Ann. Funct. Anal. 2 (2011), no. 2, 101–113
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

INVARIANT APPROXIMATION RESULTS IN CONE METRIC SPACES

MUJAHID ABBAS[1] AND PASQUALE VETRO[2]*

Communicated by G. L. Acedo

ABSTRACT. Some sufficient conditions for the existence of fixed point of mappings satisfying generalized weak contractive conditions is obtained. A fixed point theorem for nonexpansive mappings is also obtained. As an application, some invariant approximation results are derived in cone metric spaces.

1. INTRODUCTION AND PRELIMINARIES

Recently, Huang and Zhang [9] introduced the concept of a cone metric space, replacing the set of positive real numbers by an ordered Banach space. They obtained some fixed point theorems in cone metric spaces using the normality of cone which induces an order in Banach spaces. Rezapour and Hamlbarani [15] showed the existence of a non normal cone metric space and obtained some fixed point results in cone metric spaces (see also, [3, 10], [16] and [17]). Subsequently, Abbas and Rhoades [1], Abbas and Jungck [2] and Vetro [18] studied common fixed point theorems in cone metric spaces (see also, [3], [6]). The aim of this paper is to establish a generalized Banach contraction principle in cone metric spaces. Study of sufficient conditions for the existence of fixed point of nonexpansive mappings in cone metric spaces is also initiated. It is worth mentioning that our results do not require the assumption of normal cone. Our results extend and unify various comparable results in the literature ([4, 5, 8, 11, 12, 13, 14]).
Let E be a real Banach space. A subset P of E is called a *cone* if and only if:

(a) P is closed, non empty and $P \neq \{0\}$;
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$;

Date: Received: 23 February 2011; Accepted: 23 September 2011.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25, 54C60, 46B40.
Key words and phrases. Fixed points, orbitally continuous, invariant approximation, cone metric spaces, non normal cone.

(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$, while $x \ll y$ stands for $y - x \in intP$ (interior of P). A cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P. In [15], it was proved that there is no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$. A cone P is called regular if every increasing sequence which is order bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x$ for some $x \in E$, then there exists some $z \in E$ such that $\lim_{n\to\infty} \|x_n - z\| = 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is order bounded from below is convergent. Note that every regular cone is normal. For examples of a regular cone metric space we refer to [10].

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a *cone metric space*. The concept of a cone metric space is more general than that of a metric space.

Definition 1.2. Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. For every $c \in E$ with $0 \ll c$, we say that $\{x_n\}$ is

(i) a *Cauchy* sequence if there is an $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $d(x_n, x_m) \ll c$.
(ii) a *convergent* sequence if there is an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $d(x_n, x) \ll c$ for some x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$. Further, if P is a normal cone then $x_n \to x$ if and only if $d(x_n, x) \to 0$ and $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$, e.g., see ([9], Lemma 3 and Lemma 4). A set A in a cone metric space X is closed if for every sequence $\{x_n\}$ in A which converges to some x in X implies that $x \in A$. For every $A \subset X$, we define

$$\overline{A} := \{x \in X : \exists \, \{x_n\} \subset A \text{ with } x_n \to x\}.$$

A set A in a cone metric space X is compact if for every sequence $\{x_n\}$ in A there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $x \in A$ such that $x_{n_k} \to x$.

Remark 1.3. If E is a real Banach space with a cone P and if $a \leq ha$ where $a \in P$ and $h \in (0, 1)$, then $a = 0$.

Definition 1.4. Let X be a cone metric space. A map $f : X \to X$ is called:

(i) contraction if there exists $h \in [0,1)$ such that
$$d(fx, fy) \leq h\, d(x, y) \qquad \text{for all } x, y \in X.$$
(ii) Banach operator if there exists $k \in [0,1)$ such that
$$d(fx, f^2x) \leq k\, d(x, fx) \quad \text{for all } x \in X.$$
(iii) $g-$nonexpansive if $d(fx, fy) \leq d(gx, gy)$, for each $x, y \in X$, where g is a self map on X.

(iv) (ψ, ϕ)-contraction if there exist two mappings $\psi, \phi : P \to P$ continuous, with $\psi(t) = \phi(t) = 0$ if and only if $t = 0$, and ψ monotone increasing such that
$$\psi\left(d(fx, fy)\right) \leq \psi\left(d\left(x, y\right)\right) - \phi\left(d\left(x, y\right)\right) \tag{1.1}$$
for each $x, y \in X$.

Definition 1.5. Let f be a selfmap on a cone metric space X. An orbit of f at the point x in X is the set
$$O(x, f) = \{x, fx, f^2x, ...\}.$$
The orbit $O(x, f)$ of f at the point x is said to complete if any Cauchy subsequence $\{f^{n_i}x\}$ in orbit $O(x, f)$ converges in X. f is said to be f-orbitally complete if the orbit $O(x, f)$ is complete for every $x \in X$. We denote with $LO(x, f)$ the set of points z of X that are the limit of a subsequence $\{f^{n_i}x\}$ in orbit $O(x, f)$.
f is said to be continuous at $p \in X$ with respect to the orbit $O(x, f)$ if $f^{n_i}x \to p$ implies $f(f^{n_i}x) \to fp$ as $i \to \infty$. f is said to be orbitally continuous at $p \in X$ if $f^{n_i}x \to p$ implies $f(f^{n_i}x) \to fp$ as $i \to \infty$ for every $x \in X$.

Definition 1.6. Let C be a nonempty subset of a cone metric space X. For $x \in X$, $z_0 \in C$ is a best approximation of x whenever
$$d(x, z_0) \leq d(x, z)$$
for all $z \in C$ (see also [17]). The set of all best approximations of x in C is denoted by $P_C(x)$.

Definition 1.7. Let X be a cone metric space, C be a subset of X and $F = \{f_\alpha : \alpha \in C\}$ a family of functions from $[0,1]$ into C, having property that $f_\alpha(1) = \alpha$, for each $\alpha \in C$. Such a family F is said to be contractive if there exits a mapping $\varphi : (0,1) \to (0,1)$ such that for all α,$\beta \in C$, $t \in (0,1)$, we have
$$d(f_\alpha(t), f_\beta(t)) \leq \varphi(t) d(\alpha, \beta).$$
A family F is said to be jointly continuous if $\alpha \to \alpha_0$ in C and $t \to t_0$ in $(0,1)$ imply $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in C.

Lemma 1.8. *Let (X, d) be a cone metric space with respect to a cone P. Let $\{x_n\}$ be a sequence in X and $\{a_n\}$ be a sequence in P converging to 0. If $d(x_n, x_m) \leq a_n$ for every $n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Fix $0 \ll c$ and choose $I(0, \delta) = \{x \in E : \|x\| < \delta\}$ such that $c + I(0, \delta) \subset intP$. Since $a_n \to 0$, there exists $n_0 \in \mathbb{N}$ be such that $a_n \in I(0, \delta)$ for every $n \geq n_0$. From $c - a_n \in intP$, we deduce $d(x_n, x_m) \leq a_n \ll c$ for every $m, n \geq n_0$ and hence $\{x_n\}$ is a Cauchy sequence. $\square$

Lemma 1.9. *Let (X, d) be a cone metric space with respect to a cone P. Let $\{x_n\}$ be a sequence in X. If for every real number $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $\|d(x_n, x_m)\| < \varepsilon$ for all $m, n \geq n(\varepsilon)$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Fix $0 \ll c$ and choose $\varepsilon > 0$ such that $c + I(0, \varepsilon) \subset intP$. Let $n(\varepsilon) \in \mathbb{N}$ be such that $\|d(x_n, x_m)\| < \varepsilon$ for all $m, n \geq n(\varepsilon)$, then $c - d(x_n, x_m) \in intP$ and so $d(x_n, x_m) \ll c$ for all $m, n \geq n(\varepsilon)$. □

2. Fixed Point Theorems

In this section, a generalized Banach contraction principle in cone metric space is obtained.

Following theorem extends Theorem 2 of [13] to cone metric spaces, improve Theorem 2.3 of [15] and consequently correspondent results mentioned therein.

Let X be a cone metric space and f be a selfmap on X, for every $x, y \in X$ we define

$$M(x,y) := \{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)+d(y,fy)}{2}\}.$$

Theorem 2.1. *Let X be a cone metric space and f be a selfmap on X. Assume that there exist a point $u \in X$ and $k \in [0, 1)$ such that*

$$d(fx, fy) \leq kw, \tag{2.1}$$

where $w \in M(x, y)$, holds for any x, $y = fx$ in $O(u, f)$. Then $\{f^n u\}$ converges to some v in X provided $O(u, f)$ is complete.

Further, if f is continuous at v with respect to the orbit $O(u, f)$ or if (2.1) holds for any x in $O(u, f)$ and $y \in LO(u, f)$, then v is a fixed point of f. v is a unique fixed point for f if $\cap_{n=1}^{+\infty} f^n(X)$ is a singleton set.

Proof. We show that $\{f^n u\}$ is a Cauchy sequence in $O(u, f)$. Let $d_n = d(f^{n-1}u, f^n u)$, we have

$$M(f^{n-1}u, f^n u) := \{d_n, d_{n+1}, \frac{d(f^{n-1}u, f^{n+1}u)}{2}, \frac{d_n + d_{n+1}}{2}\}.$$

From (2.1), for $x = f^{n-1}u$, if $w = d_{n+1}$, we deduce that $f^n u = f^{n+1}u$ and so $f^m u = f^n u$ for every $m \geq n$. This implies that $\{f^n u\}$ is a Cauchy sequence in $O(u, f)$.

If $w = \frac{d(f^{n-1}u, f^{n+1}u)}{2}$ or $w = (d_n + d_{n+1})/2$, by(2.1) we deduce that

$$d_{n+1} \leq k\frac{d(f^{n-1}u, f^{n+1}u)}{2} \leq k\frac{d_n + d_{n+1}}{2},$$

and we get

$$d_{n+1} \leq \frac{k}{2-k} d_n.$$

If $w = d_n$, then

$$d_{n+1} \leq k d_n.$$

This implies that for every $n \in \mathbb{N}$

$$d_{n+1} \leq \quad kd_n \leq \cdots \leq \quad k^n d_1.$$

Moreover for any $n, m \in \mathbb{N}$, we deduce

$$\begin{aligned} d(f^n u, f^{n+m} u) &\leq d_{n+1} + \cdots + d_{n+m} \\ &\leq (1 + k + k^2 + \cdots + k^{m-1}) d_{n+1} \\ &\leq \frac{1}{1-k} d_{n+1} \leq \frac{k^n}{1-k} d_1. \end{aligned}$$

Since $k^n(1-k)^{-1}d_1 \to 0$ in P as $n \to \infty$, by Lemma 1.8 we deduce that $\{f^n u\}$ is a Cauchy sequence in $O(u, f)$ and hence converges to a point $v \in LO(u, f)$.

Suppose f is continuous at v with respect to the orbit $O(u, f)$. Then $f^n u \to v$ implies $f^{n+1}u \to fv$ as $n \to \infty$. This shows that $v = fv$.

Now assume that (2.1) holds for any $x \in O(u, f)$ and $y \in LO(u, f)$. Fix $0 \ll c$ and choose $n_m \in \mathbb{N}$ such that $d(v, f^{n+n_m}u) \ll c/m$, from

$$d(f^n u, v) \leq d(f^n u, f^{n+n_m} u) + d(f^{n+n_m} u, v) \leq \frac{k^n}{1-k} d_1 + \frac{c}{m},$$

as $m \to \infty$ we deduce that

$$d(f^n u, v) \leq \frac{k^n}{1-k} d(u, fu) \qquad \text{for } n = 1, 2, \ldots .$$

Now, we prove that

$$d(f^{n+1}u, fv) \leq \frac{k^{n+1}}{(1-k)^2} d(u, fu) \qquad \text{for } n = 1, 2, \ldots . \tag{2.2}$$

We have

$$\begin{aligned} M(f^n u, v) = &\{d(f^n u, v), d_{n+1}, d(v, fv), \\ &\frac{d(f^n u, fv) + d(v, f^{n+1}u)}{2}, \frac{d_{n+1} + d(v, fv)}{2}\}. \end{aligned}$$

From (2.1), for $x = f^n u$ and $y = v$, if $w = d(f^n u, v)$, then (2.2) holds. If $w = d_{n+1}$, then $d(f^{n+1}u, fv) \leq k\, d_{n+1} \leq k^{n+1} d_1$ and (2.2) holds.
If $w = d(v, fv)$, then

$$\begin{aligned} d(f^{n+1}u, fv) &\leq kd(v, fv) \leq kd(f^{n+1}u, fv) + kd(f^{n+1}u, v) \\ &\leq kd(f^{n+1}u, fv) + \frac{k^{n+2}}{1-k} d_1, \end{aligned}$$

and so

$$d(f^{n+1}u, fv) \leq \frac{k^{n+2}}{(1-k)^2} d_1$$

and (2.2) holds.
If $w = \frac{d(f^n u, fv)+d(v, f^{n+1}u)}{2}$, then

$$\begin{aligned} 2d(f^{n+1}u, fv) &\leq k[d(f^n u, fv) + d(v, f^{n+1}u)] \\ &\leq kd(f^{n+1}u, fv) + kd(f^{n+1}u, f^n u) + kd(v, f^{n+1}u) \\ &\leq kd(f^{n+1}u, fv) + \frac{k^{n+1}}{1-k}d_1, \end{aligned}$$

and (2.2) holds.
If $w = \frac{d_{n+1}+d(v, fv)}{2}$, then

$$\begin{aligned} 2d(f^{n+1}u, fv) &\leq k[d_{n+1} + d(v, fv)] \\ &\leq k^{n+1}d_1 + kd(f^{n+1}u, fv) + kd(v, f^{n+1}u) \\ &\leq kd(f^{n+1}u, fv) + \frac{k^{n+1}}{1-k}d_1, \end{aligned}$$

and (2.2) holds.

Now,

$$\begin{aligned} d(fv, v) &\leq d(fv, f^{n+1}u) + d(f^{n+1}u, v) \\ &\leq (2-k)\frac{k^{n+1}}{(1-k)^2}d(u, fu) \to 0, \end{aligned}$$

implies $d(v, fv) = 0$. Therefore $fv = v$.
If $\cap_{n=1}^{+\infty} f^n(X) = \{z\}$, since v is a fixed point for f, we deduce that $v = z$ and so f has a unique fixed point. Hence the result follows. □

Example 2.2. Let $X = [0, 1]$, $E = \mathbb{R}^2$ be a Banach space and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $d : X \times X \to E$ be defined by

$$d(x, y) = (|x - y|, h|x - y|)$$

where $h \geq 0$. Define $f : X \to X$ as $fx = 1 - x$. Take $u = 1/2$, then $O(u, f) = \{u\}$ and $d(fx, fy) \leq kd(x, y)$ for each x, $y = fx \in O(u, f)$, where $k \in [0, 1)$. Obviously $\{f^n u\}$ converges to u and f is continuous at u with respect to the orbit $O(u, f)$. Moreover, u is a fixed point of f.
Note that Theorem 1 of [9] is not applicable in this case.

Corollary 2.3. *Let X be a cone metric space and C be a subset of X. If $f : C \to C$ is a continuous Banach operator. Then f has a fixed point in C provided $f(C)$ is a complete subset of C.*

Proof. It follows by Theorem 2.1. □

Example 2.4. Let $a \in (1/2, (\sqrt{5} - 1)/2)$ and $X = [0, a]$, $E = \mathbb{R}^2$ be a Banach space and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $d : X \times X \to E$ be defined by

$$d(x, y) = (|x - y|, h|x - y|)$$

where $h \geq 0$. Define $f : X \to X$ as $fx = x^2$. Then for each $x \in X$

$$\begin{aligned} d(fx, f^2x) &= (x^2 - x^4, h(x^2 - x^4)) \\ &\leq a(1+a)(x - x^2, h(x - x^2)) \\ &= a(1+a)d(x, fx). \end{aligned}$$

Since $a(1+a) < 1$, the map f is a continuous Banach operator. Moreover, $f(0) = 0$. We note that f is not nonexpansive.

Corollary 2.5. *Let X be a cone metric space and f be a selfmap on X. Assume that there exists a point $u \in X$ such that*

$$d(fx, fy) \leq kd(x, y) + ad(x, fx) + bd(y, fy) + cd(y, fx) + ed(x, fy) \tag{2.3}$$

holds for any x, $y = fx$ in $O(u, f)$, where k, a, b, c, e are non negative real numbers satisfying $c > e$ and $k + a + b + c + e = 1$ or $c \geq e$ and $k + a + b + c + e < 1$. Then $\{f^n u\}$ converges to some v in X provided $O(u, f)$ is complete.

Further, if f is continuous at v with respect to the orbit $O(u, f)$ or if (2.3) holds for any x in $O(u, f)$ and $y \in LO(u, f)$ and $k + a > 0$, then v is a fixed point of f. v is a unique fixed point for f if $\cap_{n=1}^{+\infty} f^n(X)$ is a singleton set.

Proof. In (2.3) set $y = fx$ with $x \in O(u, f)$ to deduce

$$d(fx, f^2x) \leq (k + a + e)d(x, fx) + (b + e)d(fx, f^2x),$$

which implies

$$d(fx, f^2x) \leq hd(x, fx)$$

where $h = (k + a + e)/(1 - b - e) < 1$. The result follows by Theorem 2.1. □

Corollary 2.5 improves comparable results in the literature. Following theorem extends Theorem 2.1 of [8] and Theorem 1.4 of [14].

Theorem 2.6. *Let (X, d) be a complete regular cone metric space with respect to cone P and $f : X \to X$ be a (ψ, ϕ)-contraction. Assume that for each $0 \ll c$ we have $d(x, y) \ll c$ or $d(x, y) \geq c$ for all $x, y \in X$. Then f has a unique fixed point.*

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ such that $x_n = fx_{n-1}$, then we have

$$\psi\left(d(x_n, x_{n+1})\right) \leq \psi\left(d\left(x_{n-1}, x_n\right)\right) - \phi\left(d\left(x_{n-1}, x_n\right)\right).$$

By monotone property of ψ we obtain

$$d(x_n, x_{n+1}) \leq d\left(x_{n-1}, x_n\right).$$

It follows that the sequence $\{d\left(x_n, x_{n+1}\right)\}$ is monotone decreasing, then there exists z in E such that $d(x_n, x_{n+1}) \to z$ as $n \to \infty$, which further implies that $\psi(z) \leq \psi(z) - \phi(z)$, consequently $z = 0$. We now prove that $\{x_n\}$ is a Cauchy sequence. If not, there exists $c \gg 0$ in E for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d\left(x_{m_k}, x_{n_k}\right) \geq c. \tag{2.4}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ such that $d(x_{m_k}, x_{n_k-1}) \ll c$ and $d(x_{m_k}, x_{n_k}) \geq c$. From

$$\begin{aligned} c &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) < c + d(x_{n_k-1}, x_{n_k}), \end{aligned}$$

it follows

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = c.$$

Now

$$\begin{aligned} & d(x_{n_k}, x_{m_k}) - d(x_{n_k-1}, x_{n_k}) - d(x_{m_k}, x_{m_k-1}) \\ \leq\ & d(x_{n_k-1}, x_{m_k-1}) \\ \leq\ & d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}), \end{aligned}$$

gives

$$d(x_{n_k-1}, x_{m_k-1}) \to c \quad \text{as } k \to \infty.$$

Setting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (1.1) and using (2.4), we obtain

$$\begin{aligned} \psi(c) &\leq \psi(d(x_{m_k}, x_{n_k})) \\ &\leq \psi(d(x_{m_k-1}, x_{n_k-1})) - \phi(d(x_{m_k-1}, x_{n_k-1})). \end{aligned}$$

Therefore

$$\psi(c) \leq \psi(c) - \phi(c),$$

a contradiction which shows that $\{x_n\}$ is a Cauchy sequence in X, and so $x_n \to u$, for some u in X. Take $x = x_{n-1}$ and $y = u$ in (1.1), then

$$\psi(d(x_n, fu)) \leq \psi(d(x_{n-1}, u)) - \phi(d(x_{n-1}, u)),$$

and consequently

$$\psi(d(u, fu)) \leq \psi(0) - \phi(0) = 0.$$

This implies that $\psi(d(u, fu)) = 0$, and $fu = u$. For uniqueness of the fixed point, suppose that v is another fixed point of f, then by taking $x = u$, and $y = v$ in (1.1), we have

$$\psi(d(u, v)) = \psi(d(fu, fv)) \leq \psi(d(u, v)) - \phi(d(u, v))$$

which is possible only if $u = v$. □

Remark 2.7. Theorem 2.6 holds, also, if we assume that for each $0 \ll c$ we have $d(x, y) < c$ or $d(x, y) \geq c$ for all $x, y \in X$. In fact, if for every $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < c$ for all $m, n \geq n_0$, then $d(x_m, x_n) \ll 2c$ and so $\{x_n\}$ is a Cauchy sequence.

Theorem 2.8. *Let (X, d) be a complete regular cone metric space with respect to cone P and $f : X \to X$ be a (ψ, ϕ)-contraction. Assume that every bounded sequence $\{a_n\} \subset P$ has a subsequence convergent to a point of P. Then f has a unique fixed point.*

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ such that $x_n = fx_{n-1}$. As in the proof of Theorem 2.6 we deduce that $d(x_n, x_{n+1}) \to 0$. We now prove that $\{x_n\}$ is a Cauchy sequence, that is, $d(x_n, x_m) \to 0$ as $n, m \to \infty$. If not, by Lemma 1.9, there exists a real number $\varepsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\|d(x_{m_k}, x_{n_k})\| \geq \varepsilon.$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ such that $\|d(x_{m_k}, x_{n_k-1})\| < \varepsilon$ and $\|d(x_{m_k}, x_{n_k})\| \geq \varepsilon$. Let K be the normal constant of P. From

$$\begin{aligned}\varepsilon &\leq \|d(x_{m_k}, x_{n_k})\| \leq K\|d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})\| \\ &\leq K\|d(x_{m_k}, x_{n_k-1})\| + K\|d(x_{n_k-1}, x_{n_k})\| \\ &< K\varepsilon + K\|d(x_{n_k-1}, x_{n_k})\|,\end{aligned}$$

it follows

$$\limsup_{k\to\infty} \|d(x_{m_k}, x_{n_k})\| \leq K\varepsilon.$$

It implies that the sequence $\{d(x_{m_k}, x_{n_k})\}$ is bounded. Consequently, there exists a subsequence of $\{d(x_{m_k}, x_{n_k})\}$ that converges to a point $c \in P \setminus \{0\}$ and that we denote also by $\{d(x_{m_k}, x_{n_k})\}$.

Now

$$\begin{aligned}& d(x_{n_k}, x_{m_k}) - d(x_{n_k-1}, x_{n_k}) - d(x_{m_k}, x_{m_k-1}) \\ \leq\ & d(x_{n_k-1}, x_{m_k-1}) \\ \leq\ & d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}),\end{aligned}$$

gives

$$d(x_{n_k-1}, x_{m_k-1}) \to c \quad \text{as } k \to \infty.$$

From

$$\psi(d(x_{m_k}, x_{n_k})) \leq \psi(d(x_{m_k-1}, x_{n_k-1})) - \phi(d(x_{m_k-1}, x_{n_k-1})),$$

as $k \to \infty$, we obtain

$$\psi(c) \leq \psi(c) - \phi(c),$$

a contradiction which shows that $\{x_n\}$ is a Cauchy sequence in X, and so $x_n \to u$, for some u in X. Proceeding as in Theorem 2.6 we deduce that u is the unique fixed point for f. □

Example 2.9. Let $X = [0, 1]$, $E = \mathbb{R}^2$ be a real Banach space and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $d : X \times X \to E$ be defined by

$$d(x, y) = (|x - y|,\ h|x - y|)$$

where $h > 0$. Define $f : X \to X$ as

$$fx = \frac{x}{1 + x}$$

for every $x \in X$. Let $\psi, \phi : P \to P$, such that

$$\psi(x, y) = (x, y) \quad \text{and} \quad \phi(x, h\,y) = (\frac{x^2}{1 + x}, h\frac{y^2}{1 + y}) \text{ for all } (x, y) \in P.$$

Clearly ψ, ϕ both are continuous and monotone increasing mappings with $\psi(x, y) = \phi(x, y) = (0, 0)$ if and only if $(x, y) = (0, 0)$.
For each $x, y \in X$

$$\begin{aligned} d(fx, fy) &= (\left|\frac{x}{1+x} - \frac{y}{1+y}\right|, h\left|\frac{x}{1+x} - \frac{y}{1+y}\right|) \\ &= (\frac{|x-y|}{(1+x)(1+y)}, h\frac{|x-y|}{(1+x)(1+y)}) \\ &\leq (\frac{|x-y|}{1+|x-y|}, h\frac{|x-y|}{1+|x-y|}) \\ &= (|x-y|, h|x-y|) - (\frac{|x-y|^2}{1+|x-y|}, h\frac{|x-y|^2}{1+|x-y|}). \end{aligned}$$

Now

$$\begin{aligned} \psi(d(fx, fy)) &= d(fx, fy) \\ &\leq \psi(d(x, y)) - \phi(d(x, y)). \end{aligned}$$

Hence (1.1) is satisfied and by Theorem 2.8 f has a unique fixed point $x = 0$.

As an application of Theorem 2.8, we prove a result of existence of a solution of a Urysohn integral equation.
Let $X = C([a, b], \mathbb{R}^n)$, space of all continuous functions from $[a, b] \subset \mathbb{R}$ to $\mathbb{R}^n$, $P = \{(u, v) \in \mathbb{R}^2 : u, v \geq 0\}$ and

$$d(x, y) = (\sup_{t \in [a,b]} \|x(t) - y(t)\|, h \sup_{t \in [a,b]} \|x(t) - y(t)\|)$$

for every $x, y \in X$, where $h \geq 0$ is a constant. (X, d) is a complete cone metric space. Consider the Urysohn integral equation

$$x(t) = \int_a^b K(t, s, x(s))ds + g(t) \tag{2.5}$$

where $t, s \in [a, b]$, $K : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}^n$ are continuous.
Suppose that
(i) there exists a continuous function $p : [a, b] \times [a, b] \to \mathbb{R}_+$ and a (ψ, ϕ)-contraction $\sigma : P \to P$ such that

$$\begin{aligned} &(\|K(t, s, u) - K(t, s, v)\|, h\|K(t, s, u) - K(t, s, v)\|) \\ \leq\ & p(t, s)\sigma(d(u, v)) \end{aligned}$$

for $t, s \in [a, b]$, and $u, v \in X$,
(ii) $\sup_{t \in [a,b]} \int_a^b p(t, s)ds \leq 1$,
then the integral equation (2.5) has a unique solution x^* in $C([a, b], \mathbb{R}^n)$.

Proof. Define $f : (C[a,b], \mathbb{R}^n) \to C([a,b], \mathbb{R}^n)$ by

$$fx(t) = \int_a^b K(t,s,x(s))ds + g(t),\ t \in [a,b]$$

and put $H_t(x(s), y(s)) = K(t,s,x(s)) - K(t,s,y(s))$. For every $x, y \in X$, we have

$$\begin{aligned}
&\psi(d(fx, fy)) \\
= \ &\psi((\sup_{t\in[a,b]} \| \int_a^b H_t(x(s),y(s))ds\|, h \sup_{t\in[a,b]} \| \int_a^b H_t(x(s),y(s))ds\|)) \\
\le \ &\psi((\sup_{t\in[a,b]} \int_a^b \|H_t(x(s),y(s))\|ds, h \sup_{t\in[a,b]} \int_a^b \|H_t(x(s),y(s))\|ds)) \\
\le \ &\psi(\sup_{t\in[a,b]} \int_a^b p(t,s)\sigma(d(x,y))ds) \\
= \ &\psi(\sigma(d(x,y)) \sup_{t\in[a,b]} \int_a^b p(t,s)ds) \\
\le \ &\psi(\sigma(d(x,y))) \\
\le \ &\psi(d(x,y)) - \phi(d(x,y)).
\end{aligned}$$

That is f is a (ψ, ϕ)-contraction and conclusion follows from Theorem 2.8. □

3. Invariant Approximation Results

In this section, existence of fixed points of nonexpansive mappings defined on cone metric spaces is proved and as an application invariant approximation results are obtained.

Theorem 3.1. *Let (X,d) be a cone metric space and C be a subset of X. Let $g : C \to C$ be a nonexpansive mapping and $F = \{f_\alpha : \alpha \in \overline{g(C)}\}$ a family of contractive jointly continuous maps from $[0,1]$ into $g(C)$. If $\overline{g(C)}$ is a compact subset of C. Then g has a fixed point in C.*

Proof. For each $n \in N$, let $\lambda_n = \dfrac{n}{n+1}$. Define $g_n : C \to C$ such that $g_n x = f_{gx}(\lambda_n)$. Now,

$$\begin{aligned}
d(g_n x, g_n^2 x) &= d(f_{gx}(\lambda_n), f_{g(f_{gx}(\lambda_n))}(\lambda_n)) \\
&\le \varphi(\lambda_n) d(gx, g(f_{gx}(\lambda_n))) \\
&\le \varphi(\lambda_n) d(x, g_n x)
\end{aligned}$$

implies that each g_n is a continuous Banach operator. Moreover the orbits $O(u, g_n)$ are complete for all $u \in C$ and $n \in \mathbb{N}$, since $O(u, g_n) \subset \overline{g(C)}$. Using Corollary 2.3, we obtain a sequence $\{p_n\}$ in $g(C)$ such that $g_n(p_n) = p_n$ for each n, so there exists a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to q \in C$ and $gp_{n_k} \to gq$. Since

$$\begin{aligned} q &= \lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} g_{n_k}(p_{n_k}) \\ &= \lim_{k\to\infty} f_{gp_{n_k}}(\lambda_{n_k}) = f_{gq}(1) = gq, \end{aligned}$$

it follows the result. □

Note that above Theorem extends results of [7] to cone metric spaces.

Theorem 3.2. *Let X be a cone metric space and $g : X \to X$ be a map. Assume that C and $\{x\}$ are g-invariant subsets of X. If $P_C(x)$ is a nonempty compact set with a contractive jointly continuous family of mappings from $[0,1]$ into $P_C(x)$ and g is nonexpansive on $P_C(x) \cup \{x\}$, then $P_C(x)$ contains a g-invariant point.*

Proof. Note that $g(P_C(x)) \subseteq P_C(x)$, in fact if $y \in P_C(x)$ then

$$d(x, gy) = d(gx, gy) \leq d(x, y),$$

and so $gy \in P_C(x)$. Let $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lambda_n \to 1$. Define $g_n : P_C(x) \to P_C(x)$ such that $g_n z = f_{gz}(\lambda_n)$. For $y, z \in P_C(x)$, we have

$$\begin{aligned} d(g_n y, g_n z) &= d(f_{gy}(\lambda_n), f_{gz}(\lambda_n)) \\ &\leq \varphi(\lambda_n) d(gy, gz) \\ &\leq \varphi(\lambda_n) d(y, z). \end{aligned}$$

Thus for each n, g_n is a contraction mapping on a complete space $P_C(x)$. We obtain a sequence $\{x_n\}$ in $P_C(x)$ such that $g_n(x_n) = x_n$. Since $P_C(x)$ is compact there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0 \in P_C(x)$. Now, we show that $gx_0 = x_0$. For this, consider

$$\begin{aligned} x_0 &= \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} g_{n_k}(x_{n_k}) \\ &= \lim_{k\to\infty} f_{gx_{n_k}}(\lambda_{n_k}) = f_{gx_0}(1) = gx_0. \end{aligned}$$

Hence $x_0 \in P_C(x)$ is a g-invariant point. □

Theorem 3.3. *Let X be a cone metric space, f and g commuting mappings on X. The following are equivalent:*

(i) *x_0 is a unique invariant point of fg,*
(ii) *x_0 is a unique invariant point of f and g,*
(iii) *x_0 is a unique invariant point of f or g.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Suppose that x_0 is a unique invariant point for fg. Then $fgx_0 = x_0$, which implies that $gfgx_0 = gx_0$. Since f and g commute, $gfgx_0 = fggx_0 = gx_0$ and gx_0 is an invariant point of fg which implies $gx_0 = x_0$. The same holds for f.

(ii) $\Rightarrow$ *(i)* and *(ii)* $\Rightarrow$ *(iii)*. Obvious.

(iii) $\Rightarrow$ *(ii)*. Suppose that x_0 is a unique invariant point for f. Then $gx_0 = gfx_0$. Since f and g commute, $gx_0 = gfx_0 = fgx_0$, and gx_0 is now an invariant point for f. This implies that $gx_0 = x_0$. □

Acknowledgement. The authors thank the referee for appreciation, valuable comments and suggestions. The second author is supported by Università degli Studi di Palermo (Local University Project R.S. ex 60%).

References

1. M. Abbas and B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. **22** (2009), 511–515.
2. M. Abbas and G. Jungck, *Common fixed point results of noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 418–420.
3. M. Arshad, A. Azam and P. Vetro, *Some common fixed point results in cone metric spaces*, Fixed Point Theory Appl. **2009**, Article ID 493965, 11 pages.
4. C. Di Bari and P. Vetro, *Weakly φ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **58** (2009), 125–132.
5. C. Di Bari and P. Vetro, *φ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **57** (2008), 279–285.
6. D. Ilic and V. Rakocevic, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.
7. W.G. Dotson, *On fixed points of nonexpansive mappings in nonconvex sets*, Proc. Amer. Math. Soc. **38** (1973), 155–156.
8. P.N. Dutta and Binayak S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed Point Theory Appl. **2008**, Art. ID 406368, 8 pp.
9. L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1467–1475.
10. R.H. Haghi and Sh. Rezapour, *Fixed points of multifunctions on regular cone metric spaces*, Expo. Math. **28** (2010), no. 1, 71—77.
11. L.A. Khan and A.R. Khan, *An extension of Brosowski-Meinardus theorem on invariant approximation*, Approx. Theory Appl. **11** (1995), no. 4, 1–5.
12. T.D. Narang and M. Sharma, *On invariant approximation of nonexpansive mappings*, J. Korea Soc. Math. Educ. Ser. B: Appl. Math. **10** (2003), no. 2, 127–132.
13. S. Park, *A unified approach to fixed points of contractive maps*, J. Korean Math. Soc. **16** (1980), no. 2, 95–105.
14. B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001), no. 4, 2683–2693.
15. Sh. Rezapour and R. Hamlbarani, *Some note on the paper " Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. **345** (2008), no. 2, 719–724.
16. Sh. Rezapour, *A review on topological properties of cone metric spaces*, Analysis, Topology and Applications 2008, Vrnjacka Banja, Serbia, from May 30 to June 4, 2008.
17. Sh. Rezapour, *Best approximations in cone metric spaces*, Math. Morav. **11** (2007), 85–88.
18. P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **56** (2007), 464—468.

[1] Department of Mathematics, Lahore University of Management Sciences, 54792-Lahore, Pakistan.

E-mail address: mujahid@lums.edu.pk

[2] Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi, 34 - 90123 Palermo, Italy.

E-mail address: vetro@math.unipa.it