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HÖLDER TYPE INEQUALITIES ON HILBERT C^*-MODULES AND ITS REVERSES

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Dedicated to Professor Tsuyoshi Ando in celebration of his distinguished achievements in Matrix Analysis and Operator Theory

Communicated by J. Chmieliński

ABSTRACT. In this paper, we show Hilbert C^*-module versions of Hölder–McCarthy inequality and its complementary inequality. As an application, we obtain Hölder type inequalities and its reverses on a Hilbert C^*-module.

1. INTRODUCTION

The Hölder inequality is one of the most important inequalities in functional analysis. If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are n-tuples of nonnegative numbers, and $1/p + 1/q = 1$, then

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \qquad \text{for all } p > 1$$

and

$$\sum_{i=1}^{n} a_i b_i \ge \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \qquad \text{for all } p < 0 \text{ or } 0 < p < 1.$$

Non-commutative versions of the Hölder inequality and its reverses have been studied by many authors. Ando [1] showed the Hadamard product version of a Hölder type. Ando and Hiai [2] discussed the norm Hölder inequality and the matrix Hölder inequality. Mond and Shisha [15], Fujii, Izumino, Nakamoto and

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Seo [7], and Izumino and Tominaga [11] considered the vector state version of a Hölder type and its reverses. Bourin, Lee, Fujii and Seo [3] showed the geometric operator mean version, and so on.

In this paper, as a generalization of the vector state version due to [7], we show Hilbert C^*-module versions of Hölder–McCarthy inequality and its complementary inequality. As an application, we obtain Hölder type inequalities and its reverses on a Hilbert C^*-module.

2. PRELIMINARY

Let $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H, and $\mathscr{A}$ be a unital C^*-algebra of $\mathcal{B}(H)$ with the unit element e. For $a \in \mathscr{A}$, we denote the *absolute value* of a by $|a| = (a^*a)^{\frac{1}{2}}$. For positive elements $a, b \in \mathscr{A}$ and $t \in [0,1]$, the *t-geometric mean* of a and b in the sense of Kubo–Ando theory [12] is defined by

$$a \sharp_t b = a^{\frac{1}{2}} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^t a^{\frac{1}{2}}$$

for $a > 0$, i.e., a is invertible. In the case of non-invertible, since $a \sharp_t b$ satisfies the upper semicontinuity, we define $a \sharp_t b = \lim_{\varepsilon \to +0}(a + \varepsilon e) \sharp_t (b + \varepsilon e)$ in the strong operator topology. Hence $a \sharp_t b \in \mathscr{A}''$ in general, where $\mathscr{A}''$ is the bi-commutant of $\mathscr{A}$. In the case of $t = 1/2$, we denote $a \sharp_{1/2} b$ by $a \sharp b$ simply. The operator geometric mean has the symmetric property: $a \sharp_t b = b \sharp_{1-t} a$, and $a \sharp_t b = a^{1-t} b^t$ for commuting a and b.

A complex linear space $\mathscr{X}$ is said to be an *inner product $\mathscr{A}$-module* (or a pre-Hilbert $\mathscr{A}$-module) if $\mathscr{X}$ is a right $\mathscr{A}$-module together with a C^*-valued map $(x, y) \mapsto \langle x, y\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle \quad (x, y, z \in \mathscr{X}, \alpha, \beta \in \mathbb{C})$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a \quad (x, y \in \mathscr{X}, a \in \mathscr{A})$,
(iii) $\langle y, x\rangle = \langle x, y\rangle^* \quad (x, y \in \mathscr{X})$,
(iv) $\langle x, x\rangle \geq 0 \ (x \in \mathscr{X})$ and if $\langle x, x\rangle = 0$, then $x = 0$.

The linear structures of $\mathscr{A}$ and $\mathscr{X}$ are assumed to be compatible. If $\mathscr{X}$ satisfies all conditions for an inner-product $\mathscr{A}$-module except for the second part of (iv), then we call $\mathscr{X}$ a *semi-inner product $\mathscr{A}$-module*.

Let $\mathscr{X}$ be an inner product $\mathscr{A}$-module over a unital C^*-algebra $\mathscr{A}$. We define the norm of $\mathscr{X}$ by $\| x \| := \sqrt{\| \langle x, x\rangle \|}$ for $x \in \mathscr{X}$, where the latter norm denotes the C^*-norm of $\mathscr{A}$. If $\mathscr{X}$ is complete with respect to this norm, then $\mathscr{X}$ is called a *Hilbert $\mathscr{A}$-module*. An element x of the Hilbert $\mathscr{A}$-module is called *nonsingular* if the element $\langle x, x\rangle \in \mathscr{A}$ is invertible. For more details on Hilbert C^*-modules, see [13, 14].

In [6], from a viewpoint of operator geometric mean, we showed the following new Cauchy–Schwarz inequality:

Theorem 2.1 (Cauchy–Schwarz inequality)**.** *Let $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module over a unital C^*–algebra $\mathscr{A}$. If $x, y \in \mathscr{X}$ such that the inner product $\langle x, y\rangle$ has a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ with a partial isometry $u \in \mathscr{A}$,* *then*

$$|\langle x,y\rangle| \ \leq \ u^*\langle x,x\rangle u \ \sharp \ \langle y,y\rangle. \tag{2.1}$$

Under the assumption that $\mathscr{X}$ is an inner product $\mathscr{A}$-module and y is nonsingular, the equality in (2.1) holds if and only if $xu = yb$ for some $b \in \mathscr{A}$.

Next we review the basic concepts of adjointable operators on a Hilbert C^*-module. Let $\mathscr{X}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathscr{A}$. Let $End_{\mathscr{A}}(\mathscr{X})$ denote the set of all bounded $\mathbb{C}$-linear $\mathscr{A}$-homomorphism from $\mathscr{X}$ to $\mathscr{X}$. Let $T \in End_{\mathscr{A}}(\mathscr{X})$. We say that T is *adjointable* if there exists a $T^* \in End_{\mathscr{A}}(\mathscr{X})$ such that $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all $x,y \in \mathscr{X}$. Let $\mathcal{L}(\mathscr{X})$ denote the set of all adjointable operators from $\mathscr{X}$ to $\mathscr{X}$. Moreover, we define its norm by

$$\| T \| = \sup\{\| \langle Tx,Tx\rangle \|^{\frac{1}{2}} : \| x \| \leq 1\}.$$

Then $\mathcal{L}(\mathscr{X})$ is a C^*-algebra. The symbol I stands for the identity operator in $\mathcal{L}(\mathscr{X})$. The following lemma due to Paschke [16] is very important:

Lemma 2.2. *Let $\mathscr{X}$ be a Hilbert C^*-module and let T be a bounded $\mathscr{A}$-linear operator on $\mathscr{X}$. The following conditions are equivalent:*

(1) *T is a positive element of $\mathcal{L}(\mathscr{X})$;*
(2) *$\langle x,Tx\rangle \geq 0$ for all x in $\mathscr{X}$.*

In [8], we showed the following generalized Cauchy–Schwarz inequality on a Hilbert C^*-module by virtue of (2.1) and Lemma 2.2:

Theorem 2.3 (generalized Cauchy–Schwarz inequality)**.** *Let T be a positive operator in $\mathcal{L}(\mathscr{X})$. If $x,y \in \mathscr{X}$ such that $\langle x,Ty\rangle$ has a polar decomposition $\langle x,Ty\rangle = u|\langle x,Ty\rangle|$ with a partial isometry $u \in \mathscr{A}$, then*

$$|\langle x,Ty\rangle| \leq u^*\langle x,Tx\rangle u \ \sharp \ \langle y,Ty\rangle. \tag{2.2}$$

Under the assumption that $\langle y,Ty\rangle$ is invertible, the equality in (2.2) holds if and only if $T^{\frac{1}{2}}(xu) = T^{\frac{1}{2}}(yb)$ for some $b \in \mathscr{A}$.

3. Hölder–McCarthy inequality

In this section, we show two Hilbert C^*-module versions of Hölder–McCarthy inequality and its complementary inequality. For convenience, we use the notation $\natural_t$ for the binary operation

$$a \ \natural_t \ b = a^{\frac{1}{2}} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^t a^{\frac{1}{2}} \qquad \text{for } t \notin [0,1],$$

whose formula is the same as $\sharp_t$.

Theorem 3.1. *Let T be a positive operator in $\mathcal{L}(\mathscr{X})$ and x a nonsingular element of $\mathscr{X}$.*

(1) *If $p \geq 1$, then $\langle x,Tx\rangle \leq \langle x,x\rangle \ \sharp_{1/p} \ \langle x,T^px\rangle$.*
(2) *If $p \leq -1$ or $1/2 \leq p \leq 1$, then $\langle x,x\rangle \ \natural_{1/p} \ \langle x,T^px\rangle \leq \langle x,Tx\rangle$.*

Proof. For a nonsingular element x of $\mathscr{X}$, Put

$$\Phi_x(X) = \langle x\langle x,x\rangle^{-\frac{1}{2}}, Xx\langle x,x\rangle^{-\frac{1}{2}}\rangle \qquad \text{for} \quad X \in \mathcal{L}(\mathscr{X}).$$

Then Φ_x is a unital positive linear map from $\mathcal{L}(\mathscr{X})$ to $\mathscr{A}$.

Suppose that $p \geq 1$. Since $t^{1/p}$ is operator concave, it follows from [4, 5] that $\Phi_x(T^{1/p}) \leq \Phi_x(T)^{1/p}$ and this implies

$$\langle x,x\rangle^{-\frac{1}{2}}\langle x,T^{1/p}x\rangle\langle x,x\rangle^{-\frac{1}{2}} \leq \left(\langle x,x\rangle^{-\frac{1}{2}}\langle x,Tx\rangle\langle x,x\rangle^{-\frac{1}{2}}\right)^{1/p}$$

and

$$\begin{aligned}\langle x,T^{1/p}x\rangle &\leq \langle x,x\rangle^{\frac{1}{2}}\left(\langle x,x\rangle^{-\frac{1}{2}}\langle x,Tx\rangle\langle x,x\rangle^{-\frac{1}{2}}\right)^{1/p}\langle x,x\rangle^{\frac{1}{2}} \qquad (3.1)\\ &= \langle x,x\rangle \ \sharp_{1/p}\ \langle x,Tx\rangle.\end{aligned}$$

Replacing T by T^p in (3.1), we have (1).

Suppose that $p \leq -1$ or $1/2 \leq p \leq 1$. Since $-1 \leq 1/p < 0$ or $1 \leq 1/p \leq 2$, we have $\Phi_x(T)^{\frac{1}{p}} \leq \Phi_x(T^{\frac{1}{p}})$ by the operator convexity of $t^{1/p}$ and this implies

$$\left(\langle x,x\rangle^{-\frac{1}{2}}\langle x,Tx\rangle\langle x,x\rangle^{-\frac{1}{2}}\right)^{\frac{1}{p}} \leq \langle x,x\rangle^{-\frac{1}{2}}\langle x,T^{\frac{1}{p}}x\rangle\langle x,x\rangle^{-\frac{1}{2}}.$$

Hence it follows that

$$\langle x,x\rangle \ \natural_{1/p}\ \langle x,Tx\rangle \ \leq \ \langle x,T^{\frac{1}{p}}x\rangle \qquad (3.2)$$

and replacing T by T^p in (3.2) we have (2). □

Remark 3.2. The inequality (2) of Theorem 3.1 does not hold for $0 < p < 1/2$ in general. In fact, we give a simple counterexample to the case of $p = 1/3$ as follows: Put

$$\mathscr{X} = M_4(\mathbb{C}) = M_2(M_2(\mathbb{C})) \quad \text{and} \quad \mathscr{A} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$$

and

$$\Phi(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}) = \begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}$$

for $X,Y,Z,W \in M_2(\mathbb{C})$. Then $\mathscr{X}$ is a Hilbert $\mathscr{A}$-module with an inner product $\langle x,y\rangle = \Phi(x^*y)$ for $x,y \in \mathscr{X}$. Let

$$z = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $T = T_z$ is defined by $T_zy = zy$ for all $y \in \mathscr{X}$, then T is a positive operator in $\mathcal{L}(\mathscr{X})$ and

$$\left(\langle x,x\rangle^{-1/2}\langle x,Tx\rangle\langle x,x\rangle^{-1/2}\right)^3 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} \oplus \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

and

$$\langle x,x\rangle^{-1/2}\langle x,T^3x\rangle\langle x,x\rangle^{-1/2} = \begin{pmatrix} 29 & 22 \\ 22 & 17 \end{pmatrix} \oplus \begin{pmatrix} 17 & 17 \\ 17 & 17 \end{pmatrix},$$

so that

$$\langle x,x\rangle^{-1/2}\langle x,T^3x\rangle\langle x,x\rangle^{-1/2}-\left(\langle x,x\rangle^{-1/2}\langle x,Tx\rangle\langle x,x\rangle^{-1/2}\right)^3$$
$$=\begin{pmatrix}16 & 14\\ 14 & 12\end{pmatrix}\oplus\begin{pmatrix}13 & 13\\ 13 & 13\end{pmatrix}\not\geq 0\oplus 0.$$

Next, we show a complementary part of Theorem 3.1. For this, we need the *generalized Kantorovich constant* $K(\alpha,\beta,p)$ for $0<\alpha<\beta$, which is defined by

$$K(\alpha,\beta,p)=\frac{\alpha\beta^p-\beta\alpha^p}{(p-1)(\beta-\alpha)}\left(\frac{p-1}{p}\frac{\beta^p-\alpha^p}{\alpha\beta^p-\beta\alpha^p}\right)^p \tag{3.3}$$

for any real number $p\in\mathbb{R}$, see also [10, Definition 2.2]. The constant $K(\alpha,\beta,p)$ satisfies $0<K(\alpha,\beta,p)\leq 1$ for $0\leq p\leq 1$ and $K(\alpha,\beta,p)\geq 1$ for $p\notin[0,1]$. For more details on the generalized Kantorovich constant, see [10, Chapter 2.7].

Theorem 3.3. *Let T be a positive invertible operator in $\mathcal{L}(\mathscr{X})$ such that $\alpha I\leq T\leq\beta I$ for some scalars $0<\alpha<\beta$, and x a nonsingular element of $\mathscr{X}$.*

(1) *If $p\geq 1$, then*

$$\langle x,x\rangle\ \sharp_{1/p}\ \langle x,T^px\rangle\leq K(\alpha,\beta,p)^{1/p}\langle x,Tx\rangle.$$

(2) *If $p\leq -1$ or $1/2\leq p\leq 1$, then*

$$\langle x,Tx\rangle\leq K(\alpha^p,\beta^p,1/p)\langle x,x\rangle\ \natural_{1/p}\ \langle x,T^px\rangle,$$

where the generalized Kantorovich constant $K(\alpha,\beta,p)$ is defined by (3.3).

Proof. For a nonsingular element x of $\mathscr{X}$, put $\Phi_x(X)=\langle x\langle x,x\rangle^{-\frac{1}{2}},Xx\langle x,x\rangle^{-\frac{1}{2}}\rangle$ for $X\in\mathcal{L}(\mathscr{X})$. Then $\Phi_x:\mathcal{L}(\mathscr{X})\mapsto\mathscr{A}$ is a unital positive linear map.

Suppose that $p\geq 1$. It follows from [10, Lemma 4.3] that

$$\Phi_x(T^p)\leq K(\alpha,\beta,p)\Phi_x(T)^p.$$

This implies

$$\langle x,x\rangle\ \sharp_{1/p}\ \langle x,T^px\rangle\leq K(\alpha,\beta,p)^{1/p}\langle x,Tx\rangle$$

and we have (1).

In the case of $p\leq -1$ or $1/2\leq p\leq 1$, since $-1\leq 1/p<0$ or $1\leq 1/p\leq 2$, it follows that $\Phi_x(T^{1/p})\leq K(\alpha,\beta,1/p)\Phi_x(T)^{1/p}$. Similarly we have the desired inequality (2). □

Next, we discuss Hölder–McCarthy type inequalities on a Hilbert C^*-module outside intervals of Theorem 3.1.

Corollary 3.4. *Let T be a positive invertible operator in $\mathcal{L}(\mathscr{X})$ such that $\alpha I\leq T\leq\beta I$ for some scalars $0<\alpha<\beta$, and x a nonsingular element of $\mathscr{X}$. If $-1<p<0$ or $0<p<1/2$, then*

$$K(\alpha^p,\beta^p,1/p)^{-1}\langle x,Tx\rangle\leq\langle x,x\rangle\ \natural_{1/p}\ \langle x,T^px\rangle\leq K(\alpha^p,\beta^p,1/p)\langle x,Tx\rangle,$$

where the generalized Kantorovich constant $K(\alpha,\beta,p)$ is defined by (3.3).

Proof. For a unital positive linear map Φ_x from $\mathcal{L}(\mathscr{X})$ to $\mathscr{A}$, it follows from [10, Lemma 4.3] that for $-1 < p < 0$ or $0 < p < 1/2$

$$K(\alpha,\beta,1/p)^{-1}\Phi_x(T)^{1/p} \le \Phi_x(T^{1/p}) \le K(\alpha,\beta,1/p)\Phi_x(T)^{1/p}.$$

Hence we have this theorem as in the proof of Theorem 3.3. □

Similarly we have the following Hölder–McCarthy type inequality on a Hilbert C^*-module and its complementary inequality as follows:

Theorem 3.5. *Let T be a positive invertible operator in $\mathcal{L}(\mathscr{X})$ such that $\alpha I \le T \le \beta I$ for some scalars $0 < \alpha < \beta$. Then for $0 < p < 1$*

$$K(\alpha,\beta,p)\langle x,x\rangle \sharp_p \langle x,Tx\rangle \le \langle x,T^p x\rangle \le \langle x,x\rangle \sharp_p \langle x,Tx\rangle$$

for every nonsingular element $x \in \mathscr{X}$, where $K(\alpha,\beta,p)$ is defined by (3.3).

4. Hölder inequality

As an application of Theorem 3.1 and Theorem 3.3, we show Hölder type inequalities on a Hilbert C^*-module and its reverses.

Theorem 4.1. *Let A and B be positive invertible operators in $\mathcal{L}(\mathscr{X})$ and x a nonsingular element of $\mathscr{X}$, and $\frac{1}{p}+\frac{1}{q}=1$.*

(1) *If $p > 1$, then*

$$\langle x, B^q \sharp_{1/p} A^p\, x\rangle \le \langle x,B^q x\rangle \sharp_{1/p} \langle x,A^p x\rangle \tag{4.1}$$

or

$$\langle x, A^p \sharp_{1/q} B^q\, x\rangle \le \langle x,A^p x\rangle \sharp_{1/q} \langle x,B^q x\rangle. \tag{4.2}$$

(2) *If $p \le -1$ or $\frac{1}{2} \le p < 1$, then*

$$\langle x, B^q \natural_{1/p} A^p\, x\rangle \ge \langle x,B^q x\rangle \natural_{1/p} \langle x,A^p x\rangle \tag{4.3}$$

or

$$\langle x, A^p \natural_{1/q} B^q\, x\rangle \ge \langle x,A^p x\rangle \natural_{1/q} \langle x,B^q x\rangle. \tag{4.4}$$

Proof. Replacing x and T by $B^{\frac{q}{2}}x$ and $(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})^{\frac{1}{p}}$ in (1) of Theorem 3.1 respectively, we have (4.1) of Theorem 4.1. By (4.1) and the symmetric property of t-geometric mean, we have (4.2). The latter (4.3) and (4.4) are proved similarly. □

By Theorem 3.5, we have the following weighted version of Cauchy type inequality on a Hilbert C^*-module.

Theorem 4.2. *Let A and B be positive invertible operators in $\mathcal{L}(\mathscr{X})$ such that $\alpha I \le A, B \le \beta I$ for some scalars $0 < \alpha < \beta$. Then for $0 < p < 1$*

$$K(\frac{\alpha^2}{\beta^2},\frac{\beta^2}{\alpha^2},p)\langle x,B^2x\rangle \sharp_p \langle x,A^2x\rangle \le \langle x,A^2 \sharp_p B^2x\rangle \le \langle x,B^2x\rangle \sharp_p \langle x,A^2x\rangle$$

for every nonsingular element $x \in \mathscr{X}$.

Proof. Replace x and T by Bx and $B^{-1}A^2B^{-1}$ in Theorem 3.5 respectively. Since $\frac{\alpha^2}{\beta^2}I \le B^{-1}A^2B^{-1} \le \frac{\beta^2}{\alpha^2}$, the theorem follows. □

If we put $p = 1/2$ in Theorem 4.2, then we have the following Pólya-Szegö type inequality on a Hilbert C^*-module which is regarded as a reverse of Cauchy type inequality, also see [8, Theorem 3.3].

Corollary 4.3. *Let A and B be positive invertible operators in $\mathcal{L}(\mathscr{X})$ such that $\alpha I \le A, B \le \beta I$ for some scalars $0 < \alpha < \beta$. Then*

$$\langle x, Ax\rangle \sharp \langle x, Bx\rangle \le \frac{\alpha+\beta}{2\sqrt{\alpha\beta}}\langle x, A \sharp Bx\rangle$$

for every nonsingular element $x \in \mathscr{X}$.

Next, we show a complementary version of Theorem 4.1.

Theorem 4.4. *Let A and B be positive invertible operators in $\mathcal{L}(\mathscr{X})$ such that $\alpha I \le A, B \le \beta I$ for some scalars $0 < \alpha < \beta$, and x a nonsingular element of $\mathscr{X}$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

(1) *If $p > 1$, then*

$$\langle x, B^q x\rangle \sharp_{1/p} \langle x, A^p x\rangle \le K\left(\frac{\alpha}{\beta^{q-1}}, \frac{\beta}{\alpha^{q-1}}, p\right)^{\frac{1}{p}} \langle x, B^q \sharp_{1/p} A^p\ x\rangle.$$

(2) *If $p \le -1$ or $1/2 \le p < 1$, then*

$$\langle x, B^q x\rangle \natural_{1/p} \langle x, A^p x\rangle \ge K\left(\frac{\alpha^p}{\beta^q}, \frac{\beta^p}{\alpha^q}, \frac{1}{p}\right)^{-1} \langle x, B^q \natural_{1/p} A^p x\rangle.$$

Proof. Replace x and T by $B^{\frac{q}{2}}x$ and $(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})^{\frac{1}{p}}$ in (1) of Theorem 3.3 respectively. Since $\alpha/\beta^{q-1}I \le (B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}})^{\frac{1}{p}} \le \beta/\alpha^{q-1}I$, we have (1) of Theorem 4.4. The latter (2) are proved similarly. □

Next, we discuss Hölder type inequalities in a complementary interval of Theorem 4.1.

Corollary 4.5. *Let A and B be positive invertible operators in $\mathcal{L}(\mathscr{X})$ such that $\alpha I \le A, B \le \beta I$ for some scalars $0 < \alpha < \beta$, and x a nonsingular element of $\mathscr{X}$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $-1 < p < 0$ or $0 < p < \frac{1}{2}$, then*

$$\begin{aligned} K\left(\frac{\alpha^p}{\beta^q}, \frac{\beta^p}{\alpha^q}, \frac{1}{p}\right)^{-1} \langle x, B^q \natural_{1/p} A^p x\rangle &\le \langle x, B^q x\rangle \natural_{1/p} \langle x, A^p x\rangle \\ &\le K\left(\frac{\alpha^p}{\beta^q}, \frac{\beta^p}{\alpha^q}, \frac{1}{p}\right) \langle x, B^q \natural_{1/p} A^p x\rangle. \end{aligned}$$

Proof. Replacing x and T by $B^{\frac{q}{2}}x$ and $\left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}}$ in Corollary 3.4 respectively, we have this theorem. □

5. Weighted Cauchy–Schwarz inequality

In this section, we discuss weighted Cauchy–Schwarz inequality on a Hilbert C^*-module. We cite [9] for the case of the Hilbert space operator.

For $T \in \mathcal{L}(\mathscr{X})$, we denote the range of T and the kernel of T by $R(T)$ and $N(T)$, respectively. A closed submodule $\mathscr{M}$ of $\mathscr{X}$ is said to be *complemented* if

$\mathscr{X} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. Suppose that the closures of the ranges of T and T^* are both complemented. Then it follows from [13, page 30] that T has a polar decomposition $T = U|T|$ with a partial isometry $U \in \mathcal{L}(\mathscr{X})$ and $N(U) = N(|T|)$. Also, we showed in [8, Lemma 6.1] that

$$|T^*|^q = U|T|^q U^* \qquad \text{for any positive number } q. \tag{5.1}$$

As a generalization of Theorem 2.3, we have the following inequality.

Theorem 5.1 (Weighted Cauchy–Schwarz Inequality)**.** *Let T be an operator in $\mathcal{L}(\mathscr{X})$ such that the closures of the ranges of T and T^* are both complemented. If $x, y \in \mathscr{X}$ such that $\langle Tx, y\rangle$ has a polar decomposition $\langle Tx, y\rangle = u|\langle Tx, y\rangle|$ with a partial isometry $u \in \mathscr{A}$, then the following inequality holds*

$$|\langle Tx, y\rangle| \le u^*\langle x, |T|^{2\alpha}x\rangle u \ \sharp \ \langle y, |T^*|^{2\beta}y\rangle \tag{5.2}$$

for any $\alpha, \beta \in [0,1]$ and $\alpha + \beta = 1$. In particular,

$$|\langle Tx, y\rangle| \le u^*\langle x, |T|^2 x\rangle u \ \sharp \ \langle y, UU^* y\rangle$$

and

$$|\langle Tx, y\rangle| \le u^*\langle x, U^*Ux\rangle u \ \sharp \ \langle y, |T^*|^2 y\rangle.$$

Moreover, under the assumption that $\langle y, |T^*|^{2\beta}y\rangle$ is invertible for $\beta \in [0,1]$, the equality in (5.2) holds if and only if $Txu = |T^*|^{2\beta}yb$ for some $b \in \mathscr{A}$.

Proof. In the case of $\alpha = 0$ or 1, it follows from Theorem 2.1 that

$$|\langle Tx, y\rangle| = |\langle |T|x, U^*y\rangle| \le u^*\langle x, |T|^2x\rangle u \ \sharp \ \langle y, UU^*y\rangle$$

and

$$\begin{aligned} |\langle Tx, y\rangle| &= |\langle x, |T|U^*y\rangle| = |\langle x, U^*U|T|U^*y\rangle| = |\langle Ux, |T^*|y\rangle| \\ &\le u^*\langle Ux, Ux\rangle u \ \sharp \ \langle |T^*|y, |T^*|y\rangle = u^*\langle x, U^*Ux\rangle u \ \sharp \ \langle y, |T^*|^2 y\rangle \end{aligned}$$

by (5.1).

In the case of $0 < \alpha < 1$, we have

$$\begin{aligned} |\langle Tx, y\rangle| = |\langle U|T|x, y\rangle| &= |\langle |T|^{\alpha}x, |T|^{\beta}U^*y\rangle| \quad \text{by } \alpha + \beta = 1 \\ &\le u^*\langle x, |T|^{2\alpha}x\rangle u \ \sharp \ \langle y, U|T|^{2\beta}U^*y\rangle \quad \text{by Theorem 2.1} \\ &= u^*\langle x, |T|^{2\alpha}x\rangle u \ \sharp \ \langle y, |T^*|^{2\beta}y\rangle. \quad \text{by (5.1)}. \end{aligned}$$

Next, we consider the equality conditions in (5.2). Since $\langle Tx, y\rangle = \langle |T|^{\alpha}x, |T|^{\beta}U^*y\rangle$ and $\langle y, |T^*|^{2\beta}y\rangle$ is invertible for $\beta \in [0,1]$, it follows from Theorem 2.1 that the equality in (5.2) holds if and only if $|T|^{\alpha}xu = |T|^{\beta}U^*yb$ for some $b \in \mathscr{A}$. Since $|T|x = 0$ if and only if $|T|^{1/2}x = 0$, it follows that $N(|T|) = N(|T|^q)$ for any positive real numbers $q > 0$. If $|T|^{\beta}\left(|T|^{\alpha}xu - |T|^{\beta}U^*yb\right) = 0$, then $|T|^q\left(|T|^{\alpha}xu - |T|^{\beta}U^*yb\right) = |T|^{\alpha+q}xu - |T|^{\beta+q}U^*yb = 0$ for any $q > 0$ and this implies $|T|^{\alpha}xu - |T|^{\beta}U^*yb = 0$. Therefore we have the following implications:

$$\begin{aligned} &|T|^{\alpha}xu = |T|^{\beta}U^*yb \Longleftrightarrow |T|^{\alpha+\beta}xu = |T|^{2\beta}U^*yb \Longleftrightarrow U|T|xu = U|T|^{2\beta}U^*yb \\ &\Longleftrightarrow Txu = |T^*|^{2\beta}yb \qquad \text{by (5.1)}. \end{aligned}$$

□

If we put $\alpha = \beta = \frac{1}{2}$ in Theorem 5.1, then we have the following inequality.

Theorem 5.2. *Let T be an operator in $\mathcal{L}(\mathscr{X})$ such that the closures of the ranges of T and T^* are both complemented. If $x, y \in \mathscr{X}$ such that $\langle Tx, y\rangle$ has a polar decomposition $\langle Tx, y\rangle = u|\langle Tx, y\rangle|$ with a partial isometry $u \in \mathscr{A}$, then*

$$|\langle Tx, y\rangle| \leq u^*\langle x, |T|x\rangle u \sharp \langle y, |T^*|y\rangle. \tag{5.3}$$

Moreover, under the assumption that $\langle y, |T^*|y\rangle$ is invertible, the equality in (5.3) holds if and only if $Txu = |T^*|yb$ for some $b \in \mathscr{A}$.

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