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SINGULAR VALUES AND EIGENVALUES OF MATRICES IN $\mathfrak{so}_n(\mathbb{C})$ AND $\mathfrak{sp}_n(\mathbb{C})$

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This paper is dedicated to Professor Tsuyoshi Ando for his significant contributions to linear algebra, operator theory and functional analysis

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ABSTRACT. We give a complete relation between the singular values and eigenvalues of a complex skew symmetric matrix in terms of multiplicative majorization and double occurrences of singular values and eigenvalues. Similar studies are given for matrices in the algebras $\mathfrak{sp}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{R})$.

1. INTRODUCTION

Let $A \in \mathbb{C}_{n\times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ arranged in descending order $|\lambda_1| \geq \cdots \geq |\lambda_n|$ according to their moduli. The singular values of A are the nonnegative square roots of the eigenvalues of the positive semi-definite matrix A^*A and are denoted by $s_1 \geq \cdots \geq s_n$. Weyl [9] established the multiplicative majorization relation between the eigenvalues and singular values of A and Horn [3] established the converse (see Ando's paper [1] for some majorization results).

Theorem 1.1. (Weyl-Horn) Let $A \in \mathbb{C}_{n\times n}$ with singular values $s_1 \geq \cdots \geq s_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$ ordered as $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} s_j, \quad k = 1, \dots, n-1, \tag{1.1}$$

$$\prod_{j=1}^{n} |\lambda_j| = \prod_{j=1}^{n} s_j. \tag{1.2}$$

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Conversely, if $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $s_1 \geq \cdots \geq s_n$ satisfy (1.1) and (1.2), then there exists $A \in \mathbb{C}_{n\times n}$ such that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues and $s_1, \ldots, s_n$ are the singular values of A, respectively.

See [7] for a simple proof of Horn's theorem. Thompson [8] studied the real counterpart, i.e., $A \in \mathbb{R}_{n\times n}$. In this case the eigenvalues of A must occur in complex conjugate pairs; and such is the only additional condition.

Our goal in Section 2 is to study the analogy of Theorem 1.1 for complex skew symmetric matrix $A \in \mathbb{C}_{m\times m}$, i.e., $A^\top = -A$. This skew symmetry yields

$$\det(A - tI) = \det(A - tI)^\top = \det(-A - tI),$$

i.e., the eigenvalues of A occur in pairs but opposite in sign, counting multiplicities. Moreover the singular values $s_1, s_1, \ldots, s_{[m/2]}, s_{[m/2]}, (0)$ of A also occur in pairs. Here (0) refers to a zero singular value when m is odd [4, p.217]. Indeed, according to Hua decomposition [5, Theorem 7, p.481], there exists $U \in \mathrm{U}(m)$ such that

$$UAU^\top = \begin{cases} s_1 J \oplus s_2 J \oplus \cdots \oplus s_n J & \text{if } m \text{ is even} \\ s_1 J \oplus s_2 J \oplus \cdots \oplus s_n J \oplus (0) & \text{if } m \text{ is odd,} \end{cases}$$

where $n := [m/2]$ and

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $\mathrm{U}(m)$ denotes the unitary group. Unlike unitary equivalence $A \mapsto UAV$ $(U, V \in \mathrm{U}(m))$ is used for the origin Weyl-Horn setting, unitary congruence $A \mapsto UAU^\top$ $(U \in \mathrm{U}(m))$ is only allowed to handle the skew symmetric case. In Section 3 we have analogous study for the complex symplectic Lie algebra $\mathfrak{sp}_n(\mathbb{C})$. The doubly occurrence of the eigenvalues remains but the singular values are arbitrary. The real case $\mathfrak{sp}_n(\mathbb{R})$ is also studied.

2. Skew symmetric matrices

Denote by $\mathfrak{so}_m(\mathbb{C})$ the set of $m \times m$ complex skew symmetric matrices. The following theorem asserts that for the even case Weyl-Horn's multiplicative majorization together with double occurrence of the eigenvalues and singular values are the necessary and sufficient conditions. For the odd case, multiplicative weak majoriation plays the role of multiplicative majorization.

Theorem 2.1. (1) Let $A \in \mathfrak{so}_{2n}(\mathbb{C})$ with singular values $s_1 \geq s_1 \geq \cdots \geq s_n \geq s_n$ and eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ ordered as $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then (1.1) and (1.2) hold. Conversely, if $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $s_1 \geq \cdots \geq s_n$ satisfy (1.1) and (1.2), then there exists $A \in \mathfrak{so}_{2n}(\mathbb{C})$ with eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ and singular values $s_1, s_1, \ldots, s_n, s_n$.

(2) Let $A \in \mathfrak{so}_{2n+1}(\mathbb{C})$ with singular values $s_1 \geq s_1 \geq \cdots \geq s_n \geq s_n \geq 0$ and eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n, 0$. Then

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} s_j, \quad k = 1, \ldots, n. \tag{2.1}$$

Conversely, if $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $s_1 \geq \cdots \geq s_n$ satisfy (2.1), then there exists $A \in \mathfrak{so}_{2n+1}(\mathbb{C})$ with eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n, 0$ and singular values $s_1, s_1, \ldots, s_n, s_n, 0$.

Proof. The necessity parts of both cases follow from Theorem 1.1. We now prove the sufficiency.

Even case: Let

$$B := \begin{pmatrix} S & \\ & -S \end{pmatrix},$$

where $S := \operatorname{diag}(s_1, \ldots, s_n)$. Since $s_1, s_2, \ldots, s_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfy (1.1) and (1.2), by Theorem 1.1, we can find $U_1, V_1 \in \mathrm{U}(n)$ such that

$$A_1 := U_1 S V_1 = \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix}.$$

Let $U := U_1 \oplus V_1^\top$ and $V := V_1 \oplus U_1^\top$. Then $U, V \in \mathrm{U}(2n)$. Then

$$A_2 := UBV = \begin{pmatrix} U_1 S V_1 & 0 \\ 0 & -V_1^\top S U_1^\top \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & -A_1^\top \end{pmatrix}.$$

Clearly A_2 has eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ and singular values $s_1, s_1, \ldots, s_n, s_n$. However A_2 is not skew symmetric in general. We will prove that A_2 is unitarily similar to a skew symmetric matrix. Let $W := \frac{1}{\sqrt{2}} \begin{pmatrix} iI_n & I_n \\ I_n & iI_n \end{pmatrix} \in \mathrm{U}(2n)$. Then

$$A := W A_2 W^* = \frac{1}{2} \begin{pmatrix} A_1 - A_1^\top & i(A_1 + A_1^\top) \\ -i(A_1 + A_1^\top) & A_1 - A_1^\top \end{pmatrix}$$

is skew symmetric and has eigenvalues $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_n$ and singular values $s_1, s_1, \ldots, s_n, s_n$.

Odd case: Let $m = 2n+1$. It is trivial if $n = 0$. For $n \geq 1$, if $s_n = 0$, then $s_1, \ldots, s_n$ and $\lambda_1, \ldots, \lambda_n$ satisfy (1.1) and (1.2). So it is reduced to the even case. Suppose $s_n \neq 0$. Let

$$A_1 := \begin{pmatrix} \hat{S} & 0 & u \\ 0 & -\hat{S} & 0 \\ 0 & -u^\top & 0 \end{pmatrix},$$

where

$$\hat{S} := \operatorname{diag}(s_1, \ldots, s_{n-1}, \hat{s}_n), \ \ \hat{s}_n := \frac{\prod_{j=1}^n \lambda_j}{\prod_{j=1}^{n-1} s_j}, \ \ u := (0, \ldots, 0, \sqrt{s_n^2 - \hat{s}_n^2})^\top.$$

Direct computation shows that A_1 has singular values $s_1, s_1, \ldots, s_n, s_n, 0$. Clearly $s_1, \ldots, s_{n-1}, \hat{s}_n$ and $\lambda_1, \ldots, \lambda_n$ satisfy (1.1) and (1.2). Then by Theorem 1.1, there are $U, V \in \mathrm{U}(n)$ such that $A_2 := U\hat{S}V$ is upper triangular with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let

$$\begin{aligned} A_3 &:= (U \oplus V^\top \oplus (1)) A_1 (V \oplus U^\top \oplus (1)) \\ &= \begin{pmatrix} A_2 & 0 & Uu \\ 0 & -A_2^\top & 0 \\ 0 & -(Uu)^\top & 0 \end{pmatrix}. \end{aligned}$$

Note that A_3 has eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n, 0$ and singular values $s_1, s_1, \ldots, s_n, s_n, 0$. We will prove that A_3 is unitarily similar to a skew symmetric matrix. Define

$$W := \frac{e^{-i\pi/4}}{\sqrt{2}} \begin{pmatrix} iI_n & I_n \\ I_n & iI_n \end{pmatrix} \oplus (1).$$

Then

$$A := WA_3W^* = \frac{1}{2} \begin{pmatrix} A_2 - A_2^\top & i(A_2 + A_2^\top) & e^{i\pi/4}Uu \\ -i(A_2 + A_2^\top) & A_2 - A_2^\top & e^{-i\pi/4}Uu \\ -e^{i\pi/4}(Uu)^\top & -e^{-i\pi/4}(Uu)^\top & 0 \end{pmatrix}$$

is clearly skew symmetric and has the same eigenvalues and singular values as A_3. $\square$

We remark that the real counterpart of Theorem 2.1 is trivial since by the spectral decomposition of a real skew symmetric $A \in \mathfrak{so}(m)$, the eigenvalues of A are simply $\pm is_1, \ldots, \pm is_n, (0)$.

3. Matrices in the symplectic algebras $\mathfrak{sp}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{R})$

Consider the complex symplectic Lie algebra [6, p.128-129] which is simple for $n \geq 1$:

$$\begin{aligned} \mathfrak{sp}_n(\mathbb{C}) &:= \mathfrak{sp}(n) \oplus i\mathfrak{sp}(n) \\ &= \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^\top \end{pmatrix} : A_1, A_2, A_3 \in \mathbb{C}_{n\times n}, A_2^\top = A_2, A_3^\top = A_3 \right\}. \end{aligned}$$

The compact group $K = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n)$ [6] consists of the matrices

$$\begin{pmatrix} U & -\overline{V} \\ V & \overline{U} \end{pmatrix} \in \mathrm{U}(2n).$$

It is known [2, Proposition 3.1] that for any $B \in \mathfrak{sp}_n(\mathbb{C})$, there is $U \in K$ such that $UBU^* \in \mathfrak{b} \subset \mathfrak{sp}_n(\mathbb{C})$, where

$$\mathfrak{b} := \left\{ \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^\top \end{pmatrix},\ A_1 \in \mathbb{C}_{n\times n} \text{ is upper triangular}, A_2^\top = A_2 \right\} \tag{3.1}$$

is a Borel subalgebra of $\mathfrak{sp}_n(\mathbb{C})$. The eigenvalues of $A \in \mathfrak{sp}_n(\mathbb{C})$ occur in pairs but opposite in sign as we can see it from (3.1). However, unlike the complex skew symmetric case in the previous section, the singular values of A do not generally occur in pairs, e.g.,

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sp}_1(\mathbb{C})$$

has distinct singular values.

We first have the following simple lemma.

Lemma 3.1. Let

$$S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix},\ U = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},\ V = \begin{pmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix},$$

where $s_1, s_2 \geq 0$. Then

$$USV = \begin{pmatrix} (s_1+s_2)\sin\alpha\cos\alpha & s_1\cos^2\alpha - s_2\sin^2\alpha \\ -s_1\sin^2\alpha + s_2\cos^2\alpha & -(s_1+s_2)\sin\alpha\cos\alpha \end{pmatrix}.$$

In particular, if we choose $\sin\alpha = \frac{\sqrt{s_2}}{\sqrt{s_1+s_2}}$ and $\cos\alpha = \frac{\sqrt{s_1}}{\sqrt{s_1+s_2}}$, then

$$USV = \begin{pmatrix} \sqrt{s_1 s_2} & s_1 - s_2 \\ 0 & -\sqrt{s_1 s_2} \end{pmatrix}.$$

Theorem 3.2. Let $A \in \mathfrak{sp}_n(\mathbb{C})$ with singular values $s_1 \geq s_2 \geq \cdots \geq s_{2n-1} \geq s_{2n}$ and eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ ordered as $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then (1.1) and (1.2) hold for them. Conversely, if $|\lambda_1| \geq |\lambda_1| \geq \cdots \geq |\lambda_n| \geq |\lambda_n|$ and $s_1 \geq s_2 \geq \cdots \geq s_{2n-1} \geq s_{2n}$ satisfy (1.1) and (1.2), then there exists $A \in \mathfrak{sp}_n(\mathbb{C})$ with eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ and singular values $s_1 \geq s_2 \geq \cdots \geq s_{2n-1} \geq s_{2n}$.

Proof. The necessity of (1.1) for $s_1 \geq s_2 \geq \cdots \geq s_{2n-1} \geq s_{2n}$ and eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ follows from Theorem 1.1.

For the sufficiency part, suppose that $\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n$ and $s_1, \ldots s_{2n}$ satisfy (1.1) and (1.2), i.e.,

$$\sqrt{s_1 s_2} \geq \cdots \geq \sqrt{s_{2n-1}s_{2n}}, \quad |\lambda_1| \geq \cdots \geq |\lambda_n|$$

satisfy (1.1) and (1.2). Let $S := \operatorname{diag}(s_1, s_3, \ldots, s_{2n-1}, s_2, s_4, \ldots, s_{2n})$,

$$U := \begin{pmatrix} \cos\alpha_1 & & & \sin\alpha_1 & & \\ & \ddots & & & \ddots & \\ & & \cos\alpha_n & & & \sin\alpha_n \\ -\sin\alpha_1 & & & \cos\alpha_1 & & \\ & \ddots & & & \ddots & \\ & & -\sin\alpha_n & & & \cos\alpha_n \end{pmatrix}$$

$$V := \begin{pmatrix} \sin\alpha_1 & & & \cos\alpha_1 & & \\ & \ddots & & & \ddots & \\ & & \sin\alpha_n & & & \cos\alpha_n \\ \cos\alpha_1 & & & -\sin\alpha_1 & & \\ & \ddots & & & \ddots & \\ & & \cos\alpha_n & & & -\sin\alpha_n \end{pmatrix}.$$

By Lemma 3.1, we can choose appropriate $\alpha_1, \ldots, \alpha_n$ such that

$$USV = \begin{pmatrix} \sqrt{s_1 s_2} & & & s_1 - s_2 & & \\ & \ddots & & & \ddots & \\ & & \sqrt{s_{2n-1}s_{2n}} & & & s_{2n-1} - s_{2n} \\ & & & -\sqrt{s_1 s_2} & & \\ & & & & \ddots & \\ & & & & & -\sqrt{s_{2n-1}s_{2n}} \end{pmatrix}.$$

Let $B_1 := \operatorname{diag}(\sqrt{s_1 s_2}, \ldots, \sqrt{s_{2n-1}s_{2n}})$. Since $\sqrt{s_1 s_2} \geq \cdots \geq \sqrt{s_{2n-1}s_{2n}} \geq 0$ and $|\lambda_1| \geq \cdots \geq |\lambda_n|$ satisfy (1.1) and (1.2), by Theorem 1.1 we can find $U_1, V_1 \in$

U(n) such that

$$A_1 := U_1 B_1 V_1 = \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix} \tag{3.2}$$

has singular values $\sqrt{s_1 s_2}, \dots, \sqrt{s_{2n-1} s_n}$ and eigenvalues $\lambda_1, \dots, \lambda_n$. Set $\Sigma :=$ diag $(s_1 - s_2, \dots, s_{2n-1} - s_n)$. Then

$$A := (U_1 \oplus V_1^\top) USV (V_1 \oplus U_1^\top) = \begin{pmatrix} U_1 B_1 V_1 & U_1 \Sigma U_1^\top \\ 0 & -(U_1 B_1 V_1)^\top \end{pmatrix}. \tag{3.3}$$

Since $U_1 \Sigma U_1^\top$ is symmetric, we conclude that $A \in \mathfrak{sp}_n(\mathbb{C})$ and has eigenvalues $\pm\lambda_1, \dots, \pm\lambda_n$. Moreover A has singular values $s_1, \dots, s_{2n}$ because of the above unitary equivalence. So A is the required matrix. □

The split real form of $\mathfrak{sp}_n(\mathbb{C})$ [6] is

$$\begin{aligned} & \mathfrak{sp}_n(\mathbb{R}) \\ = & \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^\top \end{pmatrix} : A_2^\top = A_2,\ A_3^\top = A_3,\ A_1, A_2, A_3 \in \mathbb{R}_{n\times n} \right\} \\ = & \ \mathfrak{sp}_n(\mathbb{C}) \cap \mathbb{R}_{2n\times 2n}. \end{aligned}$$

The nonreal eigenvalues of each $A \in \mathfrak{sp}_n(\mathbb{R})$ appear in quadruples $(\lambda, -\lambda, \bar\lambda, -\bar\lambda)$. The proof of the following result is similar to that of Theorem 3.2. The role of Theorem 1.1 in the proof is played by Thompson's result [8] and U_1, V_1 in (3.2) are orthogonal and A_1 is a real "upper triangular" matrix with 2×2 diagonal blocks for nonreal λ's and 1×1 block for real λ's.

Theorem 3.3. Let $A \in \mathfrak{sp}_n(\mathbb{R})$ with singular values $s_1 \ge s_2 \ge \cdots \ge s_{2n-1} \ge s_{2n}$ and eigenvalues $\pm\lambda_1, \dots, \pm\lambda_n$ (the nonreal λ's appear in quadruples) ordered as $|\lambda_1| \ge \cdots \ge |\lambda_n|$. Then (1.1) and (1.2) hold for them. Conversely, if $|\lambda_1| \ge |\lambda_1| \ge \cdots \ge |\lambda_n| \ge |\lambda_n|$ and $s_1 \ge s_2 \ge \cdots \ge s_{2n-1} \ge s_{2n}$ satisfy (1.1) and (1.2) and if the nonreal λ's appear in quadruples $(\lambda, -\lambda, \bar\lambda, -\bar\lambda)$, then there exists $A \in \mathfrak{sp}_n(\mathbb{R})$ such that $\pm\lambda_1, \dots, \pm\lambda_n$ are the eigenvalues and $s_1 \ge s_2 \ge \cdots \ge s_{2n-1} \ge s_{2n}$ are the singular values of A, respectively.

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