Ann. Funct. Anal. 5 (2014), no. 1, 24–29
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL:www.emis.de/journals/AFA/

AN OPERATOR INEQUALITY IMPLYING THE USUAL AND CHAOTIC ORDERS

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Dedicated to Professor Tsuyoshi Ando for his significant contributions to our areas

Communicated by Y. Seo

ABSTRACT. We prove that if positive invertible operators A and B satisfy an operator inequality $(B^{s/2}A^{(s-t)/2}B^tA^{(s-t)/2}B^{s/2})^{\frac{1}{2s}} \geq B$ for some $t > s > 0$, then

(1) If $t \geq 3s-2 \geq 0$, then $\log B \geq \log A$, and if $t \geq s+2$ is additionally assumed, then $B \geq A$.

(2) If $0 < s < 1/2$, then $\log B \geq \log A$, and if $t \geq s+2$ is additionally assumed, then $B \geq A$.

It is an interesting application of the Furuta inequality. Furthermore we consider some related results.

1. INTRODUCTION

An operator means a bounded linear operator acting on a Hilbert space. The usual order $A \geq B$ among selfadjoint operators on H is defined by $(Ax,x) \geq (Bx,x)$ for any $x \in H$. In particular, A is said to be positive and denoted by $A \geq 0$ if $(Ax,x) \geq 0$ for $x \in H$, and $A > 0$ if A is invertible.

The noncommutativity of operators reflects on the usual order, [8] and [13], as follows:

Löwner–Heinz inequality:

(LH)
$$A \geq B \Rightarrow A^p \geq B^p$$

if and only if $p \in [0,1]$.

Date: Received: 20 March 2013; Accepted: 28 April 2013.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A63; Secondary, 47A56.
Key words and phrases. Operator inequality, chaotic order, Furuta inequality.

In 1987, Furuta [6] proposed a beautiful extension of (LH), by which the restriction $p \in [0,1]$ in (LH) is relaxed in some sense:

Furuta inequality: If $A \geq B$, then for each $r \geq 0$,

$$(A^{r/2}B^pA^{r/2})^{\frac{1}{q}} \leq A^{\frac{p+r}{q}} \quad \text{and} \quad B^{\frac{p+r}{q}} \leq (B^{r/2}A^pB^{r/2})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with

$$(\dagger) \qquad (1+r)q \geq p+r.$$

For the Furuta inequality, we refer [2],[5],[6] and [7]. Among others, the best possibility of the domain determined by (†) is proved by Tanahashi [14].

Afterwards, the Furuta inequality was discussed under the chaotic order $\log A \geq \log B$ for $A, B > 0$, which was originally discussed by Ando [1], and the final result was obtained in [3].

Theorem FFK. The following (1) - (3) are mutually equivalent for $A, B > 0$:

(1) $\log A \geq \log B$,

(2) $A^p \geq (A^{p/2}B^pA^{p/2})^{1/2}$ for $p \geq 0$,

(3) $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{r}{p+r}}$ for $p, r \geq 0$.

From the viewpoint of Kamei's satellite theorem [12] and Uchiyama's work [15], we here mention that Theorem FFK is equivalent to the Furuta inequality.

Now we consider the following operator inequality for positive invertible operators A and B:

$$(*) \qquad (B^{s/2}A^{(s-t)/2}B^tA^{(s-t)/2}B^{s/2})^{\frac{1}{2s}} \geq B.$$

Recently, as an application of the Daleckii–Krein formula (see [2]) for the derivative of matrix valued function, one of the authors [4] proved that if matrices A, B satisfy $(*)$ for any $t > 1$ and $s = 1$, then $\log B \geq \log A$. In this situation, recalling the equivalence between Theorem FFK and the Furuta inequality, it is expected that the conclusion $\log B \geq \log A$ is built up the usual order $B \geq A$.

In this note, we prove that if positive operators A and B satisfy the operator inequality $(*)$ for a fixed $t > s > 0$, then

(1) If $t \geq 3s - 2 \geq 0$, then $\log B \geq \log A$, and if $t \geq s + 2$ is additionally assumed, then $B \geq A$.
(2) If $0 < s < 1/2$, then $\log B \geq \log A$, and if $t \geq s + 2$ is additionally assumed, then $B \geq A$.

2. A preliminary result for the chaotic order

In the consideration on Kamei's satellite theorem [12] of the Furuta inequality, we are required some operator inequalities of Furuta type implying the chaotic order. Consequently one of the authors announced the following result in [4]: If positive definite matrices $A, B > 0$ satisfy

$$(B^{1/2}A^{(1-t)/2}B^tA^{(1-t)/2}B^{1/2})^{1/2} \geq B \quad \text{for all } t > 1,$$

then $\log B \geq \log A$.

We now generalize it as follows:

Theorem 1. *For positive definite matrices $A, B > 0$, if there exist α, β such that $\alpha + \beta = 1$ and*

$$(B^{(\alpha+\beta t)/2}A^{(1-t)/2}B^{\beta+\alpha t}A^{(1-t)/2}B^{(\alpha+\beta t)/2})^{1/2} \geq B \quad \text{for all } t > 1,$$

then $\log B \geq \log A$.

Proof. Let $F(x) = x^{1/2}$, $\gamma(t) = B^{(\alpha+\beta t)/2}A^{(1-t)/2}B^{\beta+\alpha t}A^{(1-t)/2}B^{(\alpha+\beta t)/2}$ and U_t be unitaries such that $U_t^*\gamma(t)U_t = D(t) = \mathrm{diag}(d_1(t), \cdots, d_n(t))$, diagonal matrices. Here we recall the *Daleckii–Krein formula*

$$\frac{dF(\gamma(t))}{dt} = U_t \left(\left(F^{[1]}(d_i(t), d_j(t)) \right) \circ U_t^* \dot{\gamma}(t) U_t \right) U_t^*,$$

where $\circ$ stands for the Hadamard–Schur product and $F^{[1]}(x, y)$ is the *divided difference*

$$F^{[1]}(x, y) = \begin{cases} \frac{F(x)-F(y)}{x-y} & \text{if } x \neq y \\ F'(x) & \text{if } x = y \end{cases}.$$

We may assume that B itself is a diagonal matrix $\mathrm{diag}(d_j)$, so $U_1 = I$, the identity matrix. Therefore, at $t = 1$, we obtain

$$\frac{dF(\gamma)}{dt}(1) = \left(F^{[1]}(d_i^2, d_j^2)\right) \circ \dot{\gamma}(1) \text{ and } \left(F^{[1]}(d_i^2, d_j^2)\right) = \left(\frac{d_i - d_j}{d_i^2 - d_j^2}\right) = \left(\frac{1}{d_i + d_j}\right).$$

It follows that

$$\begin{aligned}
\dot{\gamma}(t) &= \frac{\beta}{2}(\log B)B^{(\alpha+\beta t)/2}A^{(1-t)/2}B^{\beta+\alpha t}A^{(1-t)/2}B^{(\alpha+\beta t)/2} \\
&\quad + \alpha B^{(\alpha+\beta t)/2}A^{(1-t)/2}(\log B)B^{\beta+\alpha t}A^{(1-t)/2}B^{(\alpha+\beta t)/2} \\
&\quad + \frac{\beta}{2}B^{(\alpha+\beta t)/2}A^{(1-t)/2}B^{\beta+\alpha t}A^{(1-t)/2}B^{(\alpha+\beta t)/2}(\log B) \\
&\quad - \frac{1}{2}B^{(\alpha+\beta t)/2}(\log A)A^{(1-t)/2}B^{\beta+\alpha t}A^{(1-t)/2}B^{(\alpha+\beta t)/2} \\
&\quad - \frac{1}{2}B^{(\alpha+\beta t)/2}A^{(1-t)/2}B^{\beta+\alpha t}(\log A)A^{(1-t)/2}B^{(\alpha+\beta t)/2} \\
\longrightarrow \dot{\gamma}(1) &= (\log B)B^2 - \frac{1}{2}B^{1/2}(\log A)B^{3/2} - \frac{1}{2}B^{3/2}(\log A)B^{1/2} \\
&= \frac{1}{2}B^{1/2}\left(B(\log B - \log A) + (\log B - \log A)B\right)B^{1/2} \\
&= \frac{1}{2}(\mathbf{L}_B + \mathbf{R}_B)(B^{1/2}(\log B - \log A)B^{1/2}) \\
&= \frac{1}{2}\left((d_i + d_j)\right) \circ (B^{1/2}(\log B - \log A)B^{1/2})
\end{aligned}$$

as $t \longrightarrow 1$, so we have

$$\begin{aligned}\frac{dF(\gamma)}{dt}(1) &= \big(F^{[1]}(d_i, d_j)\big) \circ \dot{\gamma}(1)\\ &= \left(\frac{1}{d_i + d_j}\right) \circ \left(\frac{1}{2}\big((d_i + d_j)\big) \circ (B^{1/2}(\log B - \log A)B^{1/2})\right)\\ &= \frac{1}{2}(B^{1/2}(\log B - \log A)B^{1/2}).\end{aligned}$$

On the other hand, since

$$\frac{dF(\gamma)}{dt}(1) = \lim_{t\downarrow 1} \frac{(B^{(\alpha+\beta t)/2}A^{(1-t)/2}B^{\beta+\alpha t}A^{(1-t)/2}B^{(\alpha+\beta t)/2})^{1/2} - B}{t-1} \geq 0,$$

we obtain $B^{1/2}(\log B - \log A)B^{1/2} \geq 0$, that is, $\log B \geq \log A$. □

3. Main Theorems

The operator inequality $(*)$ is a multiple version of the Furuta inequality. We here generalize Theorem 1 to the case with 2 variables. Nevertheless, the Furuta inequality is applicable to resolve it. In this section, we first propose the following theorem.

Theorem 2. *Suppose that $A, B > 0$ satisfy the inequality $(*)$, i.e.,*

$$(B^{s/2}A^{(s-t)/2}B^tA^{(s-t)/2}B^{s/2})^{\frac{1}{2s}} \geq B$$

for some $t > s > 0$. Then the following assertions hold:

(1) If $t \geq 3s - 2 \geq 0$, then $\log B \geq \log A$, and if the additional condition $t \geq s + 2$ is assumed, then $B \geq A$.

(2) If $0 < s < 1/2$, then $\log B \geq \log A$, and if the additional condition $t \geq s+2$ is assumed, then $B \geq A$.

Proof. By the Furuta inequality, we have for $p = 2s$ and $r = t - s$

$$(B^{(t-s)/2}B^{s/2}A^{(s-t)/2}B^tA^{(s-t)/2}B^{s/2}B^{(t-s)/2})^{\frac{t-s+1}{t+s}} \geq B^{t-s+1},$$

that is,

$$(B^{t/2}A^{(s-t)/2}B^tA^{(s-t)/2}B^{t/2})^{\frac{t-s+1}{t+s}} \geq B^{t-s+1}.$$

Hence we have

$$(B^{t/2}A^{(s-t)/2}B^{t/2})^{\frac{2(t-s+1)}{t+s}} \geq B^{t-s+1}.$$

Now we prove (1): As $\frac{2(t-s+1)}{t+s} > 1$ by $t \geq 3s - 2$, $B^{t/2}A^{(s-t)/2}B^{t/2} \geq B^{\frac{t+s}{2}}$, and so $A^{(s-t)/2} \geq B^{(s-t)/2}$. Consequently, we have $\log B \geq \log A$ by $t > s$ and the operator monotonicity of the logarithmic function. Moreover, if $t \geq s + 2$, then $(t - s)/2 \geq 1$ and so $B \geq A$ by the Löwner–Heinz theorem.

Next, if $s < 1/2$, then by the Löwner–Heinz inequality, we have

$$B^{s/2}A^{(s-t)/2}B^tA^{(s-t)/2}B^{s/2} \geq B^{2s}.$$

Hence it follows that $A^{(s-t)/2}B^tA^{(s-t)/2} \geq B^s$ and thus

$$B^{t/2}A^{(s-t)/2}B^tA^{(s-t)/2}B^{t/2} \geq B^{s+t},$$

that is, $(B^{t/2}A^{(s-t)/2}B^{t/2})^2 \geq B^{s+t}$. Consequently, we have $A^{(s-t)/2} \geq B^{(s-t)/2}$ and the conclusion is obtained as in the proof of (1). □

Corollary 3. *If $A, B > 0$ satisfy*

$$(B^{1/2}A^{(1-t)/2}B^tA^{(1-t)/2}B^{1/2})^{\frac{1}{2}} \geq B \quad \textit{for a fixed } t > 1,$$

then $\log B \geq \log A$. Moreover if it satisfied for some $t \geq 3$, then $B \geq A$.

Unfortunately the converse in Theorem 2 does not hold.

Example 4. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, s = 1$ and $t = 4$. Then $B \geq A$ and

$$\sigma((B^{1/2}A^{(1-t)/2}B^tA^{(1-t)/2}B^{1/2})^{1/2} - B) = \{35.2421, -0.25003\}.$$

Theoretically, if operators A and B satisfy $(*)$, then $B^{(t-1)/2} \geq A^{(t-1)/2}$ as in the proof of Theorem 2. However in general, $B \geq A$ does not imply $B^{(t-1)/2} \geq A^{(t-1)/2}$, Hence $B \geq A$ does not always imply $(*)$.

Remark 5. It must be $t > 1$ in order to imply $B \geq A$. If A commutes with B, then we have $(A^{1-t}B^{1+t})^{1/2} \geq B$, that is, $B^{t-1} \geq A^{t-1}$. Hence $B \geq A$ if $t > 1$.

For $1 > t > 0$, we prove the following theorem by applying Lyapunov equation, see [2] and [9].

Theorem 6. *If $A, B > 0$ satisfy $(*)$ for $s = 1$ and any $t \in (0, 1)$, then $\log A \geq \log B$.*

Proof. Put $X_t = B^{1/2}A^{(1-t)/2}B^tA^{(1-t)/2}B^{1/2}$. Then we have

$$(X_t^{1/2} - B)X_t^{1/2} + B(X_t^{1/2} - B) = X_t - B^2$$

and

$$X_t - B^2 =$$
$$B^{1/2}\{A^{(1-t)/2}(B^t - B)A^{(1-t)/2} + A^{(1-t)/2}B(A^{(1-t)/2} - 1) + (A^{(1-t)/2} - 1)B\}B^{1/2}.$$

Here we have

$$\lim_{t\to 1} \frac{B^t - B}{t - 1} = B \log B \text{ and } \lim_{t\to 1} \frac{A^{(1-t)/2} - 1}{t - 1} = -\frac{1}{2} \log A.$$

Hence by putting $Y = \lim_{t\to 1} \frac{X_t^{1/2} - B}{t-1}$ via the chain rule (cf. [11, Theorem 8.4]), it follows that $Y \leq 0$ by the assumption and

$$BY + YB = B^{1/2}(B \log B - \frac{1}{2}(B \log A + \log A\ B))B^{1/2}.$$

By solving this Lyapunov equation,

$$\begin{aligned} Y &= B^{1/2} \left(\int_{-\infty}^0 e^{tB}((B \log B - \frac{1}{2}(B \log A + \log A\ B))e^{tB} dt \right) B^{1/2} \\ &= B^{1/2} \left(\frac{1}{2} \log B [e^{2tB}]_{-\infty}^0 - \frac{1}{2} [e^{tB} \log A e^{tB}]_{-\infty}^0 \right) B^{1/2} \\ &= \frac{1}{2} B^{1/2} (\log B - \log A) B^{1/2}. \end{aligned}$$

Since $Y \leq 0$, we have $\log A \geq \log B$. □

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