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A STRONG VERSION OF THE BIRKHOFF–JAMES ORTHOGONALITY IN HILBERT C^*-MODULES

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This paper is dedicated to Professor T. Ando

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Abstract. In this paper we introduce a strong version of the Birkhoff–James orthogonality in Hilbert C^*-modules. More precisely, we consider elements x and y of a Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$ which satisfy $\|x\| \leq \|x + ya\|$ for all $a \in \mathcal{A}$. We show that this relation can be described as the Birkhoff–James orthogonality of appropriate elements of V, and characterized in terms of states acting on the underlying C^*-algebra $\mathcal{A}$. Some analogous relations of this type are considered as well.

1. Introduction and preliminaries

The notion of orthogonality in an arbitrary normed linear space may be introduced in various ways (e.g. see [1, 2]). Among them, the one which is frequently studied in literature is the Birkhoff–James orthogonality [7, 9, 14, 15, 16]: if x, y are elements of a normed linear space X, then x *is orthogonal to* y *in the Birkhoff–James sense*, in short $x \perp y$, if

$$\|x\| \leq \|x + \lambda y\| \quad (\lambda \in \mathbb{C}). \tag{1.1}$$

If X is an inner product space, then the Birkhoff–James orthogonality is equivalent to the usual orthogonality given by the inner product. It is easy to see that the Birkhoff–James orthogonality is nondegenerate ($x \perp x$ if and only if $x = 0$), homogenous ($x \perp y \Rightarrow (\lambda x \perp \mu y$ for all $\lambda, \mu \in \mathbb{C})$), not symmetric ($x \perp y$ need not imply $y \perp x$), and not additive ($x \perp y$ and $x \perp z$ need not imply

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$x \perp (y+z)$). Also, for every $x, y \in X$ there is $\lambda \in \mathbb{C}$ such that $x \perp (\lambda x + y)$. By the Hahn–Banach theorem, if x, y are two elements of a normed linear space X, then $x \perp y$ if and only if there is a norm one linear functional f of X such that $f(x) = \|x\|$ and $f(y) = 0$. Characterizations of the Birkhoff–James orthogonality in C^*-algebras and Hilbert C^*-modules were obtained in several papers such as [5] and [6].

In Hilbert C^*-modules the role of scalars is played by the elements of the underlying C^*-algebra. So, it is natural to generalize the notion of the Birkhoff–James orthogonality in the following way. Instead of (1.1), one can consider elements x and y of a given Hilbert $\mathcal{A}$-module V satisfying

$$\|x\| \le \|x + ya\| \quad (a \in \mathcal{A}). \tag{1.2}$$

Evidently, the condition (1.2) is stronger than (1.1), and weaker than the orthogonality with respect to the inner product.

In the second section we study the relation (1.2). We show in Theorem 2.5 that x and y satisfy (1.2) if and only if x is orthogonal to $y\langle y, x\rangle$ in the Birkhoff–James sense, which enables us to apply some results of [5] to characterize (1.2) in terms of the states acting on the underlying C^*-algebra. In particular, we consider (1.2) for elements of Hilbert $\mathbf{K}(H)$-modules (Proposition 2.10), as well as for elements of the C^*-algebra $\mathbf{B}(H)$ (Proposition 2.8).

The concluding Section 3 discusses some other possible generalizations of (1.1) which are natural in Hilbert C^*-modules. However, it turns out that most of them just describe the orthogonality with respect to the inner product.

Before stating our results, let us recall some basic facts about C^*-algebras and Hilbert C^*-modules and introduce our notation.

Throughout, $\mathbf{B}(H, K)$ stands for the linear space of all bounded linear operators between Hilbert spaces H and K. When $H = K$, we write $\mathbf{B}(H)$. By $\mathbf{K}(H)$ we denote the algebra of all compact operators on H, and by $\mathbf{T}(H)$ the algebra of all trace-class operators on H. For $A \in \mathbf{B}(H, K)$ the symbol $\|A\|$ denotes the operator norm of A. $\operatorname{Ker} A$ stands for the kernel of A. By I we denote the identity operator on H. By $\operatorname{tr}(A)$ we denote the trace of $A \in \mathbf{T}(H)$. The algebra of all complex $n \times n$ matrices is denoted by $\mathbb{M}_n(\mathbb{C})$. We shall identify $\mathbf{B}(\mathbb{C}^n)$ and $\mathbb{M}_n(\mathbb{C})$ in the usual way.

A positive element a of a C^*-algebra $\mathcal{A}$ is a self-adjoint element whose spectrum $\sigma(a)$ is contained in $[0, \infty)$. If $a \in \mathcal{A}$ is positive, we write $a \ge 0$. A partial order may be introduced on the set of self-adjoint elements of a C^*-algebra $\mathcal{A}$: if a and b are self-adjoint elements of $\mathcal{A}$ such that $a - b \ge 0$, we write $a \ge b$ or $b \le a$. If $a \ge 0$, then there exists a unique positive $b \in \mathcal{A}$ such that $a = b^2$; such an element b, denoted by $a^{\frac{1}{2}}$, is called the positive square root of a. If $0 \le a \le b$ then $0 \le a^{\frac{1}{2}} \le b^{\frac{1}{2}}$. The converse does not hold in general, but it holds in commutative C^*-algebras. Also, if $0 \le a \le b$ then $0 \le c^*ac \le c^*bc$ for all $c \in \mathcal{A}$.

An approximate unit for a C^*-algebra $\mathcal{A}$ is an increasing net $(e_i)_{i \in I}$ of positive elements in the closed unit ball of $\mathcal{A}$ such that $\lim_{i \in I} \|a - ae_i\| = 0$ for all $a \in \mathcal{A}$, or equivalently $\lim_{i \in I} \|a - e_i a\| = 0$ for all a in $\mathcal{A}$.

A linear functional φ of $\mathcal{A}$ is positive if $\varphi(a) \geq 0$ for every positive element $a \in \mathcal{A}$. A state is a positive linear functional whose norm is equal to one. The numerical range of $a \in \mathcal{A}$, denoted by $V(a)$, is the set of all $\varphi(a)$, where φ ranges over the states of $\mathcal{A}$. The center of $\mathcal{A}$ is denoted by $\mathcal{Z}(\mathcal{A})$. General references for the theory of C^*-algebras are [10, 20].

A (*right*) *Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$* (or a (*right*) *Hilbert $\mathcal{A}$-module*) is a linear space which is a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner-product $\langle \cdot\,,\,\cdot\rangle : V \times V \to \mathcal{A}$ that is sesquilinear, positive definite and respects the module action, i.e.,

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in V$, $\alpha, \beta \in \mathbb{C}$,
(2) $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in V$, $a \in \mathcal{A}$,
(3) $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in V$,
(4) $\langle x, x\rangle \geq 0$ for $x \in V$; if $\langle x, x\rangle = 0$ then $x = 0$,

and such that V is complete with respect to the norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$, $x \in V$. In fact, for every $x, y \in V$ it holds $\langle y, x\rangle\langle x, y\rangle \leq \|x\|^2\langle y, y\rangle$, wherefrom $\|\langle x, y\rangle\| \leq \|x\|\|y\|$.

Obviously, every Hilbert space is a Hilbert $\mathbb{C}$-module. Also, every C^*-algebra $\mathcal{A}$ can be regarded as a Hilbert C^*-module over itself with the inner product $\langle a, b\rangle := a^*b$, and the corresponding norm is just the norm on $\mathcal{A}$ because of the C^*-condition. (For details about Hilbert C^*-modules we refer the reader to [18, 19, 21, 24].)

If x is an element of a Hilbert $\mathcal{A}$-module V, $|x| \in \mathcal{A}$ denotes the unique positive square root of $\langle x, x\rangle \in \mathcal{A}$. In the case of a C^*-algebra we get the usual $|a| = (a^*a)^{1/2}$. Although the definition of $|x|$ has the same form as that of the norm of an element of an inner product space, there are some significant differences. For example, it is well known that the C^*-valued triangle inequality $|x+y| \leq |x| + |y|$ for elements x and y of a Hilbert C^*-module need not hold (see [12]). Actually, it was recently proved in [17] that the C^*-valued triangle inequality holds for every two elements of V if and only if $\mathcal{A}$ is commutative. The case of equality in triangle inequality was characterized in [3] for elements of $\mathbf{B}(H)$, and in [4] for elements of Hilbert C^*-modules.

In a Hilbert $\mathcal{A}$-module V we have the following version of the Cauchy–Schwarz inequality:

$$|\varphi(\langle x, y\rangle)|^2 \leq \varphi(\langle x, x\rangle)\varphi(\langle y, y\rangle) \quad (x, y \in V),$$

where φ is a positive linear functional of $\mathcal{A}$.

2. Properties and characterizations of the strong Birkhoff–James orthogonality

As we have already mentioned, for two elements x, y of a normed linear space X, it holds $x \perp y$ if and only if there is a norm one linear functional f of X such that $f(x) = \|x\|$ and $f(y) = 0$. If we have additional structures on a normed linear space X, then we obtain other characterizations of the Birkhoff–James orthogonality. One of the first results of this form is the result obtained by Bhatia and Šemrl [8] for the C^*-algebra $\mathbf{B}(H)$ of all bounded linear operators on a Hilbert space H. The following result is the content of Theorem 1.1 and Remark 3.1 of [8].

Theorem 2.1. *Let $A, B \in \mathbf{B}(H)$.*

(a) *If $\dim H < \infty$, then $A \perp B$ if and only if there is a unit vector $\xi \in H$ such that $\|A\xi\| = \|A\|$ and $(A\xi, B\xi) = 0$.*

(b) *If $\dim H = \infty$, then $A \perp B$ if and only if there is a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty}(A\xi_n, B\xi_n) = 0$.*

The characterization of the Birkhoff–James orthogonality for elements of a Hilbert C^*-module by means of the states of the underlying C^*-algebra was obtained in [5]. The following result is Theorem 2.7 of [5]. (The same result is later obtained in [6] by using a different approach.)

Theorem 2.2. *Let V be a Hilbert $\mathcal{A}$-module, and $x, y \in V$. Then $x \perp y$ if and only if there is a state φ of $\mathcal{A}$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle) = 0$.*

We now introduce a new type of orthogonality in Hilbert C^*-modules.

Definition 2.3. An element x of a Hilbert $\mathcal{A}$-module V is *strongly Birkhoff–James orthogonal to an element $y \in V$*, in short $x \perp_* y$, if

$$\|x\| \leq \|x + ya\| \quad (a \in \mathcal{A}).$$

For every $x, y \in V$ it holds

$$\langle x, y\rangle = 0 \Rightarrow x \perp_* y \Rightarrow x \perp y. \tag{2.1}$$

Indeed, if $\langle x, y\rangle = 0$, then for all $a \in \mathcal{A}$ we have

$$\|x + ya\|^2 = \|\langle x + ya, x + ya\rangle\| = \|\langle x, x\rangle + \langle ya, ya\rangle\| \geq \|\langle x, x\rangle\| = \|x\|^2,$$

i.e., $x \perp_* y$. Further, if $x \perp_* y$, then for every $\lambda \in \mathbb{C}$ we have $\|x\| \leq \|x + \lambda y e_i\|$, $i \in I$, where $(e_i)_{i\in I}$ is an approximate unit for $\mathcal{A}$. Since $\lim_{i\in I} \|ye_i - y\| = 0$ ([19], p. 5), we get $x \perp y$.

The converses in (2.1) do not hold in general, as shown in the following example.

Example 2.4. Let us take $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$, regarded as a Hilbert C^*-module over itself.

(a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $I \perp A$ since

$$\|I + \lambda A\| = \left\| \begin{bmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \right\| = \max\{|1+\lambda|, |1-\lambda|\} \geq \|I\|$$

for all $\lambda \in \mathbb{C}$. But $I \not\perp_* A$ since for $B = -A$ we have $\|I + AB\| = 0 < \|I\|$.

(b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. For any $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ we have

$$\|I + AB\| = \left\| \begin{bmatrix} 1+b_1 & b_2 \\ 0 & 1 \end{bmatrix} \right\| \geq 1 = \|I\|.$$

Therefore $I \perp_* A$, but $\langle I, A\rangle = A \neq 0$.

In the next theorem we obtain some characterizations of the strong Birkhoff–James orthogonality. First observe that $x \perp_* y$ is equivalent to $\|x\| \leq \|x + \lambda ya\|$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, that is

$$x \perp_* y \Leftrightarrow (x \perp ya \text{ for all } a \in \mathcal{A}). \tag{2.2}$$

By Theorem 2.2, it means that $x \perp_* y$ if and only if for every $a \in \mathcal{A}$ there is a state φ_a of $\mathcal{A}$ such that $\varphi_a(\langle x, x\rangle) = \|x\|^2$ and $\varphi_a(\langle x, y\rangle a) = 0$. It turns out that this can be accomplished with a single state φ.

Theorem 2.5. *Let V be a Hilbert $\mathcal{A}$-module, and $x, y \in V$. The following statements are mutually equivalent*:

(a) $x \perp_* y$;
(b) $x \perp y\langle y, x\rangle$;
(c) *there is a state φ of $\mathcal{A}$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle\langle y, x\rangle) = 0$;*
(d) *there is a state φ of $\mathcal{A}$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle a) = 0$ for all $a \in \mathcal{A}$.*

Proof. It follows from (2.2) that (a)⇒(b), and from Theorem 2.2 we have (b)⇔(c). To prove (c)⇒(d), it is enough to notice that for every $a \in \mathcal{A}$, by the Cauchy–Schwarz inequality, we have

$$|\varphi(\langle x, y\rangle a)|^2 \leq \varphi(\langle x, y\rangle\langle y, x\rangle)\varphi(a^*a).$$

The implication (d)⇒(a) follows from (2.2) and Theorem 2.2. □

Next we discuss some properties of the relation $\perp_*$. We show that, as in the case of the classical Birkhoff–James orthogonality (see [5, Theorem 2.9 (a)]), for any two elements x and y of a Hilbert $\mathcal{A}$-module V, the relation $x \perp_* y$ can be described by means of the orthogonality of appropriate elements of the underlying C^*-algebra $\mathcal{A}$.

Proposition 2.6. *Let V be a Hilbert $\mathcal{A}$-module, and $x, y \in V$.*

(a) $x \perp_* y \Leftrightarrow \langle x, x\rangle \perp_* \langle x, y\rangle$.
(b) *If $\mathcal{A}$ is unital and $x \perp_* y$, then $\langle x, y\rangle$ does not have a right inverse in $\mathcal{A}$.*
(c) $x \perp_* y \Leftrightarrow$ *($x \perp_* ya$ for all $a \in \mathcal{A}$) $\Leftrightarrow$ ($x \perp ya$ for all $a \in \mathcal{A}$).*

Proof. (a) If $x \perp_* y$, then there is a state φ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi\big(\langle x, y\rangle a\big) = 0$ for all $a \in \mathcal{A}$. Then for every $a \in \mathcal{A}$ it holds

$$\|\langle x, x\rangle\| = \|x\|^2 = |\varphi(\langle x, x\rangle + \langle x, y\rangle a)| \leq \|\langle x, x\rangle + \langle x, y\rangle a\|,$$

so $\langle x, x\rangle \perp_* \langle x, y\rangle$.

Conversely, if $\langle x, x\rangle \perp_* \langle x, y\rangle$ then $\|\langle x, x\rangle\| \leq \|\langle x, x\rangle + \langle x, y\rangle a\|$ for all $a \in \mathcal{A}$, that is, $\|x\|^2 \leq \|\langle x, x + ya\rangle\| \leq \|x\|\|x + ya\|$ for all $a \in \mathcal{A}$. It follows that $x \perp_* y$.

(b) Let e be the unit of $\mathcal{A}$. If $x \perp_* y$, then there is a state φ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle a) = 0$ for all $a \in \mathcal{A}$. Suppose that $\langle x, y\rangle$ has a right inverse $b \in \mathcal{A}$. Then for $a = b$ we have $\varphi(e) = \varphi(\langle x, y\rangle b) = 0$, which is not possible.

(c) By using an approximate unit for $\mathcal{A}$, it is easy to prove the first equivalence. We have already noticed that $x \perp_* y \Leftrightarrow (x \perp ya, \forall a \in \mathcal{A})$. □

The converse in part (b) of the previous proposition does not hold: for a counterexample we can take any $x \neq 0$ such that $\langle x, x\rangle$ is noninvertible.

Remark 2.7. Let $\mathcal{A}$ be a unital C^*-algebra with the unit e, regarded as a Hilbert C^*-module over itself.

(a) If $a \in \mathcal{A}$ is such that aa^* is not invertible then, by Theorem 1 of [22], $0 \in \sigma(aa^*) \subseteq V(aa^*)$. Hence, there is a state φ of $\mathcal{A}$ such that $\varphi(aa^*) = 0$. By Theorem 2.5 (c) we conclude that $e \perp_* a$. When $a \in \mathcal{A}$ is noninvertible, then at least one of the elements aa^* and a^*a is noninvertible. Thus, $e \perp_* a$ or $e \perp_* a^*$ for every noninvertible $a \in \mathcal{A}$. In particular, $e \perp_* a$ for every self-adjoint noninvertible $a \in \mathcal{A}$.

(b) The relation $\perp_*$ is not additive. Indeed, let $a \in \mathcal{A}$ be a nonzero noninvertible positive element. Then $\|a\|e - a \in \mathcal{A}$ is also a nonzero noninvertible positive element, so by (a) we have $e \perp_* a$ and $e \perp_* (\|a\|e - a)$, but $e \not\perp_* (a + (\|a\|e - a))$.

It is also nonsymmetric. Namely, by Theorem 2.5, $a \perp_* e \Leftrightarrow a \perp e\langle e, a\rangle \Leftrightarrow a \perp a \Leftrightarrow a = 0$ while, by (a), $e \perp_* a$ for every noninvertible self-adjoint element $a \in \mathcal{A}$.

By combining Theorem 2.1 and Theorem 2.5 we obtain the following result.

Proposition 2.8. *For every $A, B \in \mathbf{B}(H)$ the following statements hold.*

(a) *If $\dim H < \infty$, then $A \perp_* B$ if and only if there is a unit vector $\xi \in H$ such that $\|A\xi\| = \|A\|$ and $B^*A\xi = 0$.*

(b) *If $\dim H = \infty$, then $A \perp_* B$ if and only if there is a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty} B^*A\xi_n = 0$.*

In particular, for a nonzero positive operator $A \in \mathbf{B}(H)$ the following statements hold.

(c) *If $\dim H < \infty$, then $A \perp_* B$ if and only if there is a unit vector $\xi \in H$ such that $A\xi = \|A\|\xi$ and $B^*\xi = 0$.*

(d) *If $\dim H = \infty$, then $A \perp_* B$ if and only if there is a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n\to\infty}(A\xi_n - \|A\|\xi_n) = 0$ and $\lim_{n\to\infty} B^*\xi_n = 0$.*

Proof. By Theorem 2.5, $A \perp_* B \Leftrightarrow A \perp B\langle B, A\rangle = BB^*A$.

Let $\dim H < \infty$. By Theorem 2.1 (a), $A \perp BB^*A$ if and only if there exists a unit vector $\xi \in H$ such that $\|A\xi\| = \|A\|$ and $(A\xi, BB^*A\xi) = 0$. Since $\|B^*A\xi\|^2 = (A\xi, BB^*A\xi)$, the statement (a) is proved. Using Theorem 2.1 (b), we can similarly prove the statement (b).

Suppose A is nonzero and positive. We shall use [23, Lemma 2.1] which says that, whenever (ξ_n) is a sequence of unit vectors in H such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$, then $\lim_{n\to\infty}(A\xi_n - \|A\|\xi_n) = 0$. In particular, if ξ is a unit vector in H such that $\|A\xi\| = \|A\|$, then $A\xi = \|A\|\xi$.

When $\dim H < \infty$ we have proved that $A \perp_* B$ if and only if $\|A\xi\| = \|A\|$ and $B^*A\xi = 0$ for some unit vector $\xi \in H$. Since $A \geq 0$, $\|A\xi\| = \|A\| \Leftrightarrow A\xi = \|A\|\xi$. Therefore, $B^*A\xi = 0 \Leftrightarrow B^*\xi = 0$, as $A \neq 0$.

If $\dim H = \infty$ then, $A \perp_* B$ if and only if there is a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty} B^*A\xi_n = 0$. The first condition is equivalent to $\lim_{n\to\infty}(A\xi_n - \|A\|\xi_n) = 0$. If $\lim_{n\to\infty} B^*A\xi_n = 0$ then

$$\lim_{n\to\infty} B^*\xi_n = -\|A\|^{-1}B^*(\lim_{n\to\infty}(A\xi_n - \|A\|\xi_n)) = 0,$$

and if $\lim_{n\to\infty} B^*\xi_n = 0$ then

$$\lim_{n\to\infty} B^*A\xi_n = B^*(\lim_{n\to\infty}(A\xi_n - \|A\|\xi_n)) = 0,$$

so the second conditions are also equivalent. □

If A and B are elements of (the Hilbert $\mathbf{B}(H)$-module) $\mathbf{B}(H,K)$ then, by Proposition 2.6 (a), $A \perp_* B$ if and only if $A^*A \perp_* A^*B$. Since $A^*A, A^*B \in \mathbf{B}(H)$ and A^*A is positive, we may apply Proposition 2.8 to obtain the following result.

Corollary 2.9. *Let $A, B \in \mathbf{B}(H,K)$.*

(a) *If $\dim H < \infty$, then $A \perp_* B$ if and only if there is a unit vector $\xi \in H$ such that $A^*A\xi = \|A\|^2\xi$ and $B^*A\xi = 0$.*
(b) *If $\dim H = \infty$, then $A \perp_* B$ if and only if there is a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n\to\infty}(A^*A\xi_n - \|A\|^2\xi_n) = 0$ and $\lim_{n\to\infty} B^*A\xi_n = 0$.*

In the case of Hilbert C^*-modules over the C^*-algebra $\mathbf{K}(H)$ of compact operators, Theorem 2.2 and Theorem 2.5 can be formulated in the following ways.

Proposition 2.10. *Let V be a Hilbert $\mathbf{K}(H)$-module, and $x, y \in V$.*

(a) *$x \perp y$ if and only if there is a positive trace one operator $p \in \mathbf{T}(H)$ such that $\mathrm{tr}(p\langle x,x\rangle) = \|x\|^2$ and $\mathrm{tr}(p\langle x,y\rangle) = 0$.*
(b) *$x \perp_* y$ if and only if there is a positive trace one operator $p \in \mathbf{T}(H)$ such that $\mathrm{tr}(p\langle x,x\rangle) = \|x\|^2$ and $p\langle x,y\rangle = 0$.*

Proof. (a) This follows from Theorem 2.2 and the fact that every state φ of $\mathbf{K}(H)$ is of the form $a \mapsto \mathrm{tr}(pa)$ for some positive trace one operator $p \in \mathbf{T}(H)$ (see e.g. [20, Theorem 4.2.1]).

(b) If $x \perp_* y$ then, by Theorem 2.5, there is a state $\varphi : \mathbf{K}(H) \to \mathbb{C}$ such that $\varphi(\langle x,x\rangle) = \|x\|^2$ and $\varphi(\langle x,y\rangle a) = 0$ for all $a \in \mathbf{K}(H)$. Let $p \in \mathbf{T}(H)$ be a positive trace one operator such that $\varphi(a) = \mathrm{tr}(pa)$, $a \in \mathbf{K}(H)$. Then we have $\mathrm{tr}(p\langle x,x\rangle) = \|x\|^2$ and $\mathrm{tr}(p\langle x,y\rangle a) = 0$ for all $a \in \mathbf{K}(H)$. For $a = \langle y,x\rangle p$ we obtain $\mathrm{tr}(p\langle x,y\rangle\langle y,x\rangle p) = 0$. Since $p\langle x,y\rangle\langle y,x\rangle p$ is a positive operator with zero trace, it has to be 0, and then $p\langle x,y\rangle = 0$.

Conversely, suppose that there is a positive trace one operator $p \in \mathbf{T}(H)$ such that $p\langle x,y\rangle = 0$ and $\mathrm{tr}(p\langle x,x\rangle) = \|x\|^2$. Since p is a positive trace one operator, one can define a state φ on $\mathbf{K}(H)$ by setting $\varphi(a) = \mathrm{tr}(pa)$, $a \in \mathbf{K}(H)$. Then we have $\varphi(\langle x,x\rangle) = \mathrm{tr}(p\langle x,x\rangle) = \|x\|^2$ and $\varphi(\langle x,y\rangle\langle y,x\rangle) = \mathrm{tr}(p\langle x,y\rangle\langle y,x\rangle) = 0$. By Theorem 2.5 we deduce that $x \perp_* y$. □

3. On the inner product orthogonality in Hilbert C^*-modules

Our motivation for Definition 2.3 was the fact that Hilbert C^*-modules generalize Hilbert spaces in the sense that inner products take values in arbitrary C^*-algebras instead of $\mathbb{C}$. There are other logical ways how to generalize (1.1). For example, we can replace the norm $\|\cdot\|$ in (1.1) and in (1.2) by the "C^*-valued norm" $|\cdot|$ defined by $|x| = \langle x,x\rangle^{\frac{1}{2}}$. Since the function $t \mapsto t^2$ is not operator monotone, we have to differ cases $|x|^2 \le |x+ya|^2$ and $|x| \le |x+ya|$, respectively, $|x|^2 \le |x+\lambda y|^2$ and $|x| \le |x+\lambda y|$. However, as it is proved in the next theorem, most of these generalizations are too strong in the sense that they coincide with $\langle x,y\rangle = 0$. The only relation for which we do not know whether it in general coincides with $\langle x,y\rangle = 0$ is that defined by $|x| \le |x+\lambda y|$ for all $\lambda \in \mathbb{C}$.

Theorem 3.1. *Let V be a Hilbert $\mathcal{A}$-module, and $x,y \in V$. The following statements are mutually equivalent*:

(a) $\langle x,y\rangle = 0$;
(b) $|x|^2 \le |x+ya|^2$ *for all* $a \in \mathcal{A}$;
(c) $|x|^2 \le |x+\lambda y|^2$ *for all* $\lambda \in \mathbb{C}$;
(d) $|x| \le |x+ya|$ *for all* $a \in \mathcal{A}$.

Proof. The equivalence of (a) and (b) was proved in [13, Proposition 2.1]. By Theorem 2.2.6 of [20], (b) implies (d).

(d)⇒(a) This is obvious for $y = 0$, so suppose that $y \neq 0$. Let $a_0 = -\frac{1}{\|y\|^2}\langle y,x\rangle \in \mathcal{A}$. Then

$$\begin{aligned} |x+ya_0|^2 &= \left| x - \frac{1}{\|y\|^2} y\langle y,x\rangle \right|^2 \\ &= |x|^2 - \frac{2}{\|y\|^2}\langle x,y\rangle\langle y,x\rangle + \frac{1}{\|y\|^4}\langle x,y\rangle\langle y,y\rangle\langle y,x\rangle \\ &\le |x|^2 - \frac{2}{\|y\|^2}\langle x,y\rangle\langle y,x\rangle + \frac{1}{\|y\|^2}\langle x,y\rangle\langle y,x\rangle \\ &= |x|^2 - \frac{1}{\|y\|^2}\langle x,y\rangle\langle y,x\rangle \\ &\le |x|^2. \end{aligned} \tag{3.1}$$

It follows that $|x+ya_0| \le |x|$ which, together with (d), gives $|x+ya_0| = |x|$, and then $|x+ya_0|^2 = |x|^2$. In particular, we have equality in (3.1), so $\langle x,y\rangle\langle y,x\rangle = 0$, and therefore $\langle x,y\rangle = 0$.

(b)⇒(c) If $(e_i)_{i\in I}$ is an approximate unit in $\mathcal{A}$ then $|x|^2 \le |x+\lambda y e_i|^2$ for all $\lambda \in \mathbb{C}$ and $i \in I$. Then we have

$$\begin{aligned} \lim_{i\in I} |x+\lambda y e_i|^2 &= \lim_{i\in I}(\langle x,x\rangle + \lambda\langle x,y\rangle e_i + \overline{\lambda} e_i\langle y,x\rangle + |\lambda|^2 e_i\langle y,y\rangle e_i) \\ &= \langle x,x\rangle + \lambda\langle x,y\rangle + \overline{\lambda}\langle y,x\rangle + |\lambda|^2\langle y,y\rangle \\ &= |x+\lambda y|^2, \end{aligned}$$

for all $\lambda \in \mathbb{C}$, and we get (c).

(c)⇒(a) If (c) holds, then

$$\lambda\langle x,y\rangle + \overline{\lambda}\langle y,x\rangle + |\lambda|^2\langle y,y\rangle \ge 0 \quad (\lambda \in \mathbb{C}). \tag{3.2}$$

In particular, choosing real λ's we get

$$\lambda\langle x,y\rangle + \lambda\langle y,x\rangle + \lambda^2\langle y,y\rangle \ge 0 \quad (\lambda \in \mathbb{R}),$$

that is,

$$\langle x,y\rangle + \langle y,x\rangle + \lambda\langle y,y\rangle \ge 0 \quad (\lambda > 0), \tag{3.3}$$

and

$$\langle x, y\rangle + \langle y, x\rangle + \lambda\langle y, y\rangle \le 0 \quad (\lambda < 0). \tag{3.4}$$

Taking $\lim_{\lambda\to 0+}$ in (3.3) and $\lim_{\lambda\to 0-}$ in (3.4) we obtain

$$\langle x, y\rangle + \langle y, x\rangle = 0. \tag{3.5}$$

Putting $i\lambda$, $\lambda \in \mathbb{R}$, in (3.2) we get $i\lambda\langle x, y\rangle - i\lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \ge 0$ for all $\lambda \in \mathbb{R}$, and, as above,

$$\langle x, y\rangle - \langle y, x\rangle = 0. \tag{3.6}$$

Note that (3.5) and (3.6) yield $\langle x, y\rangle = 0$, so (a) holds. □

If a and b are positive commuting elements of a C^*-algebra $\mathcal{A}$, then $a \le b \Leftrightarrow a^2 \le b^2$. Thus, as a consequence of Theorem 3.1, the following result immediately follows.

Corollary 3.2. *Let V be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, and $x, y \in V$ such that $|x| \in \mathcal{Z}(\mathcal{A})$. Then $\langle x, y\rangle = 0$ if and only if $|x| \le |x + \lambda y|$ for all $\lambda \in \mathbb{C}$.*

Though we could not prove that Corollary 3.2 holds without the assumption that $|x| \in \mathcal{Z}(\mathcal{A})$, it seems very likely that it does. The following results are in favor of this assumption.

Proposition 3.3. *Let V be a Hilbert $\mathbb{M}_n(\mathbb{C})$-module, and $x, y \in V$ such that $|x| \le |x + \lambda y|$ for all $\lambda \in \mathbb{C}$. Then $\mathrm{tr}(\langle x, y\rangle) = 0$. In particular, if moreover $\langle x, y\rangle \ge 0$ then $\langle x, y\rangle = 0$.*

Proof. Denote by $\mu_i(T)$, $i = 1, \dots, n$, the eigenvalues of a self-adjoint matrix $T \in \mathbb{M}_n(\mathbb{C})$ arranged in decreasing order: $\mu_1(T) \ge \cdots \ge \mu_n(T)$.

Since $|x| \le |x + \lambda y|$ for all $\lambda \in \mathbb{C}$, by the Courant–Fischer theorem (see [25, Theorem 8.9]), we obtain $\mu_i(|x|) \le \mu_i(|x + \lambda y|)$, $i = 1, \dots, n$, for all $\lambda \in \mathbb{C}$. Thus we have $\mu_i(|x|^2) \le \mu_i(|x + \lambda y|^2)$, $i = 1, \dots, n$, for all $\lambda \in \mathbb{C}$. By Theorem 8.20 of [25], for every $\lambda \in \mathbb{C}$ there is a unitary matrix $u_\lambda \in \mathbb{M}_n(\mathbb{C})$ such that $|x|^2 \le u_\lambda^*|x+\lambda y|^2 u_\lambda$. Therefore, $\mathrm{tr}(|x|^2) \le \mathrm{tr}(|x+\lambda y|^2)$ for every $\lambda \in \mathbb{C}$, from which (similarly as in the proof of (c)⇒(a) in Theorem 3.1) it follows that $\mathrm{tr}(\langle x, y\rangle) = 0$. □

Lemma 3.4. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ satisfy $|A| \le |A + \lambda B|$ for all $\lambda \in \mathbb{C}$. If A is invertible, then $\sigma(A^{-1}B) = \{0\}$, and if B is invertible then $\sigma(B^{-1}A) = \{0\}$. In particular, either A or B is noninvertible.*

Proof. First observe that $\mathrm{Ker}\,(A + \lambda B) = \mathrm{Ker}\,A \cap \mathrm{Ker}\,B$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. Indeed, let $\lambda \ne 0$ and $\xi \in \mathrm{Ker}\,(A + \lambda B)$. From $|A| \le |A + \lambda B|$ it follows that $\||A|^{\frac{1}{2}}\xi\|^2 = (|A|\xi, \xi) \le (|A + \lambda B|\xi, \xi) = 0$, so $|A|^{\frac{1}{2}}\xi = 0$ and then $A\xi = 0$. Since $\lambda \ne 0$ and $(A + \lambda B)\xi = 0$ we get $B\xi = 0$. This proves that $\mathrm{Ker}\,(A + \lambda B) \subseteq \mathrm{Ker}\,A \cap \mathrm{Ker}\,B$. The reverse inclusion is obvious.

If A (resp. B) is invertible, then so is $A + \lambda B$ for all $\lambda \ne 0$, and also $A^{-1}(A + \lambda B) = I + \lambda A^{-1}B$ (resp. $B^{-1}(A + \lambda B) = B^{-1}A + \lambda I$) for all $\lambda \ne 0$. Then $\sigma(A^{-1}B) = \{0\}$ (resp. $\sigma(B^{-1}A) = \{0\}$). In particular, $A^{-1}B$ (resp. $B^{-1}A$) is noninvertible, wherefrom B (resp. A) is noninvertible. □

Observe that in the case of matrices we can restrict our discussion to positive A. Indeed, for $A \in \mathbb{M}_n(\mathbb{C})$ we have $A = U|A|$ for some unitary $U \in \mathbb{M}_n(\mathbb{C})$, so $|A + \lambda B| = ||A| + \lambda U^*B|$ and $A^*B = |A|(U^*B)$. We may also assume that $\|A\| = 1$.

We conclude the paper with the theorem which states that Corollary 3.2 holds in $\mathbb{M}_2(\mathbb{C})$ without assuming that $|x| \in \mathcal{Z}(\mathcal{A})$. Although the calculation is elementary, we include it for the convenience of the reader.

Recall that for $A \in \mathbb{M}_2(\mathbb{C})$ we have

$$|A| = \frac{1}{\sqrt{\mathrm{tr}\,(A^*A) + 2\sqrt{\det(A^*A)}}}\left(\sqrt{\det(A^*A)}I + A^*A\right),$$

(see e.g. [11, p. 460]).

Theorem 3.5. *If $A, B \in \mathbb{M}_2(\mathbb{C})$ are such that*

$$|A| \le |A + \lambda B| \quad (\lambda \in \mathbb{C}), \tag{3.7}$$

then $A^*B = 0$.

Proof. If $A = 0$ or $B = 0$, we are done. So suppose that A and B are nonzero. As we have already mentioned, we may assume that $A \ge 0$ and $\|A\| = 1$.

Since (3.7) implies $\|A\| \le \|A + \lambda B\|$ for all $\lambda \in \mathbb{C}$, by Theorem 2.1 there is a unit vector $\xi \in \mathbb{C}^2$ such that $\|A\xi\| = 1$ and $(A\xi, B\xi) = 0$. Since $A \ge 0$ it follows that $A\xi = \xi$ and $(B\xi, \xi) = 0$. Let $\sigma(A) = \{1, a\}$, and $\eta \in \mathbb{C}^2$, $\|\eta\| = 1$, be such that $A\eta = a\eta$ and $(\xi, \eta) = 0$. Let $U \in \mathbb{M}_2(\mathbb{C})$ be the unitary matrix which maps the standard orthonormal basis of $\mathbb{C}^2$ to the orthonormal basis $\{\xi, \eta\}$. Then we have

$$U^*AU = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \quad U^*BU = \begin{bmatrix} 0 & b_1 \\ b_2 & b_3 \end{bmatrix}.$$

Since (3.7) is equivalent with $|U^*AU| \le |U^*AU + \lambda U^*BU|$ for all $\lambda \in \mathbb{C}$, and $A^*B = 0$ if and only if $(U^*AU)^*(U^*BU) = 0$, without loss of generality we can assume that A and B are of the forms

$$A = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b_1 \\ b_2 & b_3 \end{bmatrix}.$$

We differ three cases: A is invertible, B is invertible, and both A and B are noninvertible.

(1) Suppose A is invertible. Then $a \ne 0$. From Proposition 3.3, $b_3 = \frac{1}{a}\,\mathrm{tr}(A^*B) = 0$. By Lemma 3.4, B is noninvertible, and therefore $b_1 = 0$ or $b_2 = 0$. So, the only candidates for B are $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$ for some $b \ne 0$. We shall now prove that they do not satisfy (3.7). We may assume that $b = 1$.

If $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$|A + \lambda B| = \frac{1}{\sqrt{|\lambda|^2 + (a+1)^2}}\begin{bmatrix} a+1 & \lambda \\ \overline{\lambda} & a + a^2 + |\lambda|^2 \end{bmatrix}.$$

By (3.7) we have $1 \le \frac{a+1}{\sqrt{|\lambda|^2 + (a+1)^2}}$, which is impossible for $\lambda \ne 0$.

If $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then

$$|A + \lambda B| = \frac{1}{\sqrt{|\lambda|^2 + (a+1)^2}} \begin{bmatrix} a + 1 + |\lambda|^2 & \overline{\lambda} a \\ \lambda a & a^2 + a \end{bmatrix}.$$

By (3.7) we get $a \leq \frac{a^2+a}{\sqrt{|\lambda|^2+(a+1)^2}}$, which is impossible for $\lambda \neq 0$.

(2) Suppose B is invertible. Then, by Lemma 3.4, A is noninvertible, so $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since B is invertible, $b_1 \neq 0$ and $b_2 \neq 0$. By Lemma 3.4, we have $\sigma(B^{-1}A) = \{0\}$ wherefrom $b_3 = 0$. So, we may assume that $B = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}$ for some $b \neq 0$. Then

$$|A + \lambda B| = \frac{1}{\sqrt{1 + |\lambda|^2(1+|b|)^2}} \begin{bmatrix} 1 + |\lambda|^2 |b|(1+|b|) & \lambda \\ \overline{\lambda} & |\lambda|^2(1+|b|) \end{bmatrix}.$$

It is a routine calculation to show that for $\lambda \neq 0$ we have

$$\det(|A + \lambda B| - |A|) \geq 0 \Leftrightarrow |\lambda|^2 \geq \frac{2|b|+1}{|b|^2(1+|b|)^2},$$

so (3.7) cannot be satisfied for every $\lambda \in \mathbb{C}$.

(3) Suppose A and B are both noninvertible. Then $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since B is noninvertible it follows that $b_1 = 0$ or $b_2 = 0$.

If $b_1 = 0$, then $A^*B = 0$. If $b_1 \neq 0$ then $b_2 = 0$, and therefore $B = \begin{bmatrix} 0 & 1 \\ 0 & b \end{bmatrix}$, so

$$|A + \lambda B| = \frac{1}{\sqrt{(1+|\lambda||b|)^2 + |\lambda|^2}} \begin{bmatrix} 1 + |\lambda||b| & \lambda \\ \overline{\lambda} & |\lambda|(|\lambda||b|^2 + |\lambda| + |b|) \end{bmatrix}.$$

Since $\frac{1+|\lambda||b|}{\sqrt{(1+|\lambda||b|)^2+|\lambda|^2}} < 1$ for $\lambda \neq 0$, these A and B cannot satisfy (3.7). $\square$

Remark 3.6. Since $\langle x, y \rangle = 0$ if and only if $\langle y, x \rangle = 0$, Theorem 3.1 can be extended with the following statements:

(e) $|y|^2 \leq |y + xa|^2$ for all $a \in \mathcal{A}$;
(f) $|y|^2 \leq |y + \lambda x|^2$ for all $\lambda \in \mathbb{C}$;
(g) $|y| \leq |y + xa|$ for all $a \in \mathcal{A}$.

Also, as a consequence of Theorem 3.5 we have that for $A, B \in \mathbb{M}_2(\mathbb{C})$ it holds

$$(|A| \leq |A + \lambda B| \text{ for all } \lambda \in \mathbb{C}) \Leftrightarrow (|B| \leq |B + \lambda A| \text{ for all } \lambda \in \mathbb{C}).$$

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