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STABILITY RESULTS FOR C^*-UNITARIZABLE GROUPS

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ABSTRACT. We say that a locally compact group G is C^*-unitarizable if its full group C^*-algebra $C^*(G)$ satisfies Kadison's similarity problem (SP), i.e. every bounded representation of $C^*(G)$ on a Hilbert space is similar to a *-representation. We prove that locally compact and unitarizable groups are C^*-unitarizable. For discrete groups, we prove that C^*-unitarizable passes to quotients. Moreover, a given discrete group is C^*-unitarizable whenever we can find a normal and C^*-unitarizable subgroup with amenable quotient.

1. INTRODUCTION

This paper pursues one of John von Neumann's motivations for initializing the study of operator algebras, which was to provide an abstract framework for unitary representations of locally compact groups. In particular, we would like to reach a better understanding between unitary representations of groups (and Dixmier problem [4]) and C*-representations (and Kadison similarity problem [10]) inherited in the various operator algebras that are traditionally associated to groups. Of interest in its own right, Pisier asks in [16, Remark 0.6] if the following two stability problems hold for discrete groups:

$$H \text{ normal},\ H \text{ and } G/H \text{ are unitarizable} \implies G \text{ is unitarizable?} \tag{1.1}$$

$$G_1 \text{ and } G_2 \text{ are unitarizable} \implies G_1 \times G_2 \text{ also unitarizable?} \tag{1.2}$$

where a group G is said to be unitarizable if for any continuous and uniformly bounded $\sup||\pi_g|| < \infty$ representation $\pi : G \to B(H)$ on a Hilbert space H we can find an invertible operator S such that $g \to S^{-1}\pi(g)S$ is a unitary representation

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of G. Of course all these can likewise be considered for locally compact groups. Using Mackey's classical induction machinery of representation theory, Pisier [16] managed to prove that unitarizability passes to subgroups and quotients in the discrete case, and problems (1.1) and (1.2) are thus natural from the amenable group viewpoint. Their solution will shed light on the famous Dixmier problem [4] formulated in 1950 as follows:

$$G \text{ unitarizable} \implies G \text{ amenable?} \tag{1.3}$$

The converse of Dixmier problem (1.3) was proved in [4]. Note that $\mathrm{SL}(2,\mathbb{R})$ was the first non-unitarizable group discovered in 1955 by Ehrenpreis and Mautner [7]. Although (1.3) holds for connected locally compact groups see [16, Remark 0.9] and references therein, the discrete group case remains widely open.

The full group C^*-algebra $C^*(G)$, the reduced group C^*-algebra $C^*_{\mathrm{red}}(G)$, the von Neumann group algebra $\mathrm{vN}(G)$ and the *big group algebra* $\mathcal{A}(G)$ (bidual of $C^*(G)$ as Banach spaces) are some operator algebras, naturally associated to the underlying group G, and are the C^*-algebras we deal with in this paper. In general, locally compact groups cannot be embedded in $C^*(G)$, unless G is discrete. This obstacle disappears if we consider the big algebra $\mathcal{A}(G)$, i.e. we can embed G in $\mathcal{A}(G)$, see [5].

It is reasonable to formulate Dixmier problem using an operator algebraic framework. Indeed, this was done in 1955 by Kadison [10] (also known as Kadison's similarity problem, in short, (SP)) formulated as follows: for any (unital) C^*-algebra A and $\pi : A \to B(H)$ a homomorphism

$$\pi \text{ bounded} \implies \pi \text{ similar to a C}^*\text{-representation?} \tag{1.4}$$

i.e. there is an invertible operator S such that the map $a \to S^{-1}\pi(a)S$ is a $*$-homomorphism of A. C^*-algebras without tracial states [8] (for example $B(l^2(\mathbb{N}))$) and nuclear C^*-algebras (e.g. commutative and finite dimensional ones) do satisfy the Kadison (SP).

The Thompson group F, see (3.2) below, is an *exotic* example of a group whose group von Neumann algebra $\mathrm{vN}(F)$ satisfies the (SP) because F is an ICC group (= every non-trivial conjugacy class is an infinite set) and $\mathrm{vN}(F)$ has property Γ, see [2, 13]. Besides F does not contain free groups [1]. It is thus a natural candidate to become the first non-amenable discrete group and unitarizable. Golod-Shafarevich groups are proved in [6] to be non-amenable and may also be interesting examples for Dixmier problem.

We propose the notion of C^*-unitarizable group whenever its full group algebra $C^*(G)$ satisfies the Kadison (SP). Since there is no counterexample to Kadison's similarity problem there is also no known example of a non-C^*-unitarizable group. However there are certainly non-unitarizable groups, e.g. the free group $\mathbb{F}_2$, see e.g. [15]. Besides the (SP), we also consider the weak similarity problem (WSP), see Definition 3.10 below, in the case of a von Neumann algebra when we not only consider bounded homomorphisms but also their weak* continuity. This is useful to relate the unitarizibility of the group and the (WSP) for the big algebra, for locally compact groups. When we consider the group of unitaries of a C^*-algebra, we conclude that Kadison's (SP) holds for any C^*-algebra if any discrete group is C^*-unitarizable.

Kadison's similarity problem (1.4) is equivalent to another crucial problem (the *derivation problem*) in the cohomology theory of operator algebras, as follows. A linear map $D : A \to B(H)$ is called a derivation with respect to a bounded homomorphism $\pi : A \to B(H)$ if D obeys the Leibniz rule:

$$D(ab) = D(a)\pi(b) + \pi(a)D(b), \quad \text{for all } a, b \in A. \tag{1.5}$$

We say that a C^*-algebra A satisfies the derivation problem (abbreviated as (DP)) if any derivation $D : A \to B(H)$ is inner, i.e. $D = \delta_T$ where $\delta_T(a) = T\pi(a) - \pi(a)T$. A celebrated theorem of Kirchberg [11] states that

$$A \text{ satisfies the (SP)} \iff \text{A satisfies the (DP)}.$$

This equivalent formulation of the (SP) enables us to give a partial (positive) answer to Pisier problem (1.1). We naturally also have a notion of derivation for a group representation and we say that a group G satisfies the (DP) if any derivation on G is inner, see [12].

The rest of the paper is organized as follows. In Sect. 2 we fix some notation and background material necessary in the sequel. In Sect. 3 we prove our main results. In Definition 3.1 we propose the notion of C^*-unitarizable for locally compact groups. Using the derivation problem, in Theorem 3.7 we show that a discrete group inherits the C^*-unitarizability from a C^*-unitarizable subgroup and its quotient, provided the latter is also amenable (see Pisier problem (1.1)). After the notion of a *weak* version of (SP) for von Neumann algebras as in Definition 3.10, we establish in Theorem 3.11 some new consequences for Kadison similarity problem (1.4) for the various operator algebras associated to a given locally compact group (assumed to be unitarizable). In particular we conclude that a locally compact group is C^*-unitarizable if the group is unitarizable. With all these at hand, we draw some conclusions at the end of the paper towards Dixmier problem (1.3), using Thompson group F and $F \times G$ with G amenable as possible candidates of non-amenable but unitarizable groups.

2. Background material

Let G be a locally compact group and μ the (left) Haar measure of G. Recall that the notion of amenable group was introduced by von Neumann in 1929, and says that G is amenable if there exists a left invariant mean on G, i.e. if there exists positive linear functional $m : L^\infty(G) \to \mathbb{C}$ such that $m(1) = 1$ and $m(f) = m({}_g f)$ for any $g \in G$, where ${}_g f(t) = f(g^{-1}t)$ and $L^\infty(G)$ is the Banach space of all essentially bounded functions $G \to \mathbb{C}$ with respect to the Haar measure.

Next, we let $C_c(G)$ be the space of complex valued continuous functions on G with compact support. Consider $L^2(G)$ the Hilbert space of square integrable functions with respect to μ. We also recall the convolution product as follows:

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\, d\mu(s).$$

Let Δ be the modular function on G ($\Delta \equiv 1$ for discrete groups). Then $f^*(t) = \Delta(t^{-1})\overline{f(t^{-1})}$ equips $C_c(G)$ with an involution $*$. For an integrable function $f \in C_c(G)$, $||f||_1 := \int_G |f(t)|\, d\mu(t)$ equips $C_c(G)$ with a structure of a normed algebra. The convolution algebra $L^1(G)$ is the $*$-Banach algebra obtained by completion of $C_c(G)$ for that norm. Any unitary representation π of G can be lifted to a *-representation $\overline{\pi}$ of $L^1(G)$ on the same Hilbert space. The (full) group C^*-algebra C^*(G) of G is the C^*-enveloping algebra of $L^1(G)$, i.e. the completion of $L^1(G)$ with respect to the largest C*-norm:

$$||f||_{C^*(G)} := \sup_{\overline{\pi}} \|\overline{\pi}(f)\|,$$

where π ranges over all unitary representations of G on Hilbert spaces. The reduced C^*-algebra $C^*_{\text{red}}(G)$ is the C^*-algebra generated by the left regular representation $\lambda(G)$ in $B(L^2(G))$ and defined as follows:

$$\lambda_g(f)(t) = f(g^{-1}t) \quad g, t \in G,\ f \in L^2(G).$$

The left regular representation gives rise to a natural C^*-morphism $C^*(G) \to C^*_{\text{red}}(G)$ which is an isomorphism if and only if G is amenable. Also let $\text{vN}(G)$ be the von Neumann algebra generated by $\lambda(G)$ in $B(L^2(G))$. In general, for $f \in L^1(G)$, we have:

$$||f||_{C^*_{\text{red}}(G)} \leq ||f||_{C^*(G)} \leq ||f||_{L^1(G)}.$$

Any group morphism between two discrete groups $G_1 \to G_2$ can be lifted to a C^*-algebra $*$-homomorphism $C^*(G_1) \to C^*(G_2)$ (see Rieffel's [17, Proposition 4.1]). In general this functoriality does not extent to locally compact groups.

The big group C^*-algebra $\mathcal{A}(G)$ associated to a locally compact group G is defined as follows. Let H be a infinite dimensional separable Hilbert space and let G_H be the set of all unitary representations $\pi : G \to B(H)$ on that fixed Hilbert space H and $\mathcal{A}(G) = \{J : G_H \to B(H)\}$ the set of maps from G_H to $B(H)$ satisfying some natural conditions as in [5, Page 469], where we can define an involutive algebra structure. The weak topology on $\mathcal{A}(G)$ is defined to be the smallest topology such that the functions $J \to \langle J(L)\xi, \psi\rangle$ are continuous, for all $J \in G_H$ and $\xi, \psi \in H$. As Banach algebras, $\mathcal{A}(G)$ maybe identified with $C^*(G)^{**}$ the bidual of the full C^*-algebra $C^*(G)$. For every $g \in G$, let $\hat{g} : G_H \to B(H)$ be the map defined by $\hat{g}(\pi) = \pi(g)$. Then $g \to \hat{g}$ gives an imbedding of G into $\mathcal{A}(G)$, with image $\hat{G}$ constituted by unitary operators, see [5, Theorem 2.3]. Moreover the algebras $L^1(G)$ and $C^*(G)$ are weak-* dense in $\mathcal{A}(G)$, see [5, Corollary 3.2]. The *-subalgebra $\text{Alg}(\hat{G})$ of all finite linear combinations of elements in $\hat{G}$ is dense in $\mathcal{A}(G)$ relative to any of the topologies: weak and σ-weak, see [5, Theorem 7.2].

3. Main Results

Let $\pi : A \to B(H)$ be a homomorphism of a C^*-algebra A and $U_d(A)$ its group of unitary elements equipped with the discrete topology, then the restriction $\pi|_{U_d(A)} : U_d(A) \to B(H)$ is a representation of $U_d(A)$. Furthermore, π is a bounded homomorphism (*-homomorphism, similar to a *-homomorphism, respectively) if and only if the group representation $\pi|_{U_d(A)}$ is uniformly bounded (unitary, unitarizable, respectively). Needless to say, here we use the elementary fact that any $a \in A$ is a finite linear combination of unitary operators in A. The role of the discrete topology in $U_d(A)$ is just to have a locally compact group.

We clearly have any discrete group G as a subgroup of $U_d(A)$ with A being the full group C^*-algebra $C^*(G)$, reduced group C^*-algebra $C^*_{\text{red}}(G)$ or the big group algebra $\mathcal{A}(G)$.

Next we remark that by definition we easily see that if we have a surjective homomorphism $\varphi : A \to B$ between C^*-algebras, then if A satisfies Kadison (SP), so does B. In particular we apply this for the natural surjective homomorphism $\lambda : C^*(G) \to C^*_{\text{red}}(G)$ whenever $C^*(G)$ has the (SP). We now propose the following definition.

Definition 3.1. A (locally compact) group G is C^*-unitarizable if the C^*-algebra $\mathrm{C}^*(G)$ satisfies the (SP).

If G is amenable, then $C^*(G)$ is a nuclear C^*-algebra [9] and thus $C^*(G)$ satisfies the (SP), and therefore G is a C^*-unitarizable group.

Proposition 3.2. *Let G be a discrete group. A bounded representation $\pi : C^*(G) \longrightarrow B(H)$ of $C^*(G)$ on a Hilbert space H is similar to a $*$-homomorphism if and only if the restriction $\pi|_G$ is unitarizable.*

Proof. Using Theorem 0.9 of Pisier's paper [16] to show this result. □

As an immediate consequence we obtain the following.

Corollary 3.3. *Every unitarizable discrete group G is C^*-unitarizable.*

Proposition 3.4. *Let G be a discrete group and $\pi : C^*(G) \to B(H)$ a bounded homomorphism. Then π is completely bounded if and only if its restriction $\pi|_{C^*(\Gamma)}$ on $C^*(\Gamma)$ is completely bounded, for any countable subgroup Γ of G.*

Proof. The 'only if' part is obvious. Let us prove the 'if' part. Let $\pi : C^*(G) \to B(H)$ be a bounded homomorphism and let Γ be an arbitrary countable subgroup of G. Then the restriction $\pi|_{C^*(\Gamma)}$ is completely bounded, by hypothesis. This implies that the restriction $\pi|_\Gamma$ is unitarizable [16, Theorem 0.9]. Since Γ is arbitrary, by [16, Corollary 1] we conclude that $\pi|_G$ is an unitarizable representation of G. Then using again [16, Theorem 0.9] we conclude that the extension of $\pi|_G$ to $C^*(G)$ is completely bounded. Therefore π is completely bounded because the extension of $\pi|_G$ to $C^*(G)$ is unique. □

As an immediate consequence of Proposition 3.4 we have the following

Corollary 3.5. *A discrete group G is C^*-unitarizable if and only if any countable subgroup of G is C^*-unitarizable.*

Next we present further stability result.

Proposition 3.6. *Let G be a C^*-unitarizable discrete group. Then for every normal subgroup Γ, we have G/Γ is C^*-unitarizable.*

Proof. There is a canonical homomorphism $\rho_\Gamma : l^1(G) \to l^1(G/\Gamma)$ defined by $\rho_\Gamma(\delta_g) = \delta_{g\Gamma}$. The ρ_Γ extends to a *-homomorphism (still denoted by ρ_Γ) from $C^*(G)$ onto $C^*(G/\Gamma)$. Let now $\pi : C^*(G/\Gamma) \longrightarrow B(H)$ be a bounded representation. Since G is C^*-unitarizable, there is an invertible operator $S \in B(H)$ such that in $S^{-1}\pi \circ \rho_\Gamma S$ is *-homomorphism. As ρ_Γ is surjective, $S^{-1}\pi S$ is also a $*$-homomorphism on $C^*(G/\Gamma)$. □

If H is a Hilbert space and tr denotes the trace on $B(H)$, we let $C(H)$ be finite trace class operators of $B(H)$. Note that $C(H)^* = K(H)$ (where $K(H)$ denotes the compact operators). Note that a derivation on a group G w.r.t. a representation $\pi : G \to B(H)$ is a map $D : G \to B(H)$ satisfying Leibnitz rule, cf. (1.5) above.

Theorem 3.7. *Let G be a discrete group and Γ a normal subgroup of G. Assume that Γ is C^*-unitarizable and G/Γ is amenable. Then G is C^*-unitarizable.*

Proof. Let (d, π) be derivation of $C^*(G)$ into $B(H)$, $(d \circ j, \pi \circ j)$ is a derivation of $C^*(\Gamma)$ where j the injective canonical *-homomorphism from $C^*(\Gamma)$ into $C^*(G)$, see [17]. Since Γ is C^*-unitarizable, $d \circ j$ is inner. Let $T \in B(H)$ such that $d \circ j = \delta_T$. Put $D = d - \delta_T$. Of course D is vanishes on Γ and therefore gives rise to a (well defined) derivation $\tilde{D}$ of G/Γ by $\tilde{D}(g\Gamma) = d(g) - \delta_T(g)$. Let now

$$\mathcal{B} = \{T \in B(H) : \ \pi(g)T = T\pi(g) = T \quad \text{for all } g \in \Gamma\}$$

and

$$C_0 = \text{span}\{\pi(g)S + T\pi(g), \ g \in \Gamma, \ S, T \in C(H)\}.$$

It is easy to check that $\tilde{D}(G/\Gamma) \subseteq \mathcal{B}$ and moreover $\mathcal{B}$ is w*-closed (because it is weakly-closed) and so it is a dual space of $C(H)/C_0$ where C_0 is the orthogonal space of $\mathcal{B}$, i.e. $C_0 = \{T \in B(H) : tr(TB) = 0, \text{ for all } B \in \mathcal{B}\}$.

Since G/Γ is amenable then $\tilde{D} = \delta_R$ for some operator R. It follows that $d = \delta_{T+R}$. □

We now investigate the relation we have found between the (DP) and the C^*-unitarizability for discrete groups.

Corollary 3.8. *Let G_1 be an amenable group and G_2 be a C^*-unitarizable group. Then $G_1 \times G_2$ is C^*-unitarizable. In particular, $C^*_{\text{red}}(G_1 \times G_2)$ satisfies the (SP).*

Proposition 3.9. *If G is a discrete group satisfying (DP) then G is C^*-unitarizable.*

Proof. Let $D : C^*(G) \to B(H)$ be a derivation with respect to some (continuous) representation $\pi : C^*(G) \to B(H)$ of the group algebra. Then let $d : G \to B(H)$ be defined by $d(g) = D(\delta_g)$ and $\rho : G \to B(H)$ defined by $\rho(g) = \pi(\delta_g)$. Note that δ_g is the characteristic function in $\mathbb{C}[G]$. Then we can easily see that d is a derivation of G with respect to the representation ρ. Since G has the (DP), we conclude that $d(\cdot) = \delta_T(\cdot) = [\cdot, T]$ for some $T \in B(H)$.

Hence $D = \delta_T$ on $\mathbb{C}[G]$ by the definitions. As D is continuous on $\mathbb{C}[G]$ equipped with the C^*-norm of $C^*(G)$, we conclude that $D = \delta_T$ on $C^*(G)$. Thus $C^*(G)$ satisfies the (DP) and therefore $C^*(G)$ satisfies the (SP). □

Let $N \subseteq B(K)$ be a von Neumann algebra.

Definition 3.10. We say that a von Neumann algebra N satisfies the weak similarity problem, in short (WSP), if for any bounded and weak*-continuous representation $\pi : N \to B(H)$, $S^{-1}\pi(\cdot)S$ is a *-homomorphism for some invertible operator S.

It is clear by definition that N satisfies the (WSP) if N satisfies the (SP).

Theorem 3.11. *Let G be a locally compact group. Consider the following assertions:*

(1) *G is unitarizable.*
(2) *$\mathcal{A}(G)$ has the (WSP).*
(3) *G is C^*-unitarizable.*
(4) *$C^*_{\mathrm{red}}(G)$ satisfies the (SP).*
(5) *vN(G) satisfies the (WSP)*

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Proof. (1) $\Rightarrow$ (2): Let $\pi : \mathcal{A}(G) \to B(H)$ be a w*-continuous bounded representation of $\mathcal{A}(G)$. Then it restriction $\pi|_G$ is unitarizable, thus there is an invertible operator $S \in B(H)$ such that $S^{-1}\pi(\cdot)S$ is a *-homomorphism on Alg(G) (the algebra generated by $\hat{G} = \{\delta_g\}$ which is a free set [5]). Since $S^{-1}\pi(\cdot)S$ is w*-continuous, the (unique) extension $S^{-1}\pi(\cdot)S$ on $\mathcal{A}(G)$ is a *-homomorphism.

(2) $\Rightarrow$ (3): Let $\pi : C^*(G) \to B(H)$ be a bounded homomorphism. Consider $\tilde{\pi} : \mathcal{A}(G) \to B(H)$ the w*-continuous and bounded extension of π to $\mathcal{A}(G)$. Since $\mathcal{A}(G)$ has the (WSP), there is an invertible operator S such that $S^{-1}\tilde{\pi}(\cdot)S$ is a *-homomorphism. This completes the proof because $S^{-1}\pi(\cdot)S = S^{-1}\tilde{\pi}|_{C^*(G)}(\cdot)S$.

(3) $\Rightarrow$ (4): the left regular representation naturally gives rise to a surjective *-homomorphism $\lambda : C^*(G) \to C^*_{\mathrm{red}}(G)$. The proof easily follows from this, using the definition.

(4) $\Rightarrow$ (5): We first note that $\mathrm{vN}(G) = \overline{C^*_{\mathrm{red}}(G)}^{w^*}$. Let $\pi : \mathrm{vN}(G) \to B(H)$ be a bounded and w*- continuous homomorphism. Then the restriction of π to $C^*_{\mathrm{red}}(G)$ is similar to a *-homomorphism, and because it is w*-continuous, we conclude that π is similar to a *-homomorphism on vN(G), as required. □

As an immediate consequence of this Theorem 3.11, we conclude that if a locally compact group G is unitarizable then G is C^*-unitarizable. The converse of this is not true because of the example $G = \mathrm{SL}(2,\mathbb{R})$. Indeed this group is not unitarizable [7] but it is C^*-unitarizable because the group C^*-algebra $C^*(\mathrm{SL}(2,\mathbb{R}))$ is nuclear [3].

For a discrete group G we may say that G is C^*-nuclear if $C^*(G)$ is a nuclear C^*-algebra (i.e. $C^*(G) \otimes_{max} B = C^*(G) \otimes_{min} B$ for every C^*-algebra B where 'max' denotes the maximal tensor product of C^*-algebras and 'min' denotes the minimal one, see e.g. [2]). From Theorem 3.11 and well known results, we naturally arrive at the following diagram:

$$\begin{array}{ccc} C^*\text{-nuclear} & \Rightarrow & C^*\text{-unitarizable} \\ \Updownarrow & & \Uparrow \\ \text{amenable} & \Rightarrow & \text{unitarizable} \end{array} \tag{3.1}$$

Let N be a type II_1 factor with tr its unique normalized trace (so that N is infinite-dimensional, $tr(I) = 1$ and for any given $s \in [0,1]$ there exists a projection $p \in N$ such that $tr(p) = s$). The von Neumann algebra N has property Γ if there exists a sequence of unitaries $(u_n) \subseteq N$ such that $tr(u_n) = 0$ for all n and $||u_n a - a u_n||_{tr} \to 0$ for all $a \in N$. Here $|| \cdot ||_{tr}$ denotes the L^2 norm given by $||x||_{tr} = \sqrt{tr(xx^*)}$. A deep theorem of Christensen [2] shows that a II_1 factor with property Γ satisfies Kadison (SP). In particular if for a group G, vN(G) has property Γ so does vN($G \times \mathbb{F}_2$) where $\mathbb{F}_2$ is the 2-generator free group (note that vN($G \times \mathbb{F}_2$) = vN(G) $\otimes$ vN($\mathbb{F}_2$)). Therefore vN($G \times \mathbb{F}_2$) satisfies Kadison (SP), nevertheless $G \times \mathbb{F}_2$ is non-unitarizable because it contains $\{e\} \times \mathbb{F}_2$ which is non-unitarizable. So we can conclude that the assertion "vN(G) has (SP) implies G unitarizable" does not hold in general. Also the proof of non-unitarizability of a group is traditional obtained by finding a copy of a free group in the given group.

So for discrete groups G, we may raise the following problems:

Problem 3.12. (i) If G does not contain a copy of any free group and vN(G) satisfies the (WSP), then does it follow that G is unitarizable?
(ii) Let G be a C^*-unitarizable group. Does it follow that G is C^*-nuclear?

If Problem 3.12 (i) holds then we may exhibit many counterexamples to Kadison problem (1.4), for example, Burnside groups (see [14] for the non-unitarizability of some of the Burnside groups).

The Thompson group F is the group with the following infinite presentation:

$$F = \langle x_0, x_1, \cdots, x_i, \cdots \mid x_j x_i = x_i x_{j+1},\ i < j \rangle. \tag{3.2}$$

Jolissaint [13] proved that the Thompson group F is an ICC group and also vN(F) has property Γ therefore vN(F) satisfies the (SP) by Christensen [2] and thus F is unitarizable if Problem 3.12 (i) holds. This would lead to the first example of a non-amenable and unitarizable group (see Dixmier problem (1.3)). We then can actually yield many more such examples: any group $G := F \times \Gamma$ with Γ discrete and amenable will do the job. This is because $F \times \Gamma$ will be non-amenable (notice that the quotient $F \times G/G \simeq F$ will be non-amenable, and $F \times G$ will be unitarizable).

Theorem 3.11 and diagram (3.1) above we can conclude that if Problem 3.12 (ii) holds true then the Dixmier problem is true (for discrete groups).

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