_0^t T(u)x\,du &= \lim_{h\to0^+}\frac{T(h)\int_0^t T(u)x\,du-\int_0^t T(u)x\,du}{h}\\
&= \lim_{h\to0^+}\frac{\int_0^t T(h+u)x\,du-\int_0^t T(u)x\,du}{h}\\
&= \lim_{h\to0^+}\frac{\int_h^{t+h} T(u)x\,du-\int_0^t T(u)x\,du}{h}\\
&= T(t)x-x.
\end{aligned}$$

It can also be shown that

$$A\int_0^t T(u)x\,du=\int_0^t T(u)Ax\,du.$$

Let $\lambda\in\mathbb{C}$. Then define $S:\mathbb{R}^+\to B(X)$ by $S(t)=e^{-\lambda t}T(t)$. Then S is also a c_0-functional homomorphism with generator $A-\lambda I$. Hence by above result

$$\begin{aligned}
e^{-\lambda t}T(t)x-x &= S(t)x-x=(A-\lambda I)\int_0^t S(u)x\,du\\
&= (A-\lambda I)\int_0^t e^{-\lambda u}T(u)x\,du=\int_0^t e^{-\lambda u}T(u)(A-\lambda I)x\,du.
\end{aligned}$$

Thus

$$(e^{\lambda t} - T(t))x = \int_0^t e^{-\lambda(u-t)} T(u)(\lambda I - A)x du.$$

From this it follows that $e^{\lambda t} - T(t)$ is not a bijection whenever $\lambda I - A$ is not a bijection. This proves (1)

(2) If $\lambda \in \sigma_p(A)$, then there exits $x \in D(A)$ such that $Ax = \lambda x$

Now

$$(e^{\lambda t} - T(t))x = \int_0^t e^{-\lambda(u-t)} T(u)(\lambda I - A)x du = 0$$

if $Ax = \lambda x$. Hence $e^{\lambda t} \in \sigma_p(T(t))$. For the proofs of (3) and (4) we refer to [10].

4. Asymptotic behaviour of dynamical systems induced by operators

In a dynamical system $(\pi, \mathbb{R}, X)$ the orbit function O_x of a state $x \in X$ is an $X-$valued function on $\mathbb{R}$ given by $O_x(t) = \pi(t, x)$ for every $t \in \mathbb{R}$. The range of O_x is called orbit of x and we denote it by $Orb(x)$. The totality of all such distinct orbits is called orbit space. The orbit function plays very important role in exploration of some asymptotic properties of the system like stability and cyclicity. Stability property has its origin in solutions of differential equations and cyclicity is concerned with invariant sets and invariant subspaces of inducing operators. If $A \in B(X)$, then the origin (the zero vector) is a fixed point of continuous dynamical systems $(\pi_A^c, \mathbb{R}^+, X)$. The spectral theory plays pivotal role in study of stability at origin. If $T : \mathbb{R}^+ \to B(X)$ is a c_0-functional homomorphism, then define the function $u : \mathbb{R}^+ \to \mathbb{R}^+$ as $u(t) = \|T(t)\|$ and for $x \in X$, define $u_x : \mathbb{R}^+ \to \mathbb{R}^+$ as $u_x(t) = \|T(t)x\| = \|O_x(t)\|$. Then u and u_x are positive real-valued continuous functions of variable $t \in \mathbb{R}^+$. Asymptotic behaviour of these functions characterize the asymptotic properties of the system. We know that the function u is dominated by a scalar multiple of exponential function [25]. The growth bound of T denoted as $w_0(T)$ is defined as

$$w_0(T) = \inf\{w \in \mathbb{R} : u(t) \leq Me^{wt}, \text{ for } t \in \mathbb{R}^+\}.$$

It follows that the spectral radius $r(T(t))$ of $T(t)$ is equal to $e^{w_0(T)t}$ [10]. In light of Definition 2.1, we say that the dynamical system π_T induced by T is uniformly stable at 0 if

$$\lim_{t\to\infty} u(t) = 0.$$

It is called uniformly exponentially stable if for some $\epsilon > 0$

$$\lim_{t\to\infty} e^{\epsilon t} u(t) = 0.$$

It turns out that uniform stability and exponential stability are equivalent. If

$$\lim_{t\to\infty} u_x(t) = 0 \text{ for every } x \in X,$$

then we say that the system is strongly stable. It is clear that uniform stability implies strong stability. The c_0-functional homomorphism induced by translations on $L^2(0,\infty)$ are strongly stable but not uniformly stable. If the system is strongly stable and each $u_x \in L^p[0,\infty)$, then it turns out that it is uniformly stable. We shall record all these results in the following theorem without giving details of the proofs. The proofs can be seen in [5] and [10].

Theorem 4.1. *Let T be a c_0-functional homomorphism on a Banach space X and let $(\pi_T, \mathbb{R}^+, X)$ be the dynamical system induced by T. Then*

(1) *π_T^c is uniformly stable iff it is uniformly exponentially stable.*
(2) *$w_0(T) < 0$ iff π_T^c is uniformly stable.*
(3) *$\{u_x : x \in X\} \subset L^p[0,\infty), p \geq 1$(i.e $\int_0^\infty |u_x(t)|^p \, dt < \infty$ for every $x \in X$) iff π_T^c is uniformly stable in case it is strongly stable.*

Initial work on stability theory was started by A. Liapunov at the end of the 19^{th} century. The spectral properties of the inducing operator plays significant role. In 1892 Liapunov proved that if the eigenvalues of the matrix lie in the open left half plane, then the matrix induces uniformly stable dynamical system. In the proof he used Jordon form of the matrix A. In begining of 20^{th} century with growth of the functional analysis the stability theorem of Liapunov was extended to bounded linear operators on Banach spaces. In this extension the spectral mapping theorem plays key part. In the following theorem uniform stability is characterized in terms of the spectrum of the inducing operator.

Theorem 4.2. *(Liapunov theorem for bounded operator) : Let X be a Banach space and let $A \in B(X)$. Let $(\pi_A^c, \mathbb{R}^+, X)$ and $(\pi_A^d, \mathbb{Z}^+, X)$ be the continuous and discrete dynamical systems induced by A respectively. Then*

(1) *π_A^c is uniformly stable at 0 iff $\mathrm{Re}\lambda < \infty$ for every $\lambda \in \sigma(A)$.*
(2) *π_A^d is uniformly stable at 0 iff $|\lambda| < 1$ for every $\lambda \in \sigma(A)$.*

Proof. (1) We know that π_A^c is uniformly stable iff $\left\|e^{tA}\right\| \to 0$ as $t \to \infty$. In light of spectral radius formula [7] this is true iff $\sigma(e^{tA})$ is contained in the open unit disc of $\mathbb{C}$. By the spectral mapping theorem

$$\sigma(e^{tA}) = e^{t\sigma(A)} = \{e^{t\lambda} : \lambda \in \sigma(A)\}$$

Let $\lambda = \lambda_1 + \lambda_2 i$. Then $e^{t\lambda} = e^{t(\lambda_1+\lambda_2 i)} = e^{t\lambda_1}e^{\lambda_2 ti}$. Thus $\left|e^{t\lambda}\right| = e^{t\lambda_1} = e^{t\mathrm{Re}\lambda}$. Hence $\sigma(e^{t\lambda})$ is contained in the unit disc iff $\left|e^{t\lambda}\right| < 1$ for every $\lambda \in \sigma(A)$. This is true iff $\mathrm{Re}\lambda < 0$ for $\lambda \in \sigma(A)$. This completes the proof of (1).

(2) It is clear that π_A^d is uniformly stable iff $\lim_{n\to\infty} \|A^n\| = 0$. Thus $\|A^n\| < 1$ for $n \in \mathbb{Z}^+$. Thus $\|A^n\| < 1$ iff $\sigma(A^n)$ is contained in the open unit disc. By spectral mapping theorem $\sigma(A^n) = (\sigma(A))^n = \{\lambda^n : \lambda \in \sigma(A)\}$. Let $\lambda = re^{i\theta}$ be polar represention of λ. Then $\lambda^n = r^n e^{in\theta}$. Now $\lambda \in \sigma(A)$ iff $\lambda^n \in \sigma(A^n)$. This is true iff $|\lambda^n| < 1$. Hence $r^n < 1$. This shows that $r < 1$. Hence $\sigma(A)$ is contained in the unit disc. This completes the proof of (2). $\square$

Note. In light of Liapunov theorem we see that solution of Abstract Cauchy Problem

$$\begin{array}{c} \frac{dx}{dt} = Ax \\ x(0) = x_0 \end{array}.$$

is stable iff $\mathrm{Re}\lambda < \infty$ for every $\lambda \in \sigma(A)$. Using discrete dynamical system induced by A we can see that A has a non-trivial invariant subspace iff orbit of some non-zero vector of X has non-dense span.

5. Weighted composition operators and stability theorem for unbounded operators

Let A be an unbounded operator on X. Then determination of the stability of solutions of the autonomous Abstract Cauchy Problem is a very difficult task. In this case spectral mapping theorem fails and we can not use Liapunov stability theorem which was used in case of bounded operators. It turns out that for some unbounded operators the Liapunov theorem is valid .The concrete operators like the weighted composition operators on some function spaces are helpful in extension of the Liapunov theorem for unbounded operators which are infinitesimal generators of c_0-functional homomorphisms. Thus in this section we assume that A is an unbounded operator which generates a c_0-functional homomorphism $T : \mathbb{R}^+ \to B(X)$. For $\infty > p \geq 1$, let $L^p(\mathbb{R}, X) = \{f, f : \mathbb{R} \to X, \int \|f(t)\|^p \, dt < \infty\}$ with L^p-norm. Let $C_0(\mathbb{R}, X) = \{f : f \in C(\mathbb{R}, X)$ and f vanishes at infinity$\}$ and let $C_b(\mathbb{R}, X) = \{f : f \in C(\mathbb{R}, X)$ and f is bounded$\}$ with sup norm. Then it is well known that above function spaces are Banach spaces and $C_0(\mathbb{R}, X) \subset C_b(\mathbb{R}, X)$ [19]. We know that in $L^p(\mathbb{R}, X)$ $f = g$ iff $f = g$ *a.e.* Let $\theta : \mathbb{R} \to B(X)$ be a function. Define M_θ on $L^p(\mathbb{R}, X)$ (or on $C_b(\mathbb{R}, X)$) by

$$M_\theta f(t) = \theta(t) f(t).$$

Then M_θ is called the multiplication operator induced by the operator-valued map θ. Multiplication operator induced by scalar-valued maps can similarly be defined. It is known that M_θ is a bounded operator on $L^p(\mathbb{R}, X)(or C_b(\mathbb{R}, X))$ iff θ is a (essentially) bounded map. For details we refer to [16] and [17]. Let $g : \mathbb{R} \to \mathbb{C}$ be a bounded function and let $A \in B(X)$. For $t \in \mathbb{R}$, define $\theta_t(s) = e^{tg(s)A}$. Then θ_t is an operator-valued bounded map and hence induces a multiplication operator on $L^p(\mathbb{R}, X)(or C_b(\mathbb{R}, X))$. Clearly the map $T : \mathbb{R}^+ \to B(X)$ defined by $T(t) = M_{\theta_t}$, is a c_0-functional homomorphism and hence gives rise to a dynamical system on $L^p(\mathbb{R}, X)(or C_b(\mathbb{R}, X))$. For details of proof we refer to [15]. If $\phi : \mathbb{R} \to \mathbb{R}$ is a non-singular (or continuous) map, then define C_ϕ on $L^p(\mathbb{R}, X)(or\ C_b(\mathbb{R}, X))$ by $C_\phi f = f \circ \phi$. This map is a linear transformation. It has been proved that C_ϕ is a bounded operator on $L^p(\mathbb{R}, X)$ iff $\mu\phi^{-1}(E) \leq b\mu(E)$ for some $b > 0$ and every measurable set E of $\mathbb{R}$ [16] & [24]. An operator of the type $M_\theta C_\phi$ is known as a weighted composition operator and we denote it by θC_ϕ. These operators have been studied on different function spaces and play important role in study of dynamical systems. For further details we refer to [9, 12, 18]. Let $\phi_t : \mathbb{R} \to \mathbb{R}$ be the map defined as $\phi_t(s) = s - t$. Then C_{ϕ_t} is an isometry on $L^p(\mathbb{R}, X)$ for every $t \in \mathbb{R}$. Let $T : \mathbb{R} \to B(L^p(\mathbb{R}, X))$ be defined as $T(t) = C_{\phi_t}$. Then T is a c_0-

functional homomorphism and gives rise to a dynamical system on $L^p(\mathbb{R}, X)$. It is clear that if $\theta : \mathbb{R} \to B(X)$ is a bounded map and $\phi : \mathbb{R} \to \mathbb{R}$ induces a continuous composition operator, then θC_ϕ is a bounded operator on $L^p(\mathbb{R}, X)$. In particular if θ is a constant map, then $\theta C_\phi \in B(L^2(\mathbb{R}, X))$. If $A \in B(X)$, then we have seen in Theorem 4.2 that asymptotic properties of dynamical system are influenced by the spectral properties of the operator A. There are linear dynamical system on X whose generators are unbounded operators. For example, the generator of dynamical system induced by translations C_{ϕ_t} is the differential operator, which is unbounded [5]. The spectral mapping theorem, which we used in the proof of the Liapunov stability theorem fails. At this juncture the theory of weighted composition operators help to extend Liapunov theorem for some unbounded operators. This helps in study of the stability theory of solutions of the Abstract Cauchy Problems induced by some unbounded generators of c_0-functional homomorphisms. Let us assume that A is an unbounded operator which is infinitesimal generator of a c_0-functional homomorphism $T : \mathbb{R}^+ \to B(X)$. Thus

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \; x \in D(A),$$

and A is densely defined closed operator [13]. The solution of the Abstract Cauchy Problem (ACP)

$$\begin{gathered} \tfrac{dx}{dt} = Ax \\ x(0) = x_0 \end{gathered}$$

is the orbit function of x_0 in the dynamical system induced by T i.e $x(t) = \pi_T(t, x_0) = T(t)x_0$ is a solution of ACP for every $x_0 \in D(A)$. In case A is bounded, Liapunov theorem determined the stability of solutions at the origin of X in terms of the spectrum of A. Though the spectral mapping theorem fails for unbounded operators, the Liapunov theorem has been extended to unbounded operators with assistance of the weighted composition operators [21]. For every $t \in \mathbb{R}$ we know that C_{ϕ_t} is a bounded composition operator on $L^p(\mathbb{R}, X)$. If T is c_0-functional homomorphism generated by unbounded operator A, then for every $t \in \mathbb{R}^+$, define the map $\theta_t : \mathbb{R}^+ \to B(X)$ by $\theta_t(s) = T(t)$ for every $s \in \mathbb{R}^+$. Thus θ_t is the constant map, and hence it induces multiplication operator M_{θ_t}. Since $M_{\theta_t}$and C_{ϕ_t} commute for every $t \in \mathbb{R}^+$, we conclude that the mapping $E : \mathbb{R}^+ \to B(L^p(\mathbb{R}, X))$ defined as $E(t) = M_{\theta_t} C_{\phi_t} (= \theta_t C_{\phi_t})$ i.e

$$(E(t)f)(s) = T(t)f(s - t) \text{ for } s \in \mathbb{R}$$

is a functional homomorphism induced by T. It is called evolution homomorphism induced by T(or A). We shall record some properties of E in the following theorem.

Theorem 5.1. *Let $E : \mathbb{R}^+ \to B(L^p(\mathbb{R}, X))$ be the evolution homomorphism induced by an unbounded operator A which is the generator of the c_0-functional homomorphism $T : \mathbb{R}^+ \to B(X)$. Then*

(1) *E is a c_0-functional homomorphism.*
(2) *Infinitesimal generator B of E is the closure of the operator $M_A - \frac{d}{dt}$, M_A being multiplication operator on $L^p(\mathbb{R}, X)$ induced by A.*

(3) *The spectrum of* $E(t)$, *is invariant under rotations around* 0 *of* $\mathbb{C}$.
(4) *The spectrum of the generator* B *is invariant under translation parallel to imaginary axis.*
(5) $\sigma(E(t))\backslash\{0\} = e^{t\sigma(B)} = \{e^{t\lambda} : \lambda \in \sigma(B)\}$ *(i.e* E *has spectral mapping property).*

Outline of the proof.

(1) Since E is the product of two commuting functional homomorphisms M_{θ_t} & C_{ϕ_t}, E is a functional homomorphism [5]. Let $f \in L^p(\mathbb{R}, X)$ and let $\{t_n\}$ be a sequence in $\mathbb{R}^+$ converging to $t \in \mathbb{R}^+$. Then

$$\begin{aligned} \|E(t_n)f - E(t)f\| &= \left\|\theta_{t_n} C_{\phi_{t_n}} - \theta_{t_n} C_{\phi_t} f + \theta_{t_n} C_{\phi_t} f - \theta_t C_{\phi_t} f\right\| \\ &\leq \|T(t_n)\| \left\|C_{\phi_{t_n}} f - C_{\phi_t} f\right\| + \|T(t_n) f \circ \phi_t - T(t) f \circ \phi_t\| . \end{aligned}$$

Using the uniform boundedness principle for $\{T(t_n) : n \in \mathbb{N}\}$ we prove that E is strongly continuous.

(2) If f is in domain of M_A & in domain of $\frac{d}{dt}$, then

$$\begin{aligned} \lim_{t\to 0^+} \frac{E(t)f - f}{t} &= \lim_{t\to 0^+} \frac{\theta_t C_{\phi_t} f - f}{t} \\ &= \lim_{t\to 0^+} \frac{T(t)f \circ \phi_t - f}{t} = \frac{T(t)f \circ \phi_t - T(t)f + T(t)f - f}{t} \\ &= \lim_{t\to 0^+} \frac{T(t)(f \circ \phi_t - f)}{t} + \lim_{t\to 0^+} \frac{T(t)f - f}{t} \\ &= -\frac{d}{dt} f + M_A f. \end{aligned}$$

Since $D(\frac{d}{dt}) \cap D(M_A)$ is dense in $L^p(\mathbb{R}, X)$, the infinitesimal generator B of E is closure of $M_A - \frac{d}{dt}$.

(3) Let $u \in \mathbb{R}$ and let $w_u : \mathbb{R} \to \mathbb{C}$ be defined as $w_u(s) = e^{ius}$. Consider the multiplication operator M_{w_u} on $L^p(\mathbb{R}, X)$ induced by the scalar-valued map w_u. Then M_{w_u} is an onto isometry. It can be proved that

$$M_{w_u}^{-1} E(k) M_{w_u} = e^{-iut} E(t)$$

and

$$M_{w_u}^{-1} B M_{w_u} = B - iu.$$

From this we can conclude parts (3) and (4) (see [10] for details). Proof of (5) is a special case of a general evolution semigroup induced by evolution family of operators. For proof we refer to [5].

Note. : The evolution c_0-functional homomorphism E has spectral mapping property, whereas the c_0-functional homomorphism generated by A may fail to have this property. E consists of weighted composition operators and this saves the Liapunov theorem for unbounded operators. The following theorem extends the Liapunov theorem to unbounded operator.

Theorem 5.2. *(Liapunov theorem for unbounded operators):*

Let A be an unbounded generator of a c_0-functional homomorphism T, and let E be the corresponding evolution c_0-functional homomorphism with generator B. Then the following are equivalent

(1) π_T is uniformly stable on X.

(2) π_E is uniformly stable on $L^p(\mathbb{R}, X)$.

(3) For $\lambda \in \sigma(B)$, $\mathrm{Re}\lambda < 0$ (i.e $\sigma(B)$ is contained in the open left half-plane).

Proof. (1) $\Longrightarrow$ (2) Suppose $\|T(t)\| \to 0$ as $t \to \infty$ .Then

$$\|E(t)\| = \|\theta_t C_{\phi_t}\| \leq \|T(t)\| \, \|C_{\phi_t}\| = \|T(t)\| \, .$$

This shows that $\|E(t)\| \to 0$ as $t \to \infty$. Hence π_E is uniformly stable.

(2) $\Longrightarrow$ (1) Now

$$\begin{aligned} \|T(t)\| &= \|T(t)I\| = \left\|T(t).C_{\phi_t}C_{\phi_{-t}}\right\| \\ &\leq \|E(t)\| \left\|C_{\phi_{-t}}\right\| = \|E(t)\| \, . \end{aligned}$$

Hence $\|T(t)\| \to 0$ whenever $\|E(t)\| \to 0$ as $t \to \infty$.

(3) $\Leftrightarrow$ (1) It can be shown that $\sigma(B)$ is contained in the open left half-plane iff $\sigma(T(t))$ is contained in the open unit disc for every t. Using spectral mapping property of E, we can conclude that π_T is uniformly stable. This completes the proof of the theorem. □

Note. Stability of solutions of autonomous ACP is studied with help of a single operator generating a c_0-functional homomorphism. The characterization of stability of solutions of non-autonomous ACP requires a family of operators creating an evolution family of operators on super function spaces $L^p(\mathbb{R}, X)$ or $C_b(\mathbb{R}, X)$ [22]. The concrete operators like translations, multiplications and weighted composition operators play significant role here too. Besides stability, hyperbolicity is another asymptotic property of a dynamical system. These simple looking concrete operators of multiplication and composition are also helpful in study of hyperbolicity. An interaction of these concrete operators and hyperbolicity property of the system will be presented in another article.

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