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MAJORIZATION OF SINGULAR INTEGRAL OPERATORS WITH CAUCHY KERNEL ON L^2

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This paper is dedicated to Professor Tsuyoshi Ando

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ABSTRACT. Let a, b, c and d be functions in $L^2 = L^2(\mathbb{T}, d\theta/2\pi)$, where $\mathbb{T}$ denotes the unit circle. Let $\mathcal{P}$ denote the set of all trigonometric polynomials. Suppose the singular integral operators A and B are defined by $A = aP + bQ$ and $B = cP + dQ$ on $\mathcal{P}$, where P is an analytic projection and $Q = I - P$ is a co-analytic projection. In this paper, we use the Helson–Szegő type set $(HS)(r)$ to establish the condition of a, b, c and d satisfying $\|Af\|_2 \geq \|Bf\|_2$ for all f in $\mathcal{P}$. If a, b, c and d are bounded measurable functions, then A and B are bounded operators, and this is equivalent to that B is majorized by A on L^2, i.e., $A^*A \geq B^*B$ on L^2. Applications are then presented for the majorization of singular integral operators on weighted L^2 spaces, and for the normal singular integral operators $aP + bQ$ on L^2 when $a - b$ is a complex constant.

1. INTRODUCTION

Let m denote the normalized Lebesgue measure $d\theta/2\pi$ on the unit circle $\mathbb{T} = \{|z| = 1\}$. For $0 < p \leq \infty$, $L^p = L^p(\mathbb{T}, m)$ denotes the usual Lebesgue space on $\mathbb{T}$ and H^p denotes the usual Hardy space on $\mathbb{T}$. Let S be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\eta)}{\eta - \zeta} d\eta \quad (\text{a.e.} \zeta \in \mathbb{T})$$

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where the integral is understood in the sense of Cauchy's principal value (cf. [6], Vol. 1, p.12). If f is in L^1 then $(Sf)(\zeta)$ exists for almost all ζ on $\mathbb{T}$. Let

$$P = (I + S)/2 \quad \text{and} \quad Q = (I - S)/2,$$

where I denotes the identity operator. Then $Pz^n = 0$ if $n < 0$, and $Pz^n = z^n$ if $n \geq 0$. P is said to be an analytic projection or the Riesz projection. Let $\mathcal{P}_1 = \text{span}\{z^n : n \geq 0\}$ be the set of analytic polynomials, and let $\mathcal{P}_2 = \overline{z\mathcal{P}_1} = \text{span}\{z^n : n < 0\}$. Then $Q = I - P$, $P(f_1 + f_2) = f_1$ and $Q(f_1 + f_2) = f_2$ for all $f_1 \in \mathcal{P}_1$ and $f_2 \in \mathcal{P}_2$. Q is said to be a co-analytic projection. Let $\alpha, \beta \in L^\infty$, and W is a nonnegative function in L^1. In [8] and [16], the condition of α, β and W such that $\alpha P + \beta Q$ is contractive was given. In [9], the conditions of α, β and W such that $\alpha P + \beta Q$ is bounded and bounded below was given. In [10] and [11], for $\alpha, \beta \in L^\infty$, the norm formula of $\alpha P + \beta Q$ on the weighted L^2 space was given. In [10], [11] and [16], the another proofs of Feldman–Krupnik–Markus's theorem ([6], Vol. 2, p.213, Theorem 5.1, and p.215, Lemma 5.3) were given. In this paper, for $a, b, c, d \in L^2$, we consider the singular integral operators $A = aP + bQ$ and $B = cP + dQ$. If $a, b, c, d \notin L^\infty$, then A and B are unbounded. In Section 2, we use the Helson–Szegő type set $(HS)(r)$ to establish the condition of a, b, c and d satisfying $\|Af\|_2 \geq \|Bf\|_2$ for all f in $\mathcal{P}$. The main theorem is Theorem 2.4. If $a, b, c, d \in L^\infty$, then A and B are bounded, and this is equivalent to that B is majorized by A on L^2, i.e., $A^*A \geq B^*B$ on L^2. As an application of Theorem 2.4, we have Theorem 2.5. In Section 3, some applications are presented for the majorization of singular integral operators on weighted L^2 spaces, and for the normal singular integral operators $aP + bQ$ on L^2 when $a - b$ is a complex constant.

2. Main Theorem

In this section, we use the Helson–Szegő type set $(HS)(r)$ to establish the condition of a, b, c and d satisfying $\|Af\|_2 \geq \|Bf\|_2$ for all f in $\mathcal{P}$. The main theorem is Theorem 2.4. If $a, b, c, d \in L^\infty$, then this is equivalent to that B is majorized by A on L^2, i.e., $A^*A \geq B^*B$ on L^2. By Douglas's criterion (cf. [4], [14], p.2), this implies that there is a contraction C on L^2 such that B is factorized as $B = CA$. Let $\tilde{f}$ denote the harmonic conjugate function of $f \in L^1$. Then $Sf = i\tilde{f} + \int_{\mathbb{T}} f dm$. It is well known that the Helson–Szegő set

$$(HS) = \{e^{u+\tilde{v}} \ ; \ u, v \in L^\infty \text{ are real functions, and } \|v\|_\infty < \frac{\pi}{2}\}$$

is equal to the set of all Muckenhoupt (A_2)-weights (cf. [5], p.254).

Definition 2.1. For a function r satisfying $0 \leq r \leq 1$ and $\int_{\mathbb{T}} r dm > 0$, we define the Helson–Szegő type set $(HS)(r)$:

$$(HS)(r) = \{\gamma e^{u+\tilde{v}} \ ; \ \gamma \text{ is a positive constant, } u, v \text{ are real functions,}$$
$$u \in L^1, v \in L^\infty, \ |v| \leq \pi/2, \ r^2 e^u + e^{-u} \leq 2\cos v\}$$

If $|v| \leq \pi/2$, then $e^{\tilde{v}} \cos v \in L^1$ (cf. [5], p.161), and hence $e^{-\tilde{v}} \cos v \in L^1$. Therefore $(HS)(r) \subset \{W : W > 0, \ r^2 W \in L^1, \ W^{-1} \in L^1\}$. If $r^{-1} \in L^\infty$, then $(HS)(r) \subset (HS)$. In [9], we defined the another Helson–Szegő type set which is similar to $HS(r)$. We use $HS(r)$ to study the majorization of singular integral operators.

Lemma 2.2. *Let W be a non-negative function in L^1, and let ϕ be a function in L^1. Suppose $|\phi| \geq W$ and $\int_{\mathbb{T}}(|\phi| - W)dm > 0$. Then the following conditions* $(1) \sim (3)$ *are equivalent.*

(1) *There is a k in H^1 such that $|\phi - k| \leq W$.*
(2) *There is a non-zero k in H^1 such that $|\phi - k| \leq W$.*
(3) $\log|\phi| \in L^1$ *and there is a V in $(HS)(r)$ such that ϕ/V is in $H^{1/2}$, where* $r = |\phi|^{-1}\sqrt{|\phi|^2 - W^2}$.

Proof. (1) $\Rightarrow$ (2) : By (1), if $k = 0$, then $0 \leq |\phi| - W \leq 0$, and hence $|\phi| = W$. This is a contradiction. Therefore $k \neq 0$.
(2) $\Rightarrow$ (3) : By (2), $|\phi - k| \leq W \leq |\phi|$, and hence $0 < |k| \leq 2|\phi|$. Since $\log|k| \in L^1$, $\log|\phi| \in L^1$. Since

$$\left|1 - \frac{k}{\phi}\right|^2 \leq \frac{W^2}{|\phi|^2} = 1 - \frac{|\phi|^2 - W^2}{|\phi|^2} = 1 - r^2 \leq 1,$$

it follows that $\mathrm{Re}(k/\phi) \geq 0$. Since $\log|k/\phi| \in L^1$, it follows that there are real functions $u \in L^1$ and $v \in L^\infty, |v| \leq \pi/2$ such that $k/\phi = e^{-u-iv}$. Then $0 \leq r \leq 1$ and $|1-e^{-u-iv}|^2 \leq 1-r^2$. Hence $r^2+e^{-2u} \leq 2e^{-u}\cos v$, and so $r^2e^u+e^{-u} \leq 2\cos v$. Since $\phi = ke^{u+iv}$, it follows that $\phi e^{-u-\tilde{v}} = ke^{iv-\tilde{v}} \in H^p$, for some $p > 0$. Since $e^{-u} \leq 2\cos v$, it follows that $e^{-u-\tilde{v}} \leq 2e^{-\tilde{v}}\cos v$. Since $|v| \leq \pi/2$, $e^{-\tilde{v}}\cos v \in L^1$ (cf. [5], p.161). Hence $e^{-u-\tilde{v}} \in L^1$. Since $\phi \in L^1$, it follows that $\phi e^{-u-\tilde{v}} \in H^{1/2}$.
(3) $\Rightarrow$ (1) : By (3), if $k = \phi e^{-u-iv}$, then $k = (\phi e^{-u-\tilde{v}})e^{\tilde{v}-iv} \in H^p$, for some $p > 0$. Since $|\phi - k|^2 = |\phi|^2|1 - e^{-u-iv}|^2 = |\phi|^2e^{-u}(e^u + e^{-u} - 2\cos v)$ and $e^u + e^{-u} - 2\cos v \leq (1-r^2)e^u = |W/\phi|^2e^u$, it follows that $|\phi - k| \leq W$. Hence $|k| \leq |\phi| + W$, and so $k \in H^1$. This completes the proof. □

Lemma 2.3. *Let W_1, W_2 be real functions in L^1, and let ϕ be a function in L^1. Suppose $|\phi|^2 - W_1W_2 \geq 0$. Then the following conditions* (1) *and* (2) *are equivalent.*

(1) *For all $f_1 \in \mathcal{P}_1$ and $f_2 \in \mathcal{P}_2$,*

$$\left|\int_{\mathbb{T}} f_1\overline{f_2}\phi dm\right| \leq \frac{1}{2}\int_{\mathbb{T}}(|f_1|^2W_1 + |f_2|^2W_2)dm.$$

(2) $W_1 \geq 0,\ W_2 \geq 0$, *and either* (a) *or* (b) *holds.*
  (a) $|\phi|^2 - W_1W_2 = 0$.
  (b) $\log|\phi| \in L^1$, *and there is a V in $(HS)(r)$ such that ϕ/V is in $H^{1/2}$, where* $r = |\phi|^{-1}\sqrt{|\phi|^2 - W_1W_2}$.

Proof. (1) $\Rightarrow$ (2) : By Cotlar-Sadosky's lifting theorem [3], $W_1 \geq 0,\ W_2 \geq 0$, and there is a k in H^1 such that $|\phi - k|^2 \leq W_1W_2$. By Lemma 2.2, this implies (2).
(2) $\Rightarrow$ (1) : Suppose (a) holds. Then

$$\begin{aligned}\left|\int_{\mathbb{T}} f_1\overline{f_2}\phi dm\right| &\leq \int_{\mathbb{T}} |f_1f_2\phi|dm = \int_{\mathbb{T}} |f_1f_2|\sqrt{W_1W_2}dm \\ &\leq \frac{1}{2}\int_{\mathbb{T}}(|f_1|^2W_1 + |f_2|^2W_2)dm.\end{aligned}$$

This implies (1). Suppose $\int_{\mathbb{T}}(|\phi|^2 - W_1W_2)dm > 0$ and (b) holds. Then it follows from Lemma 2.2 that there is a k in H^1 such that $|\phi - k|^2 \le W_1W_2$. Hence for all $f_1 \in \mathcal{P}_1$ and $f_2 \in \mathcal{P}_2$,

$$\begin{aligned} |\int_{\mathbb{T}} f_1\overline{f_2}\phi dm| &= |\int_{\mathbb{T}} f_1\overline{f_2}(\phi - k)dm| \\ &\le \int_{\mathbb{T}} |f_1f_2| \cdot |\phi - k|dm \\ &\le \int_{\mathbb{T}} |f_1f_2|\sqrt{W_1W_2}dm \\ &\le \frac{1}{2}\int_{\mathbb{T}}(|f_1|^2W_1 + |f_2|^2W_2)dm. \end{aligned}$$

This implies (1). This completes the proof. □

Remark A. In Lemma 2.3, if $|\phi|^2 - W_1W_2 \le 0$ then

$$\begin{aligned} |\int_{\mathbb{T}} f_1\overline{f_2}\phi dm| &\le \int_{\mathbb{T}} |f_1f_2\phi|dm \le \int_{\mathbb{T}} |f_1f_2|\sqrt{W_1W_2}dm \\ &\le \frac{1}{2}\int_{\mathbb{T}}(|f_1|^2W_1 + |f_2|^2W_2)dm, \end{aligned}$$

and so (1) holds without the condition (2).

Theorem 2.4. *Let a, b, c, d be functions in L^2. Then the following conditions* (1) *and* (2) *are equivalent.*

(1) *For all f in $\mathcal{P}$,*

$$\int_{\mathbb{T}} |(aP + bQ)f|^2dm \ge \int_{\mathbb{T}} |(cP + dQ)f|^2dm.$$

(2) $|a| \ge |c|$, $|b| \ge |d|$, *and either* (a) *or* (b) *holds.*
  (a) $ad - bc = 0$.
  (b) $\log|a\bar{b} - c\bar{d}| \in L^1$, *and there is a V in $(HS)(r)$ such that $(a\bar{b} - c\bar{d})/V$ is in $H^{1/2}$, where $r = |ad - bc|/|a\bar{b} - c\bar{d}|$.*

Proof. (1) implies that

$$\int_{\mathbb{T}} |af_1 + bf_2|^2dm \ge \int_{\mathbb{T}} |cf_1 + df_2|^2dm.$$

Let $W_1 = |a|^2 - |c|^2, W_2 = |b|^2 - |d|^2$, and let $\phi = a\bar{b} - c\bar{d}$. Then W_1, W_2 are real functions in L^1, and ϕ is a function in L^1 such that for all $f_1 \in \mathcal{P}_1$ and $f_2 \in \mathcal{P}_2$,

$$\int_{\mathbb{T}}\{|f_1|^2W_1 + |f_2|^2W_2 + 2\mathrm{Re}(f_1\overline{f_2}\phi)\}dm \ge 0.$$

This is equivalent to the condition (1) of Theorem 2.4. Since $|\phi|^2 - W_1W_2 = |ad - bc|^2$ and

$$r^2 = \frac{|\phi|^2 - W_1W_2}{|\phi|^2} = \frac{|a\bar{b} - c\bar{d}|^2 - (|a|^2 - |c|^2)(|b|^2 - |d|^2)}{|a\bar{b} - c\bar{d}|^2} = \left|\frac{ad - bc}{a\bar{b} - c\bar{d}}\right|^2,$$

this theorem follows from Lemma 2.3. This completes the proof. □

Remark B. For a function r satisfying $0 \le r \le 1$ and $\int_{\mathbb{T}} r dm > 0$,

$$(HS)(r) = \{W \in L^1 : W > 0,\ \int_{\mathbb{T}} |f|^2 W dm \ge \int_{\mathbb{T}} |rPf|^2 W dm,\ (f \in \mathcal{P})\}.$$

Proof. Let $a = b = \sqrt{W}, c = r\sqrt{W}$ and $d = 0$. By Theorem 2.4, $W/V \in H^{1/2}$. By Neuwirth–Newman's theorem (cf. [14], p.79), W/V is a constant, so $W \in (HS)(r)$. The converse is also true. □

Theorem 2.5. *Let W be a positive function in L^1. Let a, b, c, d be in L^∞. Then the following conditions* (1) *and* (2) *are equivalent.*

(1) *For all f in $\mathcal{P}$,*

$$\int_{\mathbb{T}} |(aP + bQ)f|^2 W dm \ge \int_{\mathbb{T}} |(cP + dQ)f|^2 W dm.$$

(2) $|a| \ge |c|$, $|b| \ge |d|$, *and either* (a) *or* (b) *holds.*
  (a) $ad - bc = 0$.
  (b) $\log|a\bar{b} - c\bar{d}|W \in L^1$, *and there is a V in $(HS)(r)$ such that $(a\bar{b} - c\bar{d})W/V$ is in $H^{1/2}$, where $r = |ad - bc|/|a\bar{b} - c\bar{d}|$.*

Proof. Suppose (1) holds and (a) of (2) does not hold. Let $a_1 = a\sqrt{W},\ b_1 = b\sqrt{W},\ c_1 = c\sqrt{W},\ d_1 = d\sqrt{W}$. Then $\int_{\mathbb{T}} |a_1 d_1 - b_1 c_1| dm > 0$ and

$$\int_{\mathbb{T}} |(a_1 P + b_1 Q)f|^2 dm \ge \int_{\mathbb{T}} |(c_1 P + d_1 Q)f|^2 dm,$$

for all f in $\mathcal{P}$. By Theorem 2.4, this implies that $\log|a_1\bar{b_1} - c_1\bar{d_1}| \in L^1$, $|a_1|^2 - |c_1|^2 \ge 0$, $|b_1|^2 - |d_1|^2 \ge 0$, and there is a V in $(HS)(r)$ such that $(a_1\bar{b_1} - c_1\bar{d_1})/V$ is in $H^{1/2}$, where $r = |a_1 d_1 - b_1 c_1|/|a_1\bar{b_1} - c_1\bar{d_1}| = |ad - bc|/|a\bar{b} - c\bar{d}|$. Hence (b) of (2) holds, so (1) implies (2). The converse is also true. □

Remark C. Let W be a positive function in L^1. Let $L^2(W)$ be the weighted Lebesgue space with the norm

$$\|f\|_{2,W} = \{\int_{\mathbb{T}} |f|^2 W dm\}^{1/2}.$$

When $W = 1$, then we write $\|f\| = \|f\|_W$. Let $A = aP + bQ$, and let $B = cP + dQ$. Then the condition (1) implies that B is majorized by A on $L^2(W)$, i.e., $A^*A \ge B^*B$ on $L^2(W)$, i.e., $\|Af\|_{2,W} \ge \|Bf\|_{2,W}$ for all f in $L^2(W)$.

3. Applications of Theorem 2.4

The equivalence of (1) and (3) of the following corollary is Widom–Devinatz–Rochberg's theorem (cf. [1], [7], [6], [13], p.250, [15], p.93). Nakazi [7] removed the condition $W \in (HS)$ and established the condition of α satisfying

$$\int_{\mathbb{T}} |(\alpha P + Q)f|^2 W dm \ge \varepsilon^2 \int_{\mathbb{T}} |Pf|^2 W dm,$$

for all $f \in \mathcal{P}$.

Corollary 3.1. *([7]) Let W be in (HS) and let α be in L^∞. Then the following are equivalent.*

1. *T_α is bounded below on $H^2(W)$.*
2. *$\alpha^{-1} \in L^\infty$, and there is a V in (HS) such that $\alpha W/V$ is in $H^{1/2}$.*
3. *$\alpha^{-1} \in L^\infty$, and there is an inner function q and a real function $t \in L^1$ such that $\alpha/|\alpha| = qe^{i\tilde{t}}$ and $We^{-t} \in (HS)$.*

Proof. By (1), there is a constant $\varepsilon > 0$ such that

$$\int_{\mathbb{T}} |P(\alpha Pf)|^2 W dm = \int_{\mathbb{T}} |T_\alpha(Pf)|^2 W dm \geq \varepsilon^2 \int_{\mathbb{T}} |Pf|^2 W dm,$$

for all $f \in \mathcal{P}$. Since $W \in (HS)$, P is bounded on $L^2(W)$, so

$$\|P\|_W^2 \int_{\mathbb{T}} |(\alpha P + Q)f|^2 W dm \geq \int_{\mathbb{T}} |P(\alpha P + Q)f|^2 W dm$$

$$= \int_{\mathbb{T}} |P(\alpha Pf)|^2 W dm \geq \varepsilon^2 \int_{\mathbb{T}} |Pf|^2 W dm,$$

for all $f \in \mathcal{P}$. Let $a = \|P\|_W \alpha\sqrt{W}, b = \|P\|_W \sqrt{W}, c = \varepsilon\sqrt{W}$ and $d = 0$. Then

$$\int_{\mathbb{T}} |(aP + bQ)f|^2 dm \geq \int_{\mathbb{T}} |cP + dQ)f|^2 dm.$$

By Theorem 2.4, $|a| \geq |c|$, so $\|P\|_W|\alpha| \geq \varepsilon > 0$, and $r = |ad - bc|/|a\bar{b} - c\bar{d}| = \varepsilon/(\|P\|_W|\alpha|) \leq 1$. Since $\alpha \in L^\infty$, $r^{-1} \in L^\infty$. By Theorem 2.4, there is a V in $(HS)(r)$ such that $\|P\|_W^2 \alpha W/V = (a\bar{b} - c\bar{d})/V$ is in $H^{1/2}$. Since $V \in (HS)(r)$, $V = \gamma e^{u+\tilde{v}}$, where u and v are real functions such that $u \in L^1$, $v \in L^\infty$, $|v| \leq \pi/2$, and $r^2 e^u + e^{-u} \leq 2\cos v$. Since $r^{-1} \in L^\infty$, $u \in L^\infty$ and $\|v\|_\infty < \pi/2$, so $V \in (HS)$. This implies (2). The converse is also true. Suppose (2) holds. Since $\alpha W/V \in H^{1/2}$, there is an inner function q and real function $t \in L^1$ such that $\alpha W/V = qe^{t+i\tilde{t}}$. Thus $\alpha/|\alpha| = qe^{i\tilde{t}}$ and $We^{-t} = V/|\alpha| \in (HS)$. This implies (3). The converse is also true. This completes the proof. □

The following corollary is the Feldman–Krupnik–Markus theorem ([6], Vol. 2, p.213, Theorem 5.1, and p.215, Lemma 5.3). $\|\alpha P + \beta Q\|_W$ and $\|P\|_W$ denote the operator norms of each operators on $L^2(W)$. In [11], this theorem was generalized to the case when α and β are functions in L^∞.

Corollary 3.2. *([6]) Let α and β be constants. Let*

$$\gamma = \left|\frac{\alpha - \beta}{2}\right|^2 (\|P\|_W^2 - 1)$$

then

$$\|\alpha P + \beta Q\|_W = \sqrt{\gamma + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha| - |\beta|}{2}\right)^2}.$$

Proof. We assume that $\|\alpha P + \beta Q\|_W^2 \neq \alpha\bar{\beta}$. Let $a = b = \sqrt{W}\|\alpha P + \beta Q\|_W$, $c = \alpha\sqrt{W}$ and $d = \beta\sqrt{W}$. Then for all f in $\mathcal{P}$,

$$\int_{\mathbb{T}} |(aP + bQ)f|^2 dm \geq \int_{\mathbb{T}} |(cP + dQ)f|^2 dm.$$

By Theorem 2.4, $\log |\|\alpha P + \beta Q\|_W^2 - \alpha\bar{\beta}| = \log |a\bar{b} - c\bar{d}| \in L^1$, and there is a V in $(HS)(r)$ such that $(\|\alpha P + \beta Q\|_W^2 - |\alpha\bar{\beta}|)W/V = (a\bar{b} - c\bar{d})/V$ is in $H^{1/2}$, where

$$r = \frac{|ad - bc|}{|a\bar{b} - c\bar{d}|} = \frac{|\alpha - \beta|\|\alpha P + \beta Q\|_W}{|\|\alpha P + \beta Q\|_W^2 - \alpha\bar{\beta}|}.$$

Since $\|\alpha P + \beta Q\|_W^2 \neq \alpha\bar{\beta}$, $W/V \in H^{1/2}$ and $W/V \geq 0$. By the Neuwirth–Newman theorem (cf. [14], p.79), W/V is a constant, so $W \in (HS)(r)$. By Theorem 2.4

$$\int_{\mathbb{T}} |f|^2 W dm \geq \int_{\mathbb{T}} |rPf|^2 W dm.$$

Hence $r = 1/\|P\|_W$, so $\|P\|_W$ is described by $\|\alpha P + \beta Q\|_W$. By the calculation, $\|\alpha P + \beta Q\|_W$ is described by α, β and $\|P\|_W$. This completes the proof. □

An operator A is called hyponormal if its self-commutator $A^*A - AA^*$ is positive. If $\alpha - \beta$ is a constant, then the following theorem gives the descriptions of symbols of normal (and hyponormal) operators $\alpha P + \beta Q$. Brown and Halmos ([2]) proved that the Toeplitz operator T_α is normal if and only if α satisfies the condition (2) of the following corollary for some $c \in \mathbb{C}$. In [12], normal singular integral operator $\alpha P + \beta Q$ is considered without the condition that $\alpha - \beta$ is a constant.

Corollary 3.3. *Let α and β be non-constant functions in L^∞. Suppose $\alpha - \beta$ is a non-zero constant. Then the following are equivalent.*

(1) *$\alpha P + \beta Q$ is normal.*
(2) *$\alpha = cf + \bar{f} + b$ for some $f \in zH^2$ and $b \in \mathbb{C}$, where $c = (\alpha - \beta)/(\bar{\alpha} - \bar{\beta})$.*
(3) *$\alpha P + \beta Q$ is hyponormal.*

Proof. (3) $\Rightarrow$ (1): By (3), $\|(\alpha P + \beta Q)f\|^2 \geq \|(\alpha P + \beta Q)^* f\|^2$, for all $f \in L^2$. Since $\alpha - \beta \in \mathbb{C}$, it follows that

$$(\alpha P + \beta Q)^* = ((\alpha - \beta)P + \beta I)^* = (\bar{\alpha} - \bar{\beta})P + \bar{\beta}I = \bar{\alpha}P + \bar{\beta}Q.$$

Thus $\|\alpha Pf + \beta Qf\|^2 \geq \|\bar{\alpha}Pf + \bar{\beta}Qf\|^2$. Hence 2 Re $\int_{\mathbb{T}} (\alpha\bar{\beta} - \bar{\alpha}\beta)Pf\overline{Qf}dm \geq 0$, for all $f \in L^2$. This implies that 2 Re $\int_{\mathbb{T}} (\alpha\bar{\beta} - \bar{\alpha}\beta)Pf\overline{Qf}dm = 0$, for all $f \in L^2$. Thus $\|(\alpha P + \beta Q)f\|^2 = \|(\alpha P + \beta Q)^* f\|^2$, for all $f \in L^2$. Therefore $\alpha P + \beta Q$ is normal.
(1) $\Rightarrow$ (3): Trivial.
(3) $\Leftrightarrow$ (2): There exists a complex constant c such that $\beta = \alpha + c$. By Theorem 2.4 and the above proof, if $\alpha\bar{c} - \bar{\alpha}c = 0$, then (3) and (2) are equivalent. By Theorem 2.4, if $\alpha\bar{c} - \bar{\alpha}c \neq 0$, then (3) and (2) are equivalent, because $(HS)(1)$ is the set of all positive constants, and the real function $i(\alpha\bar{c} - \bar{\alpha}c)$ belongs to $H^{1/2} \cap L^\infty = H^\infty$, so that it is a real constant. □

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