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ON REVERSING OF THE MODIFIED YOUNG INEQUALITY

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This paper is dedicated to Professor Tsuyoshi Ando

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ABSTRACT. In the present paper, by Haagerup theorem, we show that if $A \in \mathbb{M}_n$ is a non scalar strictly positive matrix and $0 < \nu < 1$ be a real number such that $\nu \neq \frac{1}{2}$, then there exists $X \in \mathbb{M}_n$ such that

$$\|A^{\nu} X A^{1-\nu}\| > \|\nu AX + (1-\nu)XA\|.$$

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. A norm $|||.|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$. For $A \in \mathbb{M}_n$, the numerical radius of A is defined and denoted by

$$\omega(A) = \max\{|x^* A x| : x \in \mathbb{C}^n, x^* x = 1\}.$$

It is known that $\omega(.)$ is a vector norm on $\mathbb{M}_n$, but is not unitarily invariant.

Throughout the paper we use the term positive for a positive semidefinite matrix, and strictly positive for a positive definite matrix. Also we use the notation $A \geq 0$ to mean that A is positive, $A > 0$ to mean it is strictly positive. In $\mathbb{M}_n$, beside the usual matrix product, the entrywise product is quite important and interesting. The entry wise product of two matrices A and B is called their Schur (or Hadamard) product and denoted by $A \circ B$. With this multiplication $\mathbb{M}_n$ becomes a commutative algebra, for which the matrix with all entries equal to one is the unit and we denote that by "J". The linear operator S_A on $\mathbb{M}_n$, is

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called the Schur multiplier operator and defined by $S_A(X) := A \circ X$. The induced norm of S_A with respect to all unitarily invariant norm will be denoted by

$$|||S_A||| = \sup_{X \neq 0} \frac{|||S_A(X)|||}{|||X|||} = \sup_{X \neq 0} \frac{|||A \circ X|||}{|||X|||},$$

and the induced norm of S_A with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}.$$

For positive real numbers a, b, the classical Young inequality says that if $p, q > 1$ such that $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.1}$$

the another form of the inequality for positive real numbers a, b is in the following form:

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b, \; 0 \leq \nu \leq 1. \tag{1.2}$$

For more details about these inequalities, their refinements and associated norm inequalities with their history of origin, the reader may refer to [2, 5, 6, 8, 9]. In [9] we showed that, if $A, B \geq 0$, and $X \in \mathbb{M}_n$. Then the inequality $\omega(AXB) \leq \omega(\frac{A^p}{p}X + X\frac{B^q}{q})$ does not holds in general as follows:

Theorem 1.1. [9, Theorem 2.3], *Let $p > q > 1$ such that $1/p + 1/q = 1$ and let $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix such that $1 \in \sigma(A)$, then there exists $X \in \mathbb{M}_n$ such that*

$$\omega(AXA) > \omega(\frac{A^p}{p}X + X\frac{A^q}{q}). \tag{1.3}$$

Also, in [10] we showed the following inequaliy for the numerical radius:

Theorem 1.2. *Let $A \in \mathbb{M}_n$ be a positive matrix. Then for all $X \in \mathbb{M}_n$, we have*

$$\omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2). \tag{1.4}$$

2. Main results

Bhatia and Kittaneh in 1990 [7] established a matrix mean inequality as follows:

$$|||A^*B||| \leq \frac{1}{2}\,|||A^*A + B^*B|||\,, \tag{2.1}$$

for matrices $A, B \in \mathbb{M}_n$.

In [5] a generalization of (2.1) was proved, for all $X \in \mathbb{M}_n$,

$$|||A^*XB||| \leq \frac{1}{2}\,|||AA^*X + XBB^*|||\,. \tag{2.2}$$

Ando in 1995 [2] obtained a matrix Young inequality:

$$|||AB||| \leq \left|\left|\left|\frac{A^p}{p} + \frac{B^q}{q}\right|\right|\right|, \tag{2.3}$$

for $p, q > 1$ with $1/p + 1/q = 1$ and positive matrices A, B.
Also, in [1], the author pointed out that the matrix Young inequality $|||AXB||| \leq |||\frac{1}{p}A^pX + \frac{1}{q}XB^q|||$ is not valid for the spectral norm $\|.\|$.

Here, we clarify it. Ando and Okubo in 1991, [4], proved the following theorem[4, Theorem 1 and Corollary 3]:

Theorem 2.1. *(Haagerup theorem) For $A \in \mathbb{M}_n$ the following assertions are equivalent:*
(i) $\|S_A\| \leq 1$.
(ii) There is $0 \leq R_1, R_2 \in \mathbb{M}_n$ such that

$$\begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix} \geq 0, \qquad R_1 \circ I \leq I \qquad \text{and} \qquad R_2 \circ I \leq I.$$

Moreover, if A is Hermitian, then $\|S_A\| = \|S_A\|_\omega$.

Lemma 2.2. *Let $0 < \nu < 1$ be a real number such that $\nu \neq \frac{1}{2}$ and $a > 0$. Then*

$$\frac{a^\nu}{\nu a + (1-\nu)} = \frac{a^{1-\nu}}{\nu + (1-\nu)a}$$

holds if and only if $a = 1$.

Proof. Assume if possible there exists $a > 0$ and $a \neq 1$, such that

$$\frac{a^\nu}{\nu a + (1-\nu)} = \frac{a^{1-\nu}}{\nu + (1-\nu)a}. \tag{2.4}$$

Then (2.4) is equivalent to

$$(1-\nu)a^{1+\nu} + \nu a^\nu - \nu a^{2-\nu} - (1-\nu)a^{1-\nu} = 0. \tag{2.5}$$

Now let $\nu = \frac{1}{p}$, where $p > 1, p \neq 2$ and let

$$f(x) = \frac{1}{p}\left((p-1)x^{\frac{p+1}{p}} + x^{\frac{1}{p}} - x^{\frac{2p-1}{p}} - (p-1)x^{\frac{p-1}{p}}\right).$$

Now replace x with x^p we have

$$k(x) = \frac{1}{p}\left((p-1)x^{p+1} + x - x^{2p-1} - (p-1)x^{p-1}\right) = \frac{xk_1(x)}{p}.$$

By the assumption and by the Rolle's theorem, the (2.5) is equivalent to

$$k_1(x) = (p-1)x^p - x^{2p-2} - (p-1)x^{p-2} + 1$$

has at least one positive root $r_1 \neq 1$. Now, apply the Rolle's theorem for

$$k_2(x) = k_1'(x) = (p-1)x^{p-3}\left(px^2 - 2x^p - (p-2)\right) = (p-1)x^{p-3}k_3(x),$$

we can say that the function

$$k_3'(x) = 2px(1 - x^{p-2})$$

has at least one positive root $r_2 \neq 1$. That is a contradiction. $\square$

Now, in the following theorem, we will show that if $A, B \geq 0$, and $X \in \mathbb{M}_n$, then $|||A^\nu X B^{1-\nu}||| \leq |||\nu AX + (1-\nu)XB|||$ does not holds in general.

Theorem 2.3. *Let $0 < \nu < 1$ be a real number such that $\nu \neq \frac{1}{2}$ and $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix. Then there exists $X \in \mathbb{M}_n$ such that*

$$\|A^\nu X A^{1-\nu}\| > \|\nu AX + (1-\nu)XA\|. \tag{2.6}$$

Proof. Without loss of generality we assume that $A = \mathrm{diag}(a_1, a_2, a_3, \ldots, a_n)$ where $a_1 = 1$ and $a_2 \neq 1$. By Lemma 2.2, it is readily seen that

$$\frac{a_2^\nu}{\nu a_2 + (1-\nu)} \neq \frac{a_2^{1-\nu}}{\nu + (1-\nu)a_2}. \tag{2.7}$$

Assume if possible for all $X \in \mathbb{M}_n$,

$$\|A^\nu X A^{1-\nu}\| \leq \|\nu AX + (1-\nu)XA\|. \tag{2.8}$$

Now, let $C = (c_{ij})$ and $E = (e_{ij})$ be $n \times n$ matrices, where $c_{ij} = \nu a_i + (1-\nu)a_j$, and $e_{ij} = a_i^\nu a_j^{1-\nu}$. Then we rewrite (2.8) in the following form

$$\|E \circ X\| \leq \|C \circ X\|, \quad (X \in \mathbb{M}_n). \tag{2.9}$$

Let D be the entrywise inverse of $C (C \circ D = J)$. We replace X by $(D \circ X)$ in (2.9), then

$$\|(E \circ D) \circ X\| \leq \|X\|, \quad (X \in \mathbb{M}_n). \tag{2.10}$$

Let $F := (E \circ D) = (f_{ij})$. Then $\|F \circ X\| \leq \|X\|$ for all $X \in \mathbb{M}_n$ and hence,

$$\|S_F\| \leq 1. \tag{2.11}$$

Now by Haagerup theorem , there exist $n \times n$ matrices $X = (x_{ij}), Y = (y_{ij}) \geq 0$ with $0 \leq x_{ii}, y_{ii} \leq 1, \quad (1 \leq i \leq n)$, such that

$$\begin{bmatrix} X & F \\ F^* & Y \end{bmatrix} \geq 0.$$

By considering $\tilde{X} := (\tilde{x}_{ij})$ such that $\tilde{x}_{ij} = x_{ij}$ if $i \neq j$ and $\tilde{x}_{ii} = 1$, and $\tilde{Y} := (\tilde{y}_{ij})$ such that $\tilde{y}_{ij} = y_{ij}$ if $i \neq j$ and $\tilde{y}_{ii} = 1$, we obtain that

$$\begin{bmatrix} \tilde{X} & F \\ F^* & \tilde{Y} \end{bmatrix} \geq \begin{bmatrix} X & F \\ F^* & Y \end{bmatrix} \geq 0.$$

Since, any principal submatrix of the above matrix is positive, we have

$$\begin{bmatrix} 1 & x & 1 & f_{12} \\ \bar{x} & 1 & f_{21} & 1 \\ 1 & f_{21} & 1 & y \\ f_{12} & 1 & \bar{y} & 1 \end{bmatrix} \geq 0 \quad \textit{where} \quad x := \tilde{x}_{12} = x_{12}, y := \tilde{y}_{12} = y_{12}.$$

By using the Schur complement Theorem [5, Theorem 1.3.3], we obtain that

$$\begin{bmatrix} 1 & f_{21} & 1 \\ f_{21} & 1 & y \\ 1 & \bar{y} & 1 \end{bmatrix} - \begin{bmatrix} \bar{x} \\ 1 \\ f_{12} \end{bmatrix} \begin{bmatrix} x & 1 & f_{12} \end{bmatrix} = \begin{bmatrix} 1-|x|^2 & f_{21}-\bar{x} & 1-\bar{x}f_{12} \\ f_{21}-x & 0 & y-f_{12} \\ 1-xf_{12} & \bar{y}-f_{12} & 1-f_{12}^2 \end{bmatrix} \geq 0.$$

Since the determinant of principle submatrices of the above matrix is positive, we have $f_{21} - x = y - f_{12} = 0$ and hence

$$B = \begin{bmatrix} 1 & f_{21} & 1 & f_{12} \\ f_{21} & 1 & f_{21} & 1 \\ 1 & f_{21} & 1 & f_{12} \\ f_{12} & 1 & f_{12} & 1 \end{bmatrix} \geq 0.$$

Let $f(\lambda)$ be the characteristic polynomial of B as follows

$$f(\lambda) = \lambda^4 - 4\lambda^3 + (4 - 2f_{12}^2 - 2f_{21}^2)\lambda^2 + (-4f_{12}f_{21} + 2f_{12}^2 + 2f_{21}^2)\lambda.$$

By (2.7) we have $f_{21} \neq f_{12}$, we obtain that the coefficient of λ is positive and hence $f(\lambda)$ has one negative root, which is a contradiction with $B \geq 0$. □

Corollary 2.4. *Let $p > q > 1$ such that $1/p + 1/q = 1$ and $n \in \mathbb{N}$. Then there exist $A, B, X \in \mathbb{M}_n$ such that $A, B > 0$ and*

$$\|AXB\| > \left\| \frac{A^p}{p} X + X \frac{B^q}{q} \right\|.$$

Lemma 2.5. [4] *For all $A \in \mathbb{M}_n$*

$$\|S_A\| \leq \|S_A\|_\omega.$$

Now, by Lemma 2.5 and Theorem 2.3 we can obtain the following theorem that shows the another form of the Young inequality for the numerical radius does not holds.

Theorem 2.6. *Let $0 < \nu < 1$ be a real number such that $\nu \neq \dfrac{1}{2}$ and $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix. Then there exists $X \in \mathbb{M}_n$ such that*

$$\omega(A^\nu X A^{1-\nu}) > \omega(\nu AX + (1-\nu)XA).$$

Proof. Without loss of generality we assume that $A = \mathrm{diag}(a_1, a_2, a_3, \ldots, a_n)$. We assume if possible for all $A, X \in \mathbb{M}_n$ such that A is a non scalar strictly positive matrix, then

$$\omega(A^\nu X A^{1-\nu}) \leq \omega(\nu AX + (1-\nu)XA).$$

If we define

$$F := \left[\frac{a_i^\nu a_j^{1-\nu}}{\nu a_i + (1-\nu)a_j} \right] \in \mathbb{M}_n,$$

then easy computations show that $\|S_F\|_\omega \leq 1$. Now by Lemma 2.5 we have $\|S_F\| \leq 1$ and hence $\|A^\nu X A^{1-\nu}\| \leq \|\nu AX + (1-\nu)XA\|$, which is a contradiction by Theorem 2.3. □

Corollary 2.7. *Let $p > q > 1$ such that $1/p + 1/q = 1$ and $n \in \mathbb{N}$. Then there exist $A, B, X \in \mathbb{M}_n$ such that $A, B > 0$ and*

$$\omega(AXB) > \omega(\frac{A^p}{p} X + X \frac{B^q}{q}).$$

Remark 2.8. By the inequality (2.2) and Theorem 1.2, the condition $\nu \neq \frac{1}{2}$ in the Theorem 2.3 and Theorem 2.6 are essential.

Theorem 2.9. *Let $p > q > 1$ such that $1/p + 1/q = 1$. Then there is $A \in \mathbb{M}_n$ such that $A > 0$ and for all $X \in \mathbb{M}_n$*

$$|||AXA||| \leq \left|\left|\left| \frac{A^p}{p}X + X\frac{A^q}{q} \right|\right|\right|$$

if and only if there is

$$F = \left[\frac{a_i a_j}{\frac{a_i^p}{p} + \frac{a_j^q}{q}} \right] \in \mathbb{M}_n,$$

such that $a_i > 0 (i = 1, \ldots, n)$ and $|||S_F||| \leq 1$.

Moreover, if A is non scalar and $1 \in \sigma(A)$, then $\|S_F\|_\omega > |||S_F|||$.

Proof. Without loss of generality, assume that

$$A = \operatorname{diag}(a_1, a_2, a_3, \ldots, a_n), \quad a_i > 0, \ (i = 1, \ldots, n)$$

Now, let $C = [c_{ij}]$ and $E = [e_{ij}]$ be $n \times n$ matrices, where

$$c_{ij} = \frac{a_i^p}{p} + \frac{a_j^q}{q}, \quad e_{ij} = a_i a_j.$$

Then we have the following form

$$|||E \circ X||| \leq |||C \circ X|||, \qquad (X \in \mathbb{M}_n). \tag{2.12}$$

Let D be the entrywise inverse of $C (C \circ D = J)$. We replace X by $(D \circ X)$ in (2.12), then

$$|||(E \circ D) \circ X||| \leq |||X|||, \qquad (X \in \mathbb{M}_n).$$

Let $F := (E \circ D) = (f_{ij})$. Then, we obtain that

$$|||F \circ X||| \leq |||X|||, \qquad (X \in \mathbb{M}_n)$$

and hence, $|||S_F||| \leq 1$. It is enough to show that if A is non scalar and $1 \in \sigma(A)$, then $\|S_F\|_\omega > 1$. Assume if possible $\|S_F\|_\omega \leq 1$. Then we have for all $X \in \mathbb{M}_n$,

$$\omega(AXA) \leq \omega(\frac{A^p}{p}X + X\frac{A^q}{q}).$$

That is a contradiction by Theorem 1.1. ∎

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