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ON THE SUZUKI NONEXPANSIVE-TYPE MAPPINGS

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ABSTRACT. It is shown that if C is a nonempty convex and weakly compact subset of a Banach space X with $M(X) > 1$ and $T : C \to C$ satisfies condition (C) or is continuous and satisfies condition (C_λ) for some $\lambda \in (0, 1)$, then T has a fixed point. In particular, our theorem holds for uniformly nonsquare Banach spaces. A similar statement is proved for nearly uniformly noncreasy spaces.

1. Introduction

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to X$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for $x, y \in C$. There is a large literature concerning fixed point theory of nonexpansive mappings and their generalizations (see [13] and references therein). Recently, Suzuki [20] defined a class of generalized nonexpansive mappings as follows.

Definition 1.1. A mapping $T : C \to X$ is said to satisfy condition (C) if for all $x, y \in C$,

$$\frac{1}{2} \|x - Tx\| \le \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \le \|x - y\| .$$

Subsequently the definition was widened in [10].

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Definition 1.2. Let $\lambda \in (0,1)$. A mapping $T : C \to X$ is said to satisfy condition (C_λ) if for all $x, y \in C$,

$$\lambda \|x - Tx\| \le \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \le \|x - y\| .$$

It is not difficult to see that if $\lambda_1 < \lambda_2$ then condition (C_{λ_1}) implies condition (C_{λ_2}). Several examples of mappings satisfying condition (C_λ) are given in [10, 20].

Two other related generalizations of a nonexpansive mapping have been proposed in [1] and [17]. Recall that a sequence (x_n) is called an approximate fixed point sequence for T (afps, for short) if $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Definition 1.3 (see [1, Def. 3.1])**.** A mapping $T : C \to X$ is said to satisfy condition $(*)$ if

(i) for each nonempty closed convex and T-invariant subset D of C, T has an afps in D, and
(ii) For each pair of closed convex T-invariant subsets D and E of C, the asymptotic center $A(E,(x_n))$ of a sequence (x_n) relative to E is T-invariant for each afps (x_n) in D.

Definition 1.4 (see [17, Def. 3.1])**.** A mapping $T : C \to X$ is said to satisfy condition (L) if

(i) for each nonempty closed convex and T-invariant subset D of C, T has an afps in D, and
(ii) For any afps (x_n) of T in C and for each $x \in C$,

$$\limsup_{n\to\infty} \|x_n - Tx\| \le \limsup_{n\to\infty} \|x_n - x\| .$$

It is easily seen that condition (L) implies condition $(*)$. One can also prove that condition (C) implies condition $(*)$ (see [20, Lemma 6]) and if $T : C \to C$ is continuous and satisfies condition (C_λ) for some $\lambda \in (0,1)$, then T has a fixed point or satisfies condition (L) (see [17, Theorem 4.7]). A natural question arises whether a large collection of fixed point theorems for nonexpansive mappings has its counterparts for mappings satisfying conditions (C_λ), (L) or $(*)$. This is a nontrivial matter since some constructions developed for nonexpansive mappings do not work properly in a general case.

Let C be a nonempty convex and weakly compact subset of a Banach space X. It was proved in [20] that every mapping $T : C \to C$ which satisfies condition (C) has a fixed point when X is UCED or satisfies the Opial property, and in [3], when X has property (D). The above results were generalized in [17] by showing that if X has normal structure, then every mapping $T : C \to C$ satisfying condition (L) has a fixed point. In particular, every continuous self-mapping of type (C_λ) has a fixed point in this case. For a treatment of a more general case of metric spaces and multivalued nonexpansive-type mappings we refer the reader to [7] and the references given there.

Our paper is organized as follows. In Section 2 we prove that the mapping $T_\gamma = (1-\gamma)I + \gamma T$, where $\gamma \in (0,1)$ is uniformly asymptotically regular with respect to all $x \in C$ and all mappings from C into C which satisfy condition (C_γ). We apply this result in Section 3 to prove basic Lemmas 3.3 and 3.4. In Section 4 we are able to adapt the proof of [18, Theorem 9] and strenghten the result. As a consequence, we show that if C is a nonempty convex and weakly compact subset of a nearly uniformly noncreasy space or a Banach space X with $M(X) > 1$, then every mapping $T : C \to C$ which satisfies condition (C) and every continuous mapping $T : C \to C$ which satisfies condition (C_λ) for some $\lambda \in (0,1)$ has a fixed point. In particular, our theorems hold for both uniformly nonsquare and uniformly noncreasy Banach spaces. In the case of uniformly nonsquare spaces it answers Question 1 in [3].

2. Asymptotic regularity

Recall that a mapping $T : M \to M$ acting on a metric space (M, d) is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$$

for all $x \in M$. Ishikawa [14] proved that if C is a bounded convex subset of a Banach space X and $T : C \to C$ is nonexpansive, then the mapping $T_\gamma = (1-\gamma)I + \gamma T$ is asymptotically regular for each $\gamma \in (0, 1)$. Edelstein and O'Brien [6] showed that T_γ is uniformly asymptotically regular over $x \in C$, and Goebel and Kirk [12] proved that the convergence is uniform with respect to all nonexpansive mappings from C into C. The Ishikawa result was extended in [20, Lemma 6] for mappings with condition (C) and in [10, Theorem 4] for mappings with condition (C_λ). In this section we prove the uniform version of that result. The proof follows in part [6, Lemma 1].

Theorem 2.1. *Let C be a bounded convex subset of a Banach space X. Fix $\lambda \in (0, 1), \gamma \in [\lambda, 1)$ and let $\mathcal{F}$ denote the collection of all mappings which satisfy condition (C_λ). Let $T_\gamma = (1-\gamma)I + \gamma T$ for $T \in \mathcal{F}$. Then for every $\varepsilon > 0$, there exists a positive integer n_0 such that $\left\|T_\gamma^{n+1}x - T_\gamma^n x\right\| < \varepsilon$ for every $n \geq n_0, x \in C$ and $T \in \mathcal{F}$.*

Proof. Without loss of generality we can assume that $\operatorname{diam} C = 1$. Suppose, contrary to our claim, that there exists $\delta > 0$ such that

$$(\forall n_0 > 0)\ (\exists n \geq n_0, x \in C, T \in \mathcal{F})\ \|T_\gamma^{n+1}x - T_\gamma^n x\| \geq \delta. \tag{2.1}$$

Fix a positive integer $M > 2/\delta$ and let $L = \lceil \frac{1}{\gamma(1-\gamma)^M} \rceil$ denote the smallest integer not less than $\frac{1}{\gamma(1-\gamma)^M}$. Then, by (2.1), there exist $N > ML, x_0 \in C$ and $T \in \mathcal{F}$ such that

$$\|T_\gamma^{N+1}x_0 - T_\gamma^N x_0\| \geq \delta.$$

Let $x_i = T_\gamma^i x_0$. Since

$$\lambda\|Tx_{i-1} - x_{i-1}\| = \frac{\lambda}{\gamma}\|T_\gamma x_{i-1} - x_{i-1}\| \leq \|x_i - x_{i-1}\|,$$

$i = 1, 2, ...,$ and T satisfies condition (C_λ), we get

$$\|Tx_i - Tx_{i-1}\| \leq \|x_i - x_{i-1}\|$$

and hence

$$\|T_\gamma x_i - T_\gamma x_{i-1}\| \le (1-\gamma)\|x_i - x_{i-1}\| + \gamma\|Tx_i - Tx_{i-1}\| \le \|x_i - x_{i-1}\|$$

for every positive integer i. Thus

$$\|x_1 - x_0\| \ge \|x_2 - x_1\| \ge \ldots \ge \|x_{N+1} - x_N\| \ge \delta \tag{2.2}$$

and

$$\left\|\frac{1}{\gamma}(x_{i+1} - x_i) - \frac{1-\gamma}{\gamma}(x_i - x_{i-1})\right\| = \|Tx_i - Tx_{i-1}\| \le \|x_i - x_{i-1}\| \tag{2.3}$$

for all $i = 1, 2, \ldots, N$. We can now follow the arguments from [6]. Notice that

$$[\delta, 1] \subset \bigcup_{i=1}^{L} [b_i, b_i + \gamma(1-\gamma)^M],$$

where $b_i = \delta + (i-1)\gamma(1-\gamma)^M$. Since $\{\|x_{Mi+1} - x_{Mi}\| : 0 \le i \le L\}$ has $L+1$ elements which belong to $[\delta, 1]$ by $N > ML$ and (2.2), it follows from the pigeonhole principle that there exists an interval $I = [b, b+\gamma(1-\gamma)^M]$ with $b \ge \delta$ and $0 \le i_1 < i_2 \le L$ such that $\|x_{Mi_1+1} - x_{Mi_1}\|, \|x_{Mi_2+1} - x_{Mi_2}\| \in I$. Hence by (2.2),

$$\|x_{i+1} - x_i\| \in I \quad \text{for} \quad i = Mi_1, Mi_1 + 1, \ldots, Mi_2. \tag{2.4}$$

In particular, $\|x_{K+M+1} - x_{K+M}\| \in I$, where $K = Mi_1$. Select a functional $f \in S_{X^*}$ such that

$$f(x_{K+M+1} - x_{K+M}) = \|x_{K+M+1} - x_{K+M}\| \ge b.$$

Then (2.3) and (2.4) imply

$$\begin{aligned}
&\frac{1}{\gamma} f(x_{K+M+1} - x_{K+M}) - \frac{1-\gamma}{\gamma} f(x_{K+M} - x_{K+M-1}) \\
&\le \left\|\frac{1}{\gamma}(x_{K+M+1} - x_{K+M}) - \frac{1-\gamma}{\gamma}(x_{K+M} - x_{K+M-1})\right\| \\
&\le \|x_{K+M} - x_{K+M-1}\| \le b + \gamma(1-\gamma)^M,
\end{aligned}$$

so that

$$\frac{b}{\gamma} - \frac{1-\gamma}{\gamma} f(x_{K+M} - x_{K+M-1}) \le b + \gamma(1-\gamma)^M$$

and hence

$$f(x_{K+M} - x_{K+M-1}) \ge b - \gamma^2(1-\gamma)^{M-1}.$$

Similarly,

$$\begin{aligned}
b + (1-\gamma)^M\gamma &\ge \frac{1}{\gamma} f(x_{K+M} - x_{K+M-1}) - \frac{1-\gamma}{\gamma} f(x_{K+M-1} - x_{K+M-2}) \\
&\ge \frac{1}{\gamma}\left(b - (1-\gamma)^M\gamma^2\left(\frac{1}{1-\gamma}\right)\right) - \frac{1-\gamma}{\gamma} f(x_{K+M-1} - x_{K+M-2}),
\end{aligned}$$

and hence

$$f(x_{K+M-1} - x_{K+M-2}) \ge b - (1-\gamma)^M\gamma^2\left(\frac{1}{1-\gamma} + \frac{1}{(1-\gamma)^2}\right) \ge b - \gamma(1-\gamma)^{M-2}.$$

In general,

$$f(x_{K+M+1-i} - x_{K+M-i}) \ge b - \gamma(1-\gamma)^{M-i}$$

for all $i = 0, 1, \dots, M$. Thus

$$\begin{aligned} f(x_{K+M+1}) &\geq f(x_{K+M}) + b \\ &\vdots \\ &\geq f(x_{K+M+1-i}) + ib - \gamma((1-\gamma)^{M-1} + \dots + (1-\gamma)^{M+1-i}) \\ &\vdots \\ &\geq f(x_{K+1}) + Mb - \gamma((1-\gamma)^{M-1} + \dots + (1-\gamma)) \\ &\geq f(x_{K+1}) + Mb - 1. \end{aligned}$$

But $b \geq \delta$ implies that $Mb \geq M\delta > 2$, and so $\|x_{K+M+1} - x_{K+1}\| \geq f(x_{K+M+1} - x_{K+1}) > 1$ contradicting the assumption that $\operatorname{diam} C = 1$. □

3. Basic lemmas

Let C be a nonempty weakly compact convex subset of a Banach space X and $T : C \to C$. It follows from the Kuratowski-Zorn lemma that there exists a minimal (in the sense of inclusion) convex and weakly compact set $K \subset C$ which is invariant under T. The first lemma below is a counterpart of the Goebel-Karlovitz lemma (see [11, 16]). It was proved by Dhompongsa and Kaewcharoen [2, Theorem 4.14] in the case of mappings which satisfy condition (C), and by Butsan, Dhompongsa and Takahashi [1, Lemma 3.2] in the case of mappings satisfying condition $(*)$. Denote by

$$r(K, (x_n)) = \inf\{\limsup_{n\to\infty} \|x_n - x\| : x \in K\}$$

the asymptotic radius of a sequence (x_n) relative to K.

Lemma 3.1. *Let K be a nonempty convex weakly compact subset of a Banach space X which is minimal invariant under $T : K \to K$. If T satisfies condition $(*)$ (condition (C), in particular), then there exists an approximate fixed point sequence (x_n) for T such that*

$$\lim_{n\to\infty} \|x_n - x\| = \inf\{r(K, (y_n)) : (y_n) \text{ is an afps in } K\}$$

for every $x \in K$.

Lloréns Fuster and Moreno Gálvez [17, Th. 4.7] proved that if $T : C \to C$ is continuous and satisfies condition (C_λ) for some $\lambda \in (0, 1)$, then T has a fixed point or satisfies condition (L). Since the set consisting of a single fixed point of T is minimal invariant under T and condition (L) implies condition $(*)$, we obtain the following corollary.

Lemma 3.2. *The conclusion of Lemma 3.1 is valid for continuous mappings which satisfy condition (C_λ) for some $\lambda \in (0, 1)$.*

Now let (x_n) be a weakly null afps sequence for T in C. Fix $t < 1$ and put $v_n = tx_n$. The following technical lemma deals with the behaviour of sequences $(T^k_\gamma v_n)_{n\in\mathbb{N}}$, $k = 1, 2, \dots$.

Lemma 3.3. *Assume that $T : C \to C$ satisfies condition (C_λ) for some $\lambda \in (0,1)$. Fix $\gamma \in [\lambda, 1)$, a positive integer N, $0 < \varepsilon < \frac{1}{10N}$ and $\frac{2}{3} + 2N\varepsilon < t < 1 - 2\varepsilon$. Suppose that (x_n) is a weakly null sequence in C such that $\operatorname{diam}(x_n) = 1$ and the following conditions are satisfied for every $n, m \in \mathbb{N}$ and $k = 1, ..., N$:*

(i) *a sequence $(T_\gamma^k v_n)_{n\in\mathbb{N}}$, where $v_n = tx_n$, converges weakly to a point $y_k \in C$,*
(ii) $\|T_\gamma^k v_n - T_\gamma^k v_m\| > \liminf_i \|T_\gamma^k v_n - T_\gamma^k v_i\| - \varepsilon$,
(iii) $\min\{\|x_n\|, \|x_n - x_m\|, \|x_n - y_k\|\} > 1 - \varepsilon$,
(iv) $\|Tx_n - x_n\| < \varepsilon$.

Then, for every $n, m \in \mathbb{N}$ and $k = 1, ..., N$,

$$t - (k+2)\varepsilon < \left\|T_\gamma^k v_n - T_\gamma^k v_m\right\| \le t, \tag{3.1}$$

$$1 - t - \varepsilon < \left\|T_\gamma^k v_n - x_n\right\| < 1 - t + k\varepsilon. \tag{3.2}$$

Proof. Fix $n, m \in \mathbb{N}$ and note that

$$t - \varepsilon < \|v_n - v_m\| = t\|x_n - x_m\| \le t,$$

and

$$1 - t - \varepsilon < \|x_n - v_n\| = (1-t)\|x_n\| \le (1-t)\operatorname{diam}(x_n) \le 1 - t.$$

Since

$$\|Tx_n - x_n\| < \varepsilon < 1 - t - \varepsilon < \|x_n - v_n\|, \ (t < 1 - 2\varepsilon),$$

it follows from condition (C_λ) that

$$\|Tx_n - Tv_n\| \le \|x_n - v_n\|.$$

Hence

$$\|T_\gamma x_n - T_\gamma v_n\| \le \gamma\|Tx_n - Tv_n\| + (1-\gamma)\|x_n - v_n\| \le \|x_n - v_n\| \le 1 - t, \tag{3.3}$$

and

$$\begin{aligned}\|T_\gamma v_n - v_n\| = \gamma\|Tv_n - v_n\| &\le \|Tv_n - Tx_n\| + \|Tx_n - x_n\| + \|x_n - v_n\| \\ &< 2\|x_n - v_n\| + \varepsilon \le 2(1-t) + \varepsilon.\end{aligned} \tag{3.4}$$

We shall also use, for each $k \le N$, the following estimation which follows from the weak lower semicontinuity of the norm:

$$\begin{aligned}1 - \varepsilon < \|x_n - y_k\| &\le \liminf_m \|x_n - T_\gamma^k v_m\| \\ &\le \|x_n - T_\gamma^k v_n\| + \liminf_m \|T_\gamma^k v_n - T_\gamma^k v_m\|.\end{aligned} \tag{3.5}$$

Now we proceed by induction on k.

For $k = 1$, notice that

$$\|T_\gamma v_n - v_n\| < 2(1-t) + \varepsilon < t - \varepsilon < \|v_n - v_m\|, \ (t > \frac{2}{3} + \frac{2}{3}\varepsilon),$$

and it follows from condition (C_λ) that

$$\|T_\gamma v_n - T_\gamma v_m\| \le \|v_n - v_m\| \le t. \tag{3.6}$$

Furthermore,

$$\|T_\gamma v_n - x_n\| \le \|T_\gamma v_n - T_\gamma x_n\| + \|T_\gamma x_n - x_n\| < 1 - t + \varepsilon, \tag{3.7}$$

by (3.3). To prove the reverse inequalities, notice that by (3.5),

$$\|T_\gamma v_n - T_\gamma v_m\| > \liminf_m \|T_\gamma v_n - T_\gamma v_m\| - \varepsilon > 1 - \varepsilon - \|x_n - T_\gamma v_n\| - \varepsilon,$$

and it follows from (3.7) that

$$\|T_\gamma v_n - T_\gamma v_m\| > 1 - \varepsilon - (1 - t + \varepsilon) - \varepsilon = t - 3\varepsilon.$$

Finally, by (3.5) and (3.6),

$$\|T_\gamma v_n - x_n\| > 1 - \varepsilon - \liminf_m \|T_\gamma v_n - T_\gamma v_m\| \geq 1 - t - \varepsilon.$$

Now suppose the lemma is true for a fixed $k < N$. Then

$$\left\|T_\gamma^{k+1} v_n - T_\gamma^{k+1} v_m\right\| \leq \left\|T_\gamma^k v_n - T_\gamma^k v_m\right\| \leq t, \tag{3.8}$$

since (as in the proof of Theorem 2.1)

$$\begin{aligned}\left\|T_\gamma T_\gamma^k v_n - T_\gamma^k v_n\right\| &\leq \left\|T_\gamma^k v_n - T_\gamma^{k-1} v_n\right\| \leq \ldots \leq \|T_\gamma v_n - v_n\| \\ &< 2(1-t) + \varepsilon < t - (k+2)\varepsilon < \left\|T_\gamma^k v_n - T_\gamma^k v_m\right\|,\end{aligned}$$

(notice that $t > \frac{2}{3} + \frac{(k+3)\varepsilon}{3}$). Furthermore, by induction assumption,

$$\|T_\gamma x_n - x_n\| < \varepsilon < 1 - t - \varepsilon < \|x_n - T_\gamma^k v_n\|,$$

and hence

$$\|T_\gamma^{k+1} v_n - T_\gamma x_n\| \leq \|T_\gamma^k v_n - x_n\|.$$

We thus get

$$\begin{aligned}\left\|T_\gamma^{k+1} v_n - x_n\right\| &\leq \|T_\gamma^{k+1} v_n - T_\gamma x_n\| + \|T_\gamma x_n - x_n\| \\ &< \|T_\gamma^k v_n - x_n\| + \varepsilon < 1 - t + (k+1)\varepsilon.\end{aligned} \tag{3.9}$$

To prove the reverse inequalities, notice that by (ii), (3.5) and (3.9),

$$\begin{aligned}\left\|T_\gamma^{k+1} v_n - T_\gamma^{k+1} v_m\right\| &> \liminf_i \|T_\gamma^{k+1} v_n - T_\gamma^{k+1} v_i\| - \varepsilon \\ &> 1 - \varepsilon - \|x_n - T_\gamma^{k+1} v_n\| - \varepsilon > t - (k+3)\varepsilon.\end{aligned}$$

Finally, by (3.5) and (3.8),

$$\left\|T_\gamma^{k+1} v_n - x_n\right\| > 1 - \varepsilon - \liminf_m \|T_\gamma^{k+1} v_n - T_\gamma^{k+1} v_m\| \geq 1 - t - \varepsilon,$$

and the proof is complete. □

We can now prove a counterpart of [5, Lemma 2] (see also [15, Theorem 1]).

Lemma 3.4. *Let K be a convex weakly compact subset of a Banach space X. Suppose that a mapping $T : K \to K$ satisfies condition (C_λ) for some $\lambda \in (0,1)$ and (x_n) is a weakly null, approximate fixed point sequence for T such that*

$$r = \lim_{n\to\infty} \|x_n - x\| = \inf\{r(K,(y_n)) : (y_n) \text{ is an afps in } K\} > 0 \tag{3.10}$$

for every $x \in K$. Then, for every $\varepsilon > 0$ and $t \in (\frac{2}{3}, 1)$, there exists a subsequence of (x_n), denoted again (x_n), and a sequence (z_n) in K such that

(i) *(z_n) is weakly convergent,*
(ii) *$\|z_n\| > r(1-\varepsilon)$,*
(iii) *$\|z_n - z_m\| \leq rt$,*

(iv) $\|z_n - x_n\| < r(1-t+\varepsilon)$
for every $m, n \in \mathbb{N}$.

Proof. Let us first notice that if $S : \frac{1}{r}K \to \frac{1}{r}K$ is defined by $Sy = \frac{1}{r}T(ry)$, then

$$\|Sy - y\| = \frac{1}{r}\|T(ry) - ry\|$$

and S satisfies condition (C_λ). It follows that a sequence (x_n) satisfies the assumptions of Lemma 3.4 if and only if a sequence $(\frac{x_n}{r})$ satisfies these assumptions with S and $\bar{r} = 1$, i.e., $(\frac{x_n}{r})$ is a weakly null afps for $S : \frac{1}{r}K \to \frac{1}{r}K$ and

$$1 = \lim_{n\to\infty} \|\frac{x_n}{r} - y\| = \inf\{r(\frac{1}{r}K, (z_n)) : (z_n) \text{ is an afps for } S \text{ in } \frac{1}{r}K\}$$

for every $y \in \frac{1}{r}K$.

Therefore it suffices to prove the lemma for $r = 1$.

We claim that for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that if $x \in K$ and $\|Tx - x\| < \delta(\varepsilon)$ then $\|x\| > 1 - \varepsilon$. Indeed, otherwise, arguing as in [5], there exists ε_0 such that we can find $w_n \in K$ with $\|Tw_n - w_n\| < \frac{1}{n}$ and $\|w_n\| \le 1 - \varepsilon_0$ for every $n \in \mathbb{N}$. Then the sequence (w_n) is an approximate fixed point sequence in K, but $\limsup_{n\to\infty} \|w_n\| \le 1 - \varepsilon_0$, which contradicts our assumption that $\limsup_{n\to\infty} \|w_n\| \ge 1$.

Fix $\varepsilon > 0$, $t \in \left(\frac{2}{3}, 1\right)$ and $\gamma \in [\lambda, 1)$. From Theorem 2.1, there exists $N > 1$ such that

$$\|T_\gamma^{N+1}x - T_\gamma^N x\| < \gamma\delta(\varepsilon) \tag{3.11}$$

for every $x \in K$. Choose $\eta > 0$ so small that $0 < \eta < \min\left\{\frac{1}{3(N+2)}, \frac{\varepsilon}{N}\right\}$ and $\frac{2}{3} + N\eta < t < 1 - 2\eta$. Put $v_n = tx_n$ and consider sequences $(T_\gamma^k v_n)_{n\in\mathbb{N}}$ for $k = 1, ..., N$. We can assume, passing to subsequences, that the double limits

$$\lim_{n,m\to\infty, n\neq m} \|T_\gamma^k v_n - T_\gamma^k v_m\|, \ k = 1, ..., N,$$

exist (see, e.g., [19, Lemma 2.5]). Then, for sufficiently large n, m $(n \neq m)$,

$$\|T_\gamma^k v_n - T_\gamma^k v_m\| > \lim_{n,m\to\infty, n\neq m} \|T_\gamma^k v_n - T_\gamma^k v_m\| - \frac{\eta}{2}$$

$$= \limsup_{n\to\infty} \limsup_{m\to\infty} \|T_\gamma^k v_n - T_\gamma^k v_m\| - \frac{\eta}{2} \ge \liminf_{i\to\infty} \|T_\gamma^k v_n - T_\gamma^k v_i\| - \eta,$$

$k = 1, ..., N$. Therefore, applying (3.10) (with $r = 1$) and passing to subsequences again, we can assume that the assumptions $(i) - (iv)$ of Lemma 3.3 are satisfied, i.e., (x_n) is weakly null, $\operatorname{diam}(x_n) = 1$, and for every $n, m \in \mathbb{N}$ and $k = 1, ..., N$,

(i) $(T_\gamma^k v_n)_{n\in\mathbb{N}}$ converges weakly to $y_k \in C$,
(ii) $\|T_\gamma^k v_n - T_\gamma^k v_m\| > \liminf_i \|T_\gamma^k v_n - T_\gamma^k v_i\| - \eta$,
(iii) $\min\{\|x_n\|, \|x_n - x_m\|, \|x_n - y_k\|\} > 1 - \eta$,
(iv) $\|Tx_n - x_n\| < \eta$.

Denote $z_n = T_\gamma^N v_n$. It follows from Lemma 3.3 that for every $n, m \in \mathbb{N}$, we have

$$\|z_n - z_m\| = \|T_\gamma^N v_n - T_\gamma^N v_m\| \le t,$$

$$\|z_n - x_n\| = \|T_\gamma^N v_n - x_n\| < 1 - t + N\eta < 1 - t + \varepsilon$$

and (z_n) is weakly convergent (to y_N).

Furthermore, by (3.11),

$$\|Tz_n - z_n\| = \frac{1}{\gamma}\|T_\gamma^{N+1}v_n - T_\gamma^N v_n\| < \delta(\varepsilon)$$

and consequently, $\|z_n\| > 1-\varepsilon$, which completes the proof. □

4. Fixed point theorems

Let X be a Banach space without the Schur property. Recall [18] that

$$d(\varepsilon, x) = \inf\{\limsup_{n\to\infty}\|x+\varepsilon y_n\| - \|x\| : (y_n) \text{ is weakly null in } S_X\},$$
$$b_1(\varepsilon, x) = \sup_{(y_n)\in\mathcal{M}_X} \liminf_{n\to\infty}\|x+\varepsilon y_n\| - \|x\|,$$

where $\mathcal{M}_X$ denotes the set of all weakly null sequences (y_n) in the unit ball B_X such that

$$\limsup_{n\to\infty}\limsup_{m\to\infty}\|y_n - y_m\| \le 1.$$

Applying tools from previous sections, we are led to the following strengthening of Theorem 9 from [18].

Theorem 4.1. *Let C be a nonempty convex weakly compact subset of a Banach space X without the Schur property. If there exists $\varepsilon \in (0,1)$ such that $b_1(1,x) < 1-\varepsilon$ or $d(1,x) > \varepsilon$ for every x in the unit sphere S_X, then every continuous mapping $T : C \to C$ which satisfies condition (C_λ) for some $\lambda \in (0,1)$, has a fixed point. The assumption about the continuity of T can be dropped if T satisfies condition (C).*

Proof. Assume that there exist a nonempty weakly compact convex set $C \subset X$ and a mapping $T : C \to C$ satisfying condition (C) or, a continuous mapping $T : C \to C$ satisfying condition (C_λ) for some λ, without a fixed point. Then, there exists a nonempty weakly compact convex minimal and T-invariant subset $K \subset C$ with $\operatorname{diam} K > 0$. By Lemma 3.1 if T satisfies condition (C) or, by Lemma 3.2 in the other case, there exists an approximate fixed point sequence (x_n) for T in K such that

$$r = \lim_{n\to\infty}\|x_n - x\| = \inf\{r(K,(y_n)) : (y_n) \text{ is an afps in } K\} > 0$$

for every $x \in K$. There is no loss of generality in assuming that (x_n) converges weakly to $0 \in K$. Let $\varepsilon > 0$ and $t = \frac{3}{4}$. Lemma 3.4 yields a subsequence of (x_n), denoted again (x_n), and a sequence (z_n) in K such that

(i) (z_n) is weakly convergent to a point $z \in K$,

and for every $n, m \in \mathbb{N}$

(ii) $\|z_n\| > r(1-\varepsilon)$,
(iii) $\|z_n - z_m\| \le \frac{3}{4}r$,
(iv) $\|z_n - x_n\| < r(\frac{1}{4}+\varepsilon)$.

Then

$$\liminf_{n\to\infty} \|z_n\| \geq r(1-\varepsilon),$$

$$\limsup_{n\to\infty} \|z_n - z\| \leq \limsup_{n\to\infty} \limsup_{m\to\infty} \|z_n - z_m\| \leq \frac{3}{4}r$$

and

$$r(\frac{1}{4}-\varepsilon) \leq \limsup_{n\to\infty} \|z_n\| - \limsup_{n\to\infty} \|z_n - z\| \leq \|z\| \leq \liminf_{n\to\infty} \|z_n - x_n\| \leq r(\frac{1}{4}+\varepsilon). \tag{4.1}$$

Now we largely follow [18, Theorem 9]. Let $u = \frac{z}{\|z\|}$ and $u_n = \frac{4}{3r}(z_n - z)$ for every n. Then $u \in S_X$, (u_n) is weakly null and

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \|u_n - u_m\| = \frac{4}{3r} \limsup_{n\to\infty} \limsup_{m\to\infty} \|z_n - z_m\| \leq 1.$$

We may assume, passing to a subsequence, that $\lim_{n\to\infty} \|u_n + u\|$ exists. Notice that

$$\begin{aligned}\|u_n + u\| &\geq \left\|\frac{4}{3r}(z_n - z) + \frac{4}{r}z\right\| - \left\|\frac{4}{r}z - \frac{z}{\|z\|}\right\| \\ &= \frac{4}{r}\left\|\frac{1}{3}z_n + \frac{2}{3}z\right\| - \left\|\frac{4}{r}\|z\| - 1\right\|,\end{aligned}$$

$$\left\|\frac{1}{3}z_n + \frac{2}{3}z\right\| \geq \|z_n\| - \frac{2}{3}\|z_n - z\|$$

and

$$\left\|\frac{4}{r}\|z\| - 1\right\| \leq 4\varepsilon.$$

Hence

$$\lim_{n\to\infty} \|u_n + u\| \geq \frac{4}{r}\left(r(1-\varepsilon) - \frac{2}{3}\frac{3}{4}r\right) - 4\varepsilon = 2 - 8\varepsilon.$$

It follows that $b_1(1,u) \geq 1 - 8\varepsilon$.

Now consider the weakly null sequence $y_n = \frac{4}{r}(z_n - z - x_n)$. Since

$$\liminf_{n\to\infty} \|y_n\| \geq \frac{4}{r}(\lim_{n\to\infty} \|x_n\| - \limsup_{n\to\infty} \|z_n - z\|) \geq 1,$$

we have

$$\begin{aligned}\limsup_{n\to\infty} \|y_n + u\| &\leq \limsup_{n\to\infty} \left\|y_n + \frac{4}{r}z\right\| + \left\|\frac{z}{\|z\|} - \frac{4}{r}z\right\| \\ &\leq \frac{4}{r}r(\frac{1}{4}+\varepsilon) + 4\varepsilon = 1 + 8\varepsilon.\end{aligned}$$

From [18, Lemma 4] we conclude that also

$$\limsup_{n\to\infty} \|\frac{y_n}{\|y_n\|} + u\| \leq \limsup_{n\to\infty} \|y_n + u\| \leq 1 + 8\varepsilon.$$

Consequently, $d(1,u) \leq 8\varepsilon$ which contradicts our assumption. □

Theorem 4.1 is our main theorem which has several consequences. In [18], the notion of nearly uniformly nonreasy spaces (NUNC, for short) was introduced. Recall that a Banach space X is NUNC if it has the Schur property or, for every $\varepsilon > 0$ there is $t > 0$ such that

$$d(\varepsilon, x) \geq t \quad or \quad b(t, x) \leq \varepsilon t \quad \text{for every } x \in S_X,$$

where

$$b(\varepsilon, x) = \sup\{\liminf_{n\to\infty} \|x + \varepsilon y_n\| - \|x\| : (y_n) \text{ is weakly null in } S_X\}.$$

Corollary 7 in [18] shows that all uniformly noncreasy spaces, introduced earlier by Prus, are NUNC.

Theorem 4.2. *Let C be a nonempty convex weakly compact subset of a nearly uniformly noncreasy Banach space X. Then every continuous mapping $T : C \to C$ which satisfies condition (C_λ) for some $\lambda \in (0, 1)$, has a fixed point. The assumption about the continuity of T can be dropped if T satisfies condition (C).*

Proof. If X has the Schur property, then every weakly compact subset of X is compact in norm. Therefore every continuous mapping $T : C \to C$ which satisfies condition (C_λ) for some $\lambda \in (0, 1)$, has a fixed point. Furthermore, if T satisfies condition (C), the continuity assumption can be dropped by [20, Theorem 2] or [20, Theorem 4].

If X does not have the Schur property, we can argue as in the proof of [18, Corollary 11]. □

Remark 4.3. Notice that Example 6 in [10] shows that the assumption about the continuity of T is necessary for $\lambda > \frac{3}{4}$. The situation is unclear for $\lambda \in (\frac{1}{2}, \frac{3}{4}]$.

Now we will study spaces with $M(X) > 1$. Recall that, for a given $a \geq 0$,

$$R(a, X) = \sup\{\liminf_{n\to\infty} \|y_n + x\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences in the unit ball B_X such that

$$D[(y_n)] = \limsup_{n\to\infty} \limsup_{m\to\infty} \|y_n - y_m\| \leq 1.$$

Notice that in our notation,

$$R(a, X) = \sup_{\|x\|\leq a} (b_1(1, x) + \|x\|). \tag{4.2}$$

The modulus $R(\cdot, X)$ was defined by Domínguez Benavides in [4] as a generalization of the coefficient $R(X)$ introduced by García Falset [8]. He also defined the coefficient

$$M(X) = \sup\left\{\frac{1 + a}{R(a, X)} : a \geq 0\right\}$$

and proved that the condition $M(X) > 1$ implies that X has the weak fixed point property for nonexpansive mappings. We generalize this result to mappings which satisfy condition (C_λ).

The following lemma is an analogue (with a minor correction) of [9, Corollary 4.3 (a), (b), (c)].

Lemma 4.4. *Let X be a Banach space. The following conditions are equivalent:*

(a) $M(X) > 1$,

(b) *there exists* $a > 0$ *such that* $R(a, X) < 1 + a$,

(c) *for every* $a > 0$, $R(a, X) < 1 + a$.

Proof. First prove that $(a) \Rightarrow (b)$. Assume that $M(X) > 1$. Then there exists $a \geq 0$ with $R(a, X) < 1 + a$. If it occurs that $a = 0$ then $R(b, X) \leq R(0, X) + b < 1 + b$ for each $b \geq 0$.

The proof of $(b) \Rightarrow (c)$ follows the arguments from [9]. We will show that if $R(a, X) = 1 + a$ for some $a > 0$, then $R(b, X) = 1 + b$ for all $b > 0$. Let us then suppose that $R(a, X) = 1 + a$ for some $a > 0$ and consider another number $b > 0$. Fix $\eta \in (0, 1)$. Since

$$R(a, X) = 1 + a > 1 + a - \eta \min\{1, a\},$$

there exist $x \in X$ with $\|x\| \leq a$ and a weakly null sequence (x_n) in B_X such that $\limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\| \leq 1$ and

$$\liminf_{n\to\infty} \|x_n + x\| > 1 + a - \eta \min\{1, a\}.$$

For each $n \in \mathbb{N}$, choose a functional $f_n \in S_{X^*}$ with

$$f_n(x_n + x) = \|x_n + x\|.$$

We can assume, passing to a subsequence, that $\lim_{n\to\infty} f_n(x_n)$ exists. Since B_{X^*} is w^*-compact, there exist a directed set $(\mathcal{A}, \preceq)$ and a subnet $(f_{n_\alpha})_{\alpha\in\mathcal{A}}$ of (f_n) which is w^*-convergent to some $f \in B_{X^*}$. Then

$$\lim_\alpha f_{n_\alpha}(x_{n_\alpha} + y) = \lim_\alpha f_{n_\alpha}(x_{n_\alpha}) + \lim_\alpha f_{n_\alpha}(y) = \lim_\alpha f_{n_\alpha}(x_{n_\alpha}) + f(y)$$

for every $y \in X$.

For a fixed $\varepsilon > 0$ find $n_0 \in \mathbb{N}$ such that

$$\|x_n + x\| > \liminf_{n\to\infty} \|x_n + x\| - \varepsilon$$

for every $n \geq n_0$. Then there exists $\alpha \in \mathcal{A}$ such that $n_\beta \geq n_0$ for every $\beta \succeq \alpha$ and consequently, since $\varepsilon > 0$ is arbitrary,

$$\liminf_\alpha \|x_{n_\alpha} + x\| = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\succeq\alpha} \|x_{n_\alpha} + x\| \geq \liminf_{n\to\infty} \|x_n + x\|.$$

Thus

$$\begin{aligned} 1 + a - \eta \min\{1, a\} &< \liminf_{n\to\infty} \|x_n + x\| \leq \liminf_\alpha \|x_{n_\alpha} + x\| \\ &= \lim_\alpha f_{n_\alpha}(x_{n_\alpha} + x) = \lim_\alpha f_{n_\alpha}(x_{n_\alpha}) + f(x). \end{aligned}$$

Since for each $n \geq 1$,

$$f_n(x_n) \leq \|x_n\| \leq 1$$

and

$$f(x) \leq \|x\| \leq a$$

we get

$$\lim_\alpha f_{n_\alpha}(x_{n_\alpha}) > 1 - \eta \min\{1, a\} \geq 1 - \eta$$

and

$$f(x) > a - \eta \min\{1, a\} \geq a(1-\eta).$$

Therefore,

$$\begin{aligned}\liminf_{n\to\infty} \|x_n + \frac{b}{a}x\| &\geq \lim_{n\to\infty} f_n(x_n + \frac{b}{a}x) = \lim_{\alpha} f_{n_\alpha}(x_{n_\alpha} + \frac{b}{a}x) \\ &= \lim_{\alpha} f_{n_\alpha}(x_{n_\alpha}) + \frac{b}{a}f(x) > 1 - \eta + b(1-\eta) = (1+b)(1-\eta).\end{aligned}$$

Hence $R(b, X) \geq (1+b)(1-\eta)$ and, by the arbitrariness of $\eta > 0$, we have $R(b, X) \geq 1 + b$, which gives $(b) \Rightarrow (c)$.

Clearly, $(c) \Rightarrow (a)$, and the lemma follows. □

Theorem 4.1 and Lemma 4.4 give the following corollary.

Theorem 4.5. *Let C be a nonempty convex weakly compact subset of a Banach space X with $M(X) > 1$. Then every mapping $T : C \to C$ which satisfies condition (C) and every continuous mapping $T : C \to C$ which satisfies condition (C_λ) for some $\lambda \in (0, 1)$, has a fixed point.*

Proof. If X has the Schur property and $T : C \to C$ satisfies condition (C), the continuity assumption can be dropped by [20, Theorem 2] as in the proof of Theorem 4.2.

Assume now that X does not have the Schur property and set $\varepsilon = 2 - R(1, X)$. Then, by Lemma 4.4 (c), $\varepsilon \in (0, 1)$. It suffices to notice that from (4.2),

$$b_1(1, x) \leq R(1, X) - 1 = 1 - (2 - R(1, X))$$

for every $x \in S_X$, and apply Theorem 4.1. □

García Falset, Lloréns Fuster and Mazcuñan Navarro [9] introduced another modulus, $RW(a, X)$, which plays an important role in fixed point theory for nonexpansive mappings. Recall that, for a given $a \geq 0$,

$$RW(a, X) = \sup \min\{\liminf_n \|x_n + x\|, \liminf_n \|x_n - x\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences in the unit ball B_X, and,

$$MW(X) = \sup\left\{\frac{1+a}{RW(a, X)} : a \geq 0\right\}.$$

It was proved in [9, Theorem 3.3] that if B_{X^*} is w^*-sequentially compact, then $M(X) \geq MW(X)$. Since B_{X^*} is w^*-sequentially compact if X is separable, we obtain the following corollary.

Corollary 4.6. *Let C be a nonempty convex weakly compact subset of Banach space X with $MW(X) > 1$. Then every mapping $T : C \to C$ which satisfies condition (C) and every continuous mapping $T : C \to C$ which satisfies condition (C_λ) for some $\lambda \in (0, 1)$, has a fixed point.*

Recall that a Banach space X is uniformly nonsquare if

$$J(X) = \sup_{x,y \in S_X} \min \{\|x+y\|, \|x-y\|\} < 2.$$

In [9], a characterization of reflexive Banach spaces with $MW(X) > 1$ is given. In particular (see [9, Corollary 5.1]), all uniformly nonsquare Banach spaces fulfill this condition. Thus we obtain the following corollary which answers Question 1 in [3].

Corollary 4.7. *Let C be a nonempty convex weakly compact subset of a uniformly nonsquare Banach space. Then every mapping $T : C \to C$ which satisfies condition (C) and every continuous mapping $T : C \to C$ which satisfies condition (C_λ) for some $\lambda \in (0, 1)$, has a fixed point.*

Remark 4.8. It is not known whether our results are valid for mappings satisfying property (L) or $(*)$.

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