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APPROXIMATION OF ANALYTIC FUNCTIONS BY SPECIAL FUNCTIONS

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ABSTRACT. We survey the recent results concerning the applications of power series method to the study of Hyers–Ulam stability of differential equations.

1. Introduction

Differential equations have been studied for more than 300 years since the 17th century when the concepts of differentiation and integration were formulated by Newton and Leibniz. By use of differential equations, we can explain many natural phenomena: gravity, projectiles, wave, vibration, nuclear physics and so on.

Let us consider a closed system which can be explained by the first order linear differential equation, namely, $y'(t) = \lambda y(t)$. The past, present, and future of this system are completely determined if we know the general solution and an initial condition of that differential equation. So we can say that this system is 'predictable.' Sometimes, because of the disturbances (or noises) of the outside, the system may not be determined by $y'(t) = \lambda y(t)$ but can only be explained by an inequality like $|y'(t) - \lambda y(t)| \le \varepsilon$. Then it is impossible to predict the exact future of the disturbed system.

Even though the system is not predictable exactly because of outside disturbances, we say the differential equation $y'(t) = \lambda y(t)$ has the Hyers–Ulam stability if the 'real' future of the system follows the solution of $y'(t) = \lambda y(t)$ with a bounded error (see [2, 3, 4, 5, 6, 17, 26, 28] for the exact definition of Hyers–Ulam

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stability). But if the error bound is 'too big,' we say that differential equation $y'(t) = \lambda y(t)$ does not have the Hyers–Ulam stability. Resonance is the case.

There is another way to explain the Hyers–Ulam stability. Usually the experiment (or the observed) data do not exactly coincide with theoretical ones. We may express natural phenomena by use of equations but because of the errors due to measurement or observance the actual experiment data can almost always be a little bit off the expectations. If we would use inequalities instead of equalities to explain natural phenomena, then these errors could be absorbed into the solutions of inequalities, i.e., those errors would be no more errors.

Considering this point of view, the Hyers–Ulam stability (of differential equations) is fundamental. (The Hyers–Ulam stability is not same as the concept of the stability of differential equations which has been studied by many mathematicians from long time ago).

The Hyers–Ulam stability of functional equations has been studied for more than 70 years. But the history of the Hyers–Ulam stability of differential equations is less than 20 years. For example, Obłoza seems to be the first author who has investigated in 1993 the Hyers–Ulam stability of linear differential equations (see [24, 25]). Thereafter, Alsina and Ger published their paper [1], which handles the Hyers–Ulam stability of the linear differential equation $y'(x) = y(x)$: If a differentiable function $y(x)$ is a solution of the inequality $|y'(x) - y(x)| \leq \varepsilon$ for any $x \in (a, \infty)$, then there exists a constant c such that $|y(x) - ce^x| \leq 3\varepsilon$ for all $x \in (a, \infty)$. We know that the general solution of the linear differential equation $y'(x) = y(x)$ is $y(x) = ce^x$, where c is a constant.

Therefore, we say that the differential equation $y'(x) = y(x)$ has the Hyers–Ulam stability. If we can get a similar result with a control function $\varphi(x)$ in place of ε, we say that the differential equation $y'(x) = y(x)$ has the Hyers–Ulam-Rassias stability.

In 2001 and 2002, Miura et al. [27] expanded Alsina and Ger's result by proving that the differential equation $y'(x) = \lambda y(x)$ has the Hyers–Ulam stability. The author wrote a paper with Miura and Takahasi which expanded the result of Hyers–Ulam stability of that differential equation. To be more precise, we may choose a constant c such that the solution of the inequality $|y'(x) - \lambda y(x)| \leq \varphi(x)$ is not too far away from $ce^{\lambda x}$ in the sense of upper norm (see [23]).

As above, still not many papers were published about the Hyers–Ulam stability of differential equations. The author has been studying this subject since Alsina and Ger's paper, and published several papers (ref. [7, 8, 9, 14, 20]).

In this paper, we will survey the recent results concerning the applications of power series method to the study of Hyers–Ulam stability of differential equations.

2. Power series method

In 2007, using the power series method, the author [10] proved the Hyers–Ulam stability of the Legendre's differential equation

$$\left(1 - x^2\right)y''(x) - 2xy'(x) + p(p+1)y(x) = 0,$$

where p is a real number, whose solutions are called Legendre functions.

The Legendre's differential equation plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting spherical symmetry.

To the best of our knowledge, the author first applied the power series method to the Hyers–Ulam stability problems. Here, we will introduce a theorem from [13]:

Theorem 2.1. *Assume that ρ and ρ_0 are positive constants with $\rho < \rho_0$. Let $y : (-\rho, \rho) \to \mathbb{C}$ be a function which can be represented by a power series $y(x) = \sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is ρ_0. Assume moreover that there exists a positive number ρ_1 for which the condition*

$$\rho_1 = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| > 0.$$

is satisfied, where

$$c_k = b_k - \frac{b_{k-2[k/2]}}{k!} \prod_{j=1}^{[k/2]} (k - 2j - p)(k - 2j + p + 1)$$

for all $k \in \{2, 3, \ldots\}$. If $\rho < \min\{1, \rho_0, \rho_1\}$, then there exist a Legendre function $y_h : (-\rho, \rho) \to \mathbb{C}$ and a constant $K > 0$ such that

$$|y(x) - y_h(x)| \le \frac{Kx^2}{1 - x^2}$$

for all $x \in (-\rho, \rho)$.

Thereafter, the author [11] applied the power series method to calculate an upper bound of error when we are approximating analytic function with Airy function about the origin, where an Airy function is a solution of the Airy's differential equation

$$y''(x) - xy(x) = 0$$

which appears in many calculations in applied mathematics.

Theorem 2.2. *Assume that ρ and ρ_0 are positive constants with $\rho < \min\{\sqrt[3]{6}, \rho_0\}$. If a function $y : (-\rho, \rho) \to \mathbb{C}$ can be represented by a power series $y(x) = \sum_{m=0}^{\infty} b_m x^m$, whose radius of convergence is ρ_0, then there exist an Airy function $y_h : (-\rho, \rho) \to \mathbb{C}$ and a constant $K > 0$ such that*

$$|y(x) - y_h(x)| \le Kx^2$$

for all $x \in (-\rho, \rho)$.

The Hermite's differential equation

$$y''(x) - 2xy'(x) + 2\lambda y(x) = 0$$

plays an important role in quantum mechanics, probability theory, statistical mechanics, and in solutions of Laplace's equation in parabolic coordinates. Every solution of the Hermite's differential equation is called an Hermite function.

Also, the author found an approximation formula of analytic functions in terms of Hermite functions (ref. [12]):

Theorem 2.3. *Let λ be a fixed real number not less than 1. Assume that ρ and ρ_0 are positive constants with $\rho < \rho_0$. If a function $y : (-\rho, \rho) \to \mathbb{C}$ can be represented by a power series $y(x) = \sum_{m=0}^{\infty} b_m x^m$, whose radius of convergence is ρ_0, then there exist an Hermite function $y_h : (-\rho, \rho) \to \mathbb{C}$ and a constant $K > 0$ such that*

$$|y(x) - y_h(x)| \leq K x^2 e^{x^2}$$

for all $x \in (-\rho, \rho)$, where K depends on λ, ρ and y.

The Kummer's differential equation

$$xy''(x) + (\beta - x)y'(x) - \alpha y(x) = 0,$$

which is also called the confluent hypergeometric differential equation, appears frequently in practical problems and applications. The Kummer's differential equation has a regular singularity at $x = 0$ and an irregular singularity at ∞.

We define $(\alpha)_m$ by $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)$ for all $m \in \mathbb{N}$. Moreover, we define

$$I_\rho = \begin{cases} (-\rho, \rho) & (\text{for } \beta < 1), \\ (-\rho, \rho)\backslash\{0\} & (\text{for } \beta > 1) \end{cases}$$

for any $0 < \rho \leq \infty$.

For a given $\kappa \geq 0$, let us denote $\mathcal{K}_\kappa$ the set of all functions $y : I_\rho \to \mathbb{C}$ with the properties:

(K_1) $y(x)$ is represented by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(K_2) $\sum_{m=0}^{\infty} |a_m x^m| \leq \kappa |\sum_{m=0}^{\infty} a_m x^m|$ for all $x \in I_\rho$, where $a_m = (m+\beta)(m+1)b_{m+1} - (m+\alpha)b_m$ for each $m \in \mathbb{N}_0$.

In the following theorem, a local Hyers–Ulam stability of the Kummer's differential equation is investigated (see [15]):

Theorem 2.4. *Let α and β be real constants such that $\beta \notin \mathbb{Z}$ and neither α nor $1+\alpha-\beta$ is a nonpositive integer. Suppose a function $y : I_\rho \to \mathbb{C}$ is representable by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least $\rho > 0$. Assume that there exist nonnegative constants $\mu \neq 0$ and ν satisfying the condition*

$$\left|\frac{(m-1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right| \leq \mu \left|\sum_{i=0}^{m-1} \frac{i!(\beta)_i a_i}{(\alpha)_{i+1}}\right| \leq \nu \left|\frac{(m+1)!(\beta)_m a_m}{(\alpha)_{m+1}}\right|$$

for all $m \in \mathbb{N}_0$, where $a_m = (m+\beta)(m+1)b_{m+1} - (m+\alpha)b_m$. (Indeed, it is sufficient for the first inequality to hold for all sufficiently large integers m.) Let us define $\rho_0 = \min\{\rho, 1/\mu\}$. If $y \in \mathcal{K}_\kappa$ and it satisfies the differential inequality

$$\left|xy''(x) + (\beta - x)y'(x) - \alpha y(x)\right| \leq \varepsilon$$

for all $x \in I_{\rho_0}$ and for some $\varepsilon \geq 0$, then there exists a solution $y_h : I_\infty \to \mathbb{C}$ of the Kummer's differential equation such that

$$|y(x) - y_h(x)| \leq \begin{cases} \dfrac{\nu}{\mu} \cdot \dfrac{2\alpha - 1}{\alpha} \kappa\varepsilon & (\textit{for } \alpha > 1), \\ \dfrac{\nu}{\mu} \left[\displaystyle\sum_{m=0}^{m_0 - 1} \left| \left| \frac{m+1}{m+\alpha} \right| - \left| \frac{m+2}{m+1+\alpha} \right| \right| + \frac{m_0 + 1}{m_0 + \alpha} \right] \kappa\varepsilon & (\textit{for } \alpha \leq 1) \end{cases}$$

for any $x \in I_{\rho_0}$, where $m_0 = max\{0, \lceil -\alpha \rceil\}$.

A function is called a Bessel function (of fractional order) if it is a solution of the Bessel's differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0,$$

where ν is a positive nonintegral number. The Bessel's differential equation plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting cylindrical symmetries.

Let us define $I_\rho = (-\rho, \rho)\backslash\{0\}$ for a positive constant ρ. For a given $\kappa \geq 0$, we denote by $\mathcal{B}_\kappa$ the set of all functions $y : I_\rho \to \mathbb{C}$ with the properties:

(B_1) $y(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(B_2) $\sum_{m=0}^{\infty} |a_m x^m| \leq \kappa \left| \sum_{m=0}^{\infty} a_m x^m \right|$ for any $x \in I_\rho$, where $a_m = b_{m-2} + (m^2 - \nu^2) b_m$ for all $m \in \mathbb{N}_0$ and set $b_{-2} = b_{-1} = 0$.

For the given positive nonintegral number ν, define

$$M_e(x) = \max\left\{ \prod_{j=i}^{k} \frac{x^2}{|\nu^2 - (2j)^2|} : 0 \leq i \leq k \leq \mu \right\},$$

$$M_o(x) = \max\left\{ \prod_{j=i}^{k} \frac{x^2}{|\nu^2 - (2j+1)^2|} : 0 \leq i \leq k \leq \mu \right\},$$

$$M(x) = \max\{M_e(x),\, M_o(x),\, 1\},$$

where $\mu = [\sqrt{\nu^2 + x^2}/2]$ and

$$L_\nu = \sum_{m=0}^{\infty} \frac{1}{(m-\nu)^2} < \infty.$$

We remark that $M(x) \to 1$ as $|x| \to 0$.

In the following theorem, a local Hyers–Ulam stability of the Bessel's differential equation is investigated (see [18] and ref. [22]):

Theorem 2.5. *Let ν and p be a positive nonintegral number and a nonnegative integer with $p < \nu < p + 1$, respectively. Assume that a function $y \in \mathcal{B}_\kappa$ satisfies the differential inequality*

$$\left| x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) \right| \leq \varepsilon$$

for all $x \in I_\rho$ *and for some* $\varepsilon \geq 0$*. If the sequence* $\{b_m\}$ *satisfies the condition*

$$b_{m+2} = O(b_m) \text{ as } m \to \infty$$

with a Landau constant $C \geq 0$*, then there exists a Bessel function* $y_h(x)$ *such that*

$$|y(x) - y_h(x)| \leq \kappa L_\nu M(x)\varepsilon$$

for any $x \in I_{\rho_0}$*, where* $\rho_0 = \min\{\rho, 1/\sqrt{C^*}\}$ *and* C^* *is a positive number larger than* C*. If* C *is sufficiently small and* ρ *is large, then*

$$M(x) \leq \max\left\{\frac{|x|^{|x|+2}}{|\nu^2 - p^2|^{|x|/2+1}}, \frac{|x|^{|x|+2}}{|\nu^2 - (p+1)^2|^{|x|/2+1}}\right\}$$

for all sufficiently large $|x|$*.*

Each solution of the Laguerre's differential equation

$$xy''(x) + (1-x)y'(x) + ny(x) = 0$$

is called a Laguerre function. The Laguerre's differential equation is a special case of the Sturm-Liouville boundary problem. This equation can deal with harmonic oscillator phenomena in quantum mechanics. For instance, the Laguerre's differential equation can describe the state of electrons in the field of the Coulomb force. Indeed, the solution of radial part of the Schrödinger equation for the electron in hydrogen atom consists of functions expressed by Laguerre polynomials.

Let $\kappa \geq 0$ and $1 < \rho \leq \infty$ be constants. We denote by $\mathcal{L}_\kappa$ the set of all functions $y : [0, \rho) \to \mathbb{C}$ with the following properties:

(L_1) $y(x)$ is expressible by a power series $\sum_{m=1}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(L_2) $\sum_{m=0}^{\infty} |a_m x^m| \leq \kappa |\sum_{m=0}^{\infty} a_m x^m|$ for any $x \in [0,1]$, where $a_m = (m+1)^2 b_{m+1} + (n-m)b_m$ for all $m \in \mathbb{N}_0$ and set $b_0 = 0$.

The author applied the power series method to the study of Hyers–Ulam stability of the Laguerre's differential equation (see [16]):

Theorem 2.6. *Let* n *be a nonnegative integer. If a function* $y \in \mathcal{L}_\kappa$ *satisfies the differential inequality*

$$\left|xy''(x) + (1-x)y'(x) + ny(x)\right| \leq \varepsilon$$

for all $x \in [0, \rho)$ *and for some* $\varepsilon > 0$*, then there exists a Laguerre function* $y_h : [0, 1) \to \mathbb{C}$ *such that*

$$|y(x) - y_h(x)| \leq \frac{\pi^2}{6}\kappa M \varepsilon x$$

for all $x \in [0, 1)$*, where* M *is defined by*

$$M = \begin{cases} 1 & (n \in \{0, 1\}), \\ \max\limits_{1 \leq k \leq \ell \leq [\sqrt{n}]} \prod\limits_{j=k}^{\ell} \dfrac{|n-j|}{j^2} & (n \geq 2). \end{cases}$$

Every solution of the Chebyshev's differential equation

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0$$

is called a Chebyshev function. The Chebyshev's differential equation has regular singular points at -1, 1, and ∞ and it plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting certain symmetries.

Let $\kappa \geq 0$ and $\rho > 0$ be constants. We denote by $\mathcal{C}_\kappa$ the set of all functions $y : (-\rho, \rho) \to \mathbb{C}$ with the following properties:

(C_1) $y(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ;

(C_2) $\sum_{m=0}^{\infty} |a_m x^m| \leq \kappa \left| \sum_{m=0}^{\infty} a_m x^m \right|$ for any $x \in (-\rho, \rho)$, where $a_m = (m+2)(m+1)b_{m+2} - (m^2 - n^2)b_m$ for all $m \in \mathbb{N}_0$ and set $b_0 = b_1 = 0$.

In the following theorem, a local Hyers–Ulam stability of the Chebyshev's differential equation is proved (see [21] and ref. [19]).

Theorem 2.7. *Let n be a positive integer and assume that a function $y \in \mathcal{C}_\kappa$ satisfies the differential inequality*

$$\left|(1-x^2)y''(x) - xy'(x) + n^2y(x)\right| \leq \varepsilon$$

for all $x \in (-\rho, \rho)$ and for some $\varepsilon > 0$. Let $\rho_0 = \min\{1, \rho\}$. Then there exists a Chebyshev function $y_h : (-\rho_0, \rho_0) \to \mathbb{C}$ such that

$$|y(x) - y_h(x)| \leq \frac{\kappa M_e \varepsilon}{2} \frac{x^2}{1-x^2}$$

for all $x \in (-\rho_0, \rho_0)$, where the constant M_e is defined by

$$M_e = \max_{0 \leq i \leq \ell \leq n_e} \frac{(2i)!}{(2\ell+1)!}(n^2-4)^{\ell - i}$$

with $n_e = [n/\sqrt{8}]$.

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