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SPG systems and semipartial geometries

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Abstract. First, the paper contains new necessary and sufficient conditions for a set of subspaces of $PG(n,q)$ to be an SPG regulus. Then a new construction method for semipartial geometries is given. As particular cases we find known classes of partial and semipartial geometries, but also new classes of semipartial geometries. The new semipartial geometries have parameters $s = q^n - 1$, $t = q^{n+1}$, $\alpha = 2q^{n-1}$, $\mu = 2q^n(q^n - 1)$, with either q any prime power and $n = 2$, or $q = 2^h$, $h \geqslant 1$, and $n \geqslant 3$.

1 Introduction

A *semipartial geometry* (SPG) is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ of points and lines, with $P \neq \emptyset \neq B$, satisfying the following axioms.

(i) Each point is incident with $1 + t$ ($t \geqslant 1$) lines and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ ($s \geqslant 1$) points and two distinct lines are incident with at most one point.

(iii) If two points are not collinear, then there are μ ($\mu > 0$) points collinear with both.

(iv) If a point x and a line L are not incident, then there are either 0 or α ($\alpha \geqslant 1$) points which are collinear with x and incident with L (i.e., there are either 0 or α points x_i and either 0 or α lines L_i respectively, such that $x \,\mathrm{I}\, L_i \,\mathrm{I}\, x_i \,\mathrm{I}\, L$).

The parameter α is called the *incidence number* of $\mathcal{S}$ and the semipartial geometry $\mathcal{S}$ is denoted by $\mathrm{spg}(s, t, \alpha, \mu)$. Semipartial geometries were introduced in 1978 by Debroey and Thas [6].

If $\mathcal{S} = (P, B, \mathrm{I})$ is an $\mathrm{spg}(s, t, \alpha, \mu)$, then it is easy to show that $|P| = v = 1 + (t+1)s(1 + t(s - \alpha + 1)/\mu)$ and $v(t+1) = b(s+1)$, with $b = |B|$; see Debroey and Thas [6].

An SPG with $\alpha = 1$ is called a *partial quadrangle* (PQ). Partial quadrangles were introduced and studied in 1975 by Cameron [4]. If in (iv) there are always α points collinear with x and incident with L, then the spg(s, t, α, μ) is called a *partial geometry* (PG). In such a case (iii) is automatically satisfied. A semipartial geometry which is not a partial geometry is called a *proper* semipartial geometry. Partial geometries were introduced in 1963 by Bose [1]. A partial geometry which is also a partial quadrangle is called a *generalized quadrangle* (GQ); generalized quadrangles were introduced in 1959 by Tits [15].

An SPG is a PG if and only if $\mu = (t+1)\alpha$. If $\mu \neq (t+1)\alpha$, then $t \geqslant s$ for any spg(s, t, α, μ); see Debroey and Thas [6]. Further, the dual of an SPG $\mathscr{S}$ is again an SPG if and only if either $s = t$ or $\mathscr{S}$ is a PG; see Debroey and Thas [6]. Finally we remark that the point graph of an spg(s, t, α, μ) is a strongly regular graph

$$\mathrm{srg}\left(1 + \frac{(t+1)s(\mu + t(s-\alpha+1))}{\mu}, s(t+1), s-1+t(\alpha-1), \mu\right).$$

For an excellent survey (with many examples) on semipartial geometries we refer to De Clerck and Van Maldeghem [8].

2 SPG reguli

An SPG *regulus* is a set $\mathscr{R}$ of r m-dimensional subspaces $\mathrm{PG}^{(1)}(m, q)$, $\mathrm{PG}^{(2)}(m, q), \ldots, \mathrm{PG}^{(r)}(m, q)$, $r > 1$, of $\mathrm{PG}(n, q)$ satisfying the following properties.

(a) $\mathrm{PG}^{(i)}(m, q) \cap \mathrm{PG}^{(j)}(m, q) = \emptyset$ for all $i \neq j$.

(b) If $\mathrm{PG}(m+1, q)$ contains $\mathrm{PG}^{(i)}(m, q), i \in \{1, 2, \ldots, r\}$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $\mathscr{R} - \{\mathrm{PG}^{(i)}(m, q)\}$. If $\mathrm{PG}(m+1, q)$ has no point in common with $\mathrm{PG}^{(j)}(m, q)$ for all $j \neq i$, then it is called a *tangent space* of $\mathscr{R}$ at $\mathrm{PG}^{(i)}(m, q)$.

(c) If the point x of $\mathrm{PG}(n, q)$ is not contained in an element of $\mathscr{R}$, then it is contained in a constant number θ ($\theta \geqslant 0$) of tangent spaces of $\mathscr{R}$.

SPG were introduced in 1983 by Thas [13]. By (a) we have $n \geqslant 2m+1$. In Thas [13] it is shown that for any SPG regulus $\mathscr{R}$ we have

$$\alpha(q-1) \text{ divides } (r-1)(q^{m+1}-1)$$

and

$$\theta = \frac{(\alpha(q^{n-m}-1) - (r-1)(q^{m+1}-1))rq^{m+1}}{\alpha((q^{n+1}-1) - r(q^{m+1}-1))}.$$

Let $\mathscr{R}$ be an SPG regulus, i.e., let $\mathscr{R}$ be a set of m-dimensional subspaces $\mathrm{PG}^{(1)}(m, q)$, $\mathrm{PG}^{(2)}(m, q), \ldots, \mathrm{PG}^{(r)}(m, q)$, $r > 1$, of $\mathrm{PG}(n, q)$ satisfying (a)–(c). Now we embed $\mathrm{PG}(n, q)$ as a hyperplane in $\mathrm{PG}(n+1, q)$, and we define an incidence structure $\mathscr{S} = (P, B, \mathrm{I})$ of points and lines as follows: points of $\mathscr{S}$ are the points in $\mathrm{PG}(n+1, q) - \mathrm{PG}(n, q)$, lines of $\mathscr{S}$ are the $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1, q)$ which contain an element of $\mathscr{R}$ but are not contained in $\mathrm{PG}(n,q)$, incidence is that of $\mathrm{PG}(n+1,q)$.

Theorem 2.1 (Thas [13]). *The incidence structure $\mathscr{S}$ arising from the* SPG *regulus $\mathscr{R}$ is a semipartial geometry with parameters*

$$s = q^{m+1} - 1, \quad t = r - 1, \quad \alpha = \alpha$$

and

$$\mu = (r-\theta)\alpha = r(r-1)(q^{m+1}-1)(q^{m+1}-\alpha)/(q^{n+1}-1-r(q^{m+1}-1)).$$

Remark. The SPG $\mathscr{S}$ is a PG if and only if $\theta = 0$; if $\mathscr{S}$ is not a PG, i.e., if $\theta \neq 0$, or, equivalently, $\alpha(q^{n-m}-1) \neq (r-1)(q^{m+1}-1)$, then $t \geqslant s$ implies $r \geqslant q^{m+1}$, see Thas [13].

Particular cases. *Case* a: $n = 2m+1$. Then the SPG regulus has no tangent spaces, hence $\alpha = r-1$ and $\theta = 0$. In this case the SPG $\mathscr{S}$ is a (Bruck) net of order $s+1 = q^{m+1}$ and degree $t+1 = r$; for the definition of net, see Bruck [3].

Case b: $n = 2m+2$. By Thas [13], in such a case a proper semipartial geomety $\mathscr{S}$ has parameters

$$s = q^{m+1} - 1, \quad t = r - 1,$$

$$\alpha = \frac{r^2(q^{m+1}-1) - r(q^{m+1}-1)(q^{m+1}+2) + q^{2m+3} - 1}{r(q^{m+2}-1) - (q^{2m+3}-1)},$$

and

$$\mu = \frac{r(r-1)(q^{m+1}-1)(r-(q^{m+1}+1))}{r(q^{m+2}-1) - (q^{2m+3}-1)}.$$

Theorem 2.2. *Let $\mathscr{R}$ be a set of r m-dimensional subspaces* $\mathrm{PG}^{(1)}(m,q)$, $\mathrm{PG}^{(2)}(m,q), \ldots, \mathrm{PG}^{(r)}(m,q)$, $r > 1$, *of* $\mathrm{PG}(2m+2,q)$ *satisfying* (a) *and* (b) *in the definition of* SPG *regulus. If $\mathscr{R}$ admits tangent spaces, then*

$$\alpha(r(q^{m+2}-1) - (q^{2m+3}-1)) \leqslant r^2(q^{m+1}-1) - r(q^{m+1}-1)(q^{m+1}+2) + (q^{2m+3}-1),$$

with equality if and only if $\mathscr{R}$ is an SPG *regulus.*

Proof. Let $V = \{x_1, x_2, \ldots, x_d\}$ be the set of all points of $\mathrm{PG}(2m+2,q)$ not contained in an element of $\mathscr{R}$, and let t_i be the number of all tangent spaces of $\mathscr{R}$ containing x_i, with $i = 1, 2, \ldots, d$. Then

$$d = \frac{q^{2m+3}-1}{q-1} - r\frac{q^{m+1}-1}{q-1}. \tag{1}$$

Now we count in two ways the number of ordered pairs (x_i, π), with $x_i \in V$, π a tangent space of $\mathscr{R}$ containing x_i. We obtain

$$\sum_{i=1}^{d} t_i = q^{m+1} r \left(\frac{q^{m+2}-1}{q-1} - \frac{(r-1)(q^{m+1}-1)}{\alpha(q-1)} \right). \tag{2}$$

Next we count in two ways the number of ordered triples (x_i, π, π'), with $x_i \in V$ and with π, π' distinct tangent spaces of $\mathscr{R}$ containing x_i. We obtain

$$\sum_{i=1}^{d} t_i(t_i - 1) = r(r-1) \left(\frac{q^{m+2}-1}{q-1} - \frac{(r-1)(q^{m+1}-1)}{\alpha(q-1)} \right)^2. \tag{3}$$

Hence from (2) and (3)

$$\sum_{i=1}^{d} t_i^2 = r \left(\frac{q^{m+2}-1}{q-1} - \frac{(r-1)(q^{m+1}-1)}{\alpha(q-1)} \right) \times \left(q^{m+1} + \frac{(r-1)(q^{m+2}-1)}{q-1} - \frac{(r-1)^2(q^{m+1}-1)}{\alpha(q-1)} \right).$$

Since $\mathscr{R}$ admits tangent spaces we have

$$\frac{q^{m+2}-1}{q-1} - \frac{(r-1)(q^{m+1}-1)}{\alpha(q-1)} > 0.$$

Now from $d \sum_{i=1}^{d} t_i^2 - (\sum_{i=1}^{d} t_i)^2 \geqslant 0$ it follows that

$$\left(\frac{q^{2m+3}-1}{q-1} - r \frac{q^{m+1}-1}{q-1} \right) \left(q^{m+1} + \frac{(r-1)(q^{m+2}-1)}{q-1} - \frac{(r-1)^2(q^{m+1}-1)}{\alpha(q-1)} \right) - rq^{2m+2} \left(\frac{q^{m+2}-1}{q-1} - \frac{(r-1)(q^{m+1}-1)}{\alpha(q-1)} \right) \geqslant 0.$$

Consequently, after some calculation,

$$\alpha(r(q^{m+2}-1) - (q^{2m+3}-1)) \leqslant r^2(q^{m+1}-1) - r(q^{m+1}-1)(q^{m+1}+2) + (q^{2m+3}-1). \tag{4}$$

We have equality in (4) if and only if $d \sum_{i=1}^{d} t_i^2 - (\sum_{i=1}^{d} t_i)^2 = 0$, if and only if $t_i = (\sum_{i=1}^{d} t_i)/d = \theta$ for all i, that is, if and only if (c) in the definition of SPG regulus is satisfied. Hence we have equality in (4) if and only if $\mathscr{R}$ is an SPG regulus. The theorem is proved.

Examples. 1. For $r = q^2 + 1$ and $m = 1$, we obtain $\alpha \leqslant q^2$. So $\mathscr{R}$ is an SPG regulus if and only if $\alpha = q^2$. In such a case $\mathscr{R}$ is a line spread of $\mathrm{PG}(3, q)$ in $\mathrm{PG}(4, q)$.

2. For q a square, $\alpha = q\sqrt{q}$ and $m = 1$, we obtain that either $r \geqslant q^2\sqrt{q} + 1$ or $r \leqslant \frac{(q\sqrt{q}+1)(q^2\sqrt{q}-1)}{q^2-1}$. So $\mathscr{R}$ is an SPG regulus if and only if $r = q^2\sqrt{q} + 1$ (as $(q\sqrt{q}+1)(q^2\sqrt{q}-1)/(q^2-1) \notin \mathbb{N}$). If H is a nonsingular Hermitian variety of $\mathrm{PG}(4, q)$, with q a square, then a line spread of H would be such an SPG regulus. However, it is still an open question whether or not line spreads of H exist; for $q = 4$ Brouwer [2] proved that H does not admit a line spread.

Case c: $\mathscr{R}$ has tangent spaces, and for all $i \in \{1, 2, \dots, r\}$, the union of all tangent spaces at $\mathrm{PG}^{(i)}(m, q)$ is a $\mathrm{PG}(n - m - 1, q)$.

In such a case the SPG regulus satisfies $r = \alpha q^{n-2m-1} + 1$; see Thas [13].

Let $\mathscr{R}$ be a set of r m-dimensional subspaces $\mathrm{PG}^{(1)}(m, q), \mathrm{PG}^{(2)}(m, q), \dots, \mathrm{PG}^{(r)}(m, q)$, $r > 1$, satisfying (a) and (b) in the definition of SPG regulus. Assume moreover that $\mathscr{R}$ has tangent spaces and that for all $i \in \{1, 2, \dots, r\}$, the union of all tangent spaces at $\mathrm{PG}^{(i)}(m, q)$ is a $\mathrm{PG}(n - m - 1, q)$. Then, by Thas [13], $r \leqslant 1 + q^{(n+1)/2}$, with equality if and only if $\mathscr{R}$ is an SPG regulus.

In Case c the semipartial geometry $\mathscr{S}$ has parameters

$$s = q^{m+1} - 1, \quad t = q^{(n+1)/2}, \quad \alpha = q^{2m-(n/2)+(3/2)}, \quad \mu = q^{m+1}(q^{m+1} - 1).$$

In Thas [13] the following class of proper semipartial geometries arising from SPG reguli is given. Let $\mathscr{R}$ be a spread of the nonsingular elliptic quadric $Q^-(2m + 3, q)$, $m \geqslant 0$, of $\mathrm{PG}(2m + 3, q)$, that is, a partition of $Q^-(2m + 3, q)$ by generators, that is, by maximal totally singular subspaces. Then $\mathscr{R}$ is an SPG regulus of $\mathrm{PG}(2m + 3, q)$, consisting of $q^{m+2}+1$ m-dimensional subspaces. The corresponding SPG has parameters

$$s = q^{m+1} - 1, \quad t = q^{m+2}, \quad \alpha = q^m, \quad \mu = q^{m+1}(q^{m+1} - 1).$$

For $m = 0$, the SPG regulus coincides with $Q^-(3, q)$ and the SPG is a partial quadrangle with parameters

$$s = q - 1, \quad t = q^2, \quad \alpha = 1, \quad \mu = q(q - 1).$$

For $m = 1$, many spreads of $Q^-(5, q)$ are known. For $m > 1$ with q even, the quadric $Q^-(2m + 3, q)$ admits spreads, but for $m > 1$ with q odd no spread of $Q^-(2m + 3, q)$ is known.

Recently, De Clerck, Delanote, Hamilton and Mathon [9] discovered by computer an SPG regulus with

$$n = 5, \quad m = 1, \quad q = 3, \quad r = 21, \quad \alpha = 2 \quad \text{and} \quad \theta = 0.$$

Consequently the corresponding SPG is a PG with parameters

$$s = 8, \quad t = 20 \quad \text{and} \quad \alpha = 2.$$

3 SPG systems

Definition 3.1. Let $Q(2n+2,q)$, $n \geqslant 1$, be a nonsingular quadric of $\mathrm{PG}(2n+2,q)$. An SPG *system of* $Q(2n+2,q)$ is a set $\mathcal{T}$ of $(n-1)$-dimensional totally singular subspaces of $Q(2n+2,q)$ such that the elements of $\mathcal{T}$ on any nonsingular elliptic quadric $Q^-(2n+1,q) \subset Q(2n+2,q)$ constitute a spread of $Q^-(2n+1,q)$.

Let $Q^+(2n+1,q)$, $n \geqslant 1$, be a nonsingular hyperbolic quadric of $\mathrm{PG}(2n+1,q)$. An SPG *system of* $Q^+(2n+1,q)$ is a set $\mathcal{T}$ of $(n-1)$-dimensional totally singular subspaces of $Q^+(2n+1,q)$ such that the elements of $\mathcal{T}$ on any nonsingular quadric $Q(2n,q) \subset Q^+(2n+1,q)$ constitute a spread of $Q(2n,q)$.

Let $H(2n+1,q)$, $n \geqslant 1$, be a nonsingular Hermitian variety of $\mathrm{PG}(2n+1,q)$, q a square. An SPG *system of* $H(2n+1,q)$ is a set $\mathcal{T}$ of $(n-1)$-dimensional totally singular subspaces of $H(2n+1,q)$ such that the elements of $\mathcal{T}$ on any nonsingular Hermitian variety $H(2n,q) \subset H(2n+1,q)$ constitute a spread of $H(2n,q)$.

Let P be a singular polar space with ambient space $\mathrm{PG}(t,q)$, having as radical the point x. Assume that the projective index of P is n, with $n \geqslant 1$, that is, n is the dimension of the maximal totally singular subspaces on P. An SPG *system of* P is a set $\mathcal{T}$ of $(n-1)$-dimensional totally singular subspaces of P, not containing x, such that the elements of $\mathcal{T}$ which are (maximal) totally singular for the polar subspace P' of P induced by any $\mathrm{PG}(d-1,q) \subset \mathrm{PG}(d,q)$ not containing x constitute a spread of P'.

Remark 3.2. Polar spaces arising from quadrics and Hermitian varieties will be often identified with their point set.

Remark 3.3. If in the previous definition we replace "spread" by "partial spread" then we obtain the definition of a *partial* SPG *system*.

Definition 3.4. The *ovoid number* $o(P)$ of a nonsingular polar space P is the number of elements of a (hypothetical) spread (resp. ovoid) of P.

Lemma 3.5. *If $\mathcal{T}$ is an* SPG *system of the nonsingular polar space $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q)\}$, then*

$$|\mathcal{T}| = o(P)(q^{n+1}-1)/(q-1) = |P|.$$

Let P be a singular polar space having as radical the point x, with projective index n, and for which the quotient $P/\{x\}$ is the nonsingular polar space $\tilde{P}$. If $\mathcal{T}$ is an SPG *system of P, then*

$$|\mathcal{T}| = o(\tilde{P})q^n.$$

Proof. We give the proof for $P = Q(2n+2,q)$. The other cases are similar.

We count in two ways the number of ordered pairs (π,ζ), with $\pi \in \mathcal{T}$ and ζ a hyperplane of $\mathrm{PG}(2n+2,q)$ containing π and intersecting $Q(2n+2,q)$ in a nonsingular elliptic quadric. We obtain

$$|\mathcal{T}|q^{n+1}(q-1)/2 = o(Q^-(2n+1,q))q^{n+1}(q^{n+1}-1)/2.$$

As $o(Q^-(2n+1,q)) = q^{n+1}+1$ and $o(Q(2n+2,q)) = q^{n+1}+1$, see Hirschfeld and Thas [11], we now have:

$$|\mathcal{T}| = o(P)(q^{n+1}-1)/(q-1) = |P|.$$

Theorem 3.6. *Let $\mathcal{T}$ be a set of $(n-1)$-dimensional totally singular subspaces of the nonsingular polar space $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q)\}$. Then $\mathcal{T}$ is an* SPG *system of P if and only if the following conditions are satisfied*:

(i) $|\mathcal{T}| = |P|$,

(ii) *if $\pi, \pi' \in \mathcal{T}$, with $\pi \neq \pi'$ and $\pi \cap \pi' \neq \emptyset$, then $\langle \pi, \pi' \rangle$ contains a generator, that is, a maximal totally singular subspace of P.*

Let $\mathcal{T}$ be a set of $(n-1)$-dimensional totally singular subspaces of the singular polar space P with projective index n and having as radical the point x. Assume also that no element of $\mathcal{T}$ contains x. The nonsingular quotient $P/\{x\}$ will be denoted by $\tilde{P}$. Then $\mathcal{T}$ is an SPG *system of $\tilde{P}$ if and only if the following conditions are satisfied*:

(i) $|\mathcal{T}| = o(\tilde{P})q^n$,

(ii) *if $\pi, \pi' \in \mathcal{T}$, with $\pi \neq \pi'$ and $\pi \cap \pi' \neq \emptyset$, then $\langle \pi, \pi' \rangle$ contains a generator of P, that is, $\langle \pi, \pi' \rangle$ contains x.*

Proof. We give the proof for $P = Q(2n+2,q)$. The other cases are similar.

Assume first that $\mathcal{T}$ is an SPG system of $Q(2n+2,q)$. Then

$$|\mathcal{T}| = o(Q(2n+2,q))(q^{n+1}-1)/(q-1) = |P|.$$

Let $\pi, \pi' \in \mathcal{T}$, with $\pi \neq \pi'$ and $\pi \cap \pi' \neq \emptyset$. If $\langle \pi, \pi' \rangle$ contains no generator of P, then there is a hyperplane of $\mathrm{PG}(2n+2,q)$ containing $\langle \pi, \pi' \rangle$ and intersecting $Q(2n+2,q)$ in a nonsingular elliptic quadric $Q^-(2n+1,q)$. In such a case, as $\pi \cap \pi' \neq \emptyset$, the elements of $\mathcal{T}$ on $Q^-(2n+1,q)$ do not form a spread of $Q^-(2n+1,q)$, a contradiction.

Conversely, assume that $\mathcal{T}$ is a set of $(n-1)$-dimensional totally singular subspaces of $Q(2n+2,q)$ for which (i) and (ii) are satisfied. Let $Q_i^-(2n+1,q)$ be any nonsingular elliptic quadric on $Q(2n+2,q)$, and let t_i be the number of elements of $\mathcal{T}$ on $Q_i^-(2n+1,q)$. Assume that $\pi, \pi' \in \mathcal{T}, \pi \neq \pi'$, and that $\pi \subset Q_i^-(2n+1,q)$, $\pi' \subset Q_i^-(2n+1,q)$. If $\pi \cap \pi' \neq \emptyset$, then $\langle \pi, \pi' \rangle$ contains a generator of $Q(2n+2,q)$, so $Q_i^-(2n+1,q)$ contains a n-dimensional subspace, clearly a contradiction. Hence $\pi \cap \pi' = \emptyset$. Now we count in two ways the number of ordered pairs (π, ζ), with $\pi \in \mathcal{T}$ and ζ a hyperplane of $\mathrm{PG}(2n+2,q)$ containing π and intersecting $Q(2n+2,q)$ in a nonsingular elliptic quadric. We obtain

$$|\mathcal{T}|q^{n+1}(q-1)/2 = \sum_{i \in I} t_i,$$

with $|I| = q^{n+1}(q^{n+1}-1)/2$, $|\mathcal{T}| = o(Q(2n+2,q))(q^{n+1}-1)/(q-1) = |P|$, and $t_i \leqslant o(Q_i^-(2n+1,q)) = q^{n+1}+1$. It follows that $t_i = o(Q_i^-(2n+1,q))$ for all $i \in I$. Consequently the elements of $\mathcal{T}$ in $Q_i^-(2n+1,q)$ constitute a spread of $Q_i^-(2n+1,q)$. We conclude that $\mathcal{T}$ is an SPG system of $Q(2n+2,q)$.

Now we consider the respective cases $n=1$, $n=2$, $n>2$.

Case $n=1$. In the nonsingular case we have $\mathcal{T} = P$, with $P \in \{Q(4,q), Q^+(3,q), H(3,q)\}$, and in the singular case we have $\mathcal{T} = P - \{x\}$, with P of projective index one.

Case $n=2$. Let $\mathcal{R}$ be a spread of the nonsingular polar space $Q(6,q)$. Such a spread exists when $q = p^h$, with $p \in \{2,3\}$; see Hirschfeld and Thas [11]. Then the set of all lines in all elements of $\mathcal{R}$ is an SPG system $\mathcal{T}$ of $Q(6,q)$.

Let $H(q)$ be the classical generalized hexagon of order q embedded in the quadric $Q(6,q)$. Two lines of $H(q)$ intersect if and only if they lie in a common plane of $Q(6,q)$. As the number of lines of $H(q)$ is equal to $(q^6-1)/(q-1)$, it follows from Theorem 3.6 that the line set $\mathcal{T}$ of $H(q)$ is an SPG system of $Q(6,q)$.

Let P be a singular polar space having as radical the point x, with $\tilde{P} = P/\{x\} \in \{Q^+(3,q), Q(4,q), Q^-(5,q), W_3(q), H(4,q)\}$ (here $W_3(q)$ is the polar space defined by a nonsingular symplectic polarity of $\mathrm{PG}(3,q)$). Let $\mathcal{R}$ be a spread of $\tilde{P}$. Such a spread exists when $\tilde{P} \in \{Q^+(3,q), Q^-(5,q), W_3(q)\}$, when $\tilde{P} = Q(4,q)$ if and only if q is even, and possibly but not likely when $\tilde{P} = H(4,q)$ with $q>4$ ($H(4,4)$ does not admit a spread); see Hirschfeld and Thas [11]. Then the lines of the ambient space of P contained in the elements of $\mathcal{R}$, but not containing x, constitute an SPG system of P.

Theorem 3.7. *For* $n=2$ *there are no other* SPG *systems than the foregoing examples.*

Proof. Let $n=2$ and let $\mathcal{T}$ be an SPG system of the polar space P over $\mathrm{GF}(q)$. Let y be a point of P, with y not the radical x of P in the singular case. If L_1, L_2, L_3 are distinct lines of $\mathcal{T}$ containing y, then, by Theorem 3.6, these lines are coplanar. Hence in the nonsingular case there are at most $q+1$ lines of $\mathcal{T}$ containing y. Consequently in this case $|\mathcal{T}| \leqslant |P|(q+1)/(q+1)$, and as $|\mathcal{T}| = |P|$ it follows that y is contained in exactly $q+1$ lines of $\mathcal{T}$. In the singular case there are at most q lines of $\mathcal{T}$ containing y. Hence in this case $|\mathcal{T}| \leqslant (|P|-1)q/(q+1) = o(P/\{x\})q^2$, and so, as $|\mathcal{T}| = o(P/\{x\})q^2$, it follows that y is contained in exactly q lines of $\mathcal{T}$. In both cases the plane containing the lines of $\mathcal{T}$ through y will be denoted by π_y.

First, let P be singular. Then for any $y \in P - \{x\}$, the plane π_y contains x. Let $z \in \pi_y$, with $z \notin xy$. If $\pi_z \neq \pi_y$, then $x \in \pi_z \cap \pi_y = yz$, a contradiction. Hence $\pi_z = \pi_y$. It easily follows that $\pi_y = \pi_z$ for all $z \in \pi_y - \{x\}$. So, for $y, z \in P - \{x\}$, either $\pi_y = \pi_z$, or $\pi_y \cap \pi_z = \{x\}$. Also, the lines of $\mathcal{T}$ in any π_y, with $y \in P - \{x\}$, are the q^2 lines of π_y not containing x. Hence the planes π_y, with $y \in P - \{x\}$, form a partial spread of $\tilde{P} = P/\{x\}$. As $|\mathcal{T}| = o(\tilde{P})q^2$, the planes π_y, with $y \in P - \{x\}$, form a spread of $\tilde{P}$. It follows that $\mathcal{T}$ is of the type described above (notice that $H(3,q)$, q a square, does not admit a spread; see Hirschfeld and Thas [11]).

Next, let P be nonsingular. Assume first that $\pi_y = \pi_z$ for $y, z \in P$, with $y \neq z$. Then $\pi_y = \pi_u$ for all $u \in \pi_y$. It follows that in such a case all lines of π_y belong to $\mathcal{T}$. Consider a point $r \in P$, $r \notin \pi_y$. Let $M \in \mathcal{T}$ with $r \in M$; then $M \cap \pi_y = \emptyset$. If $\mathcal{T}$ does not contain all lines of π_r, then all planes π_l with $l \in M$ are distinct; in such a case $P = Q(6, q)$. At least one of these planes π_l, say π_0, has a point u in common with π_y. So, as π_u contains l_0, we have $\pi_u \neq \pi_y$, a contradiction. Hence all lines of π_r belong to $\mathcal{T}$. Also, $\pi_r \cap \pi_y = \emptyset$. Hence the planes π_b, with $b \in P$, form a spread of P, and $\mathcal{T}$ consists of all lines of the elements of this spread. Notice that $H(5, q), q$ a square, does not admit a spread, that $Q^+(5, q)$ does not admit a spread, and that $Q(6, q)$, $q = p^h$ with $p \in \{2, 3\}$, admits a spread; see Hirschfeld and Thas [11].

Assume next that distinct points $y \in P$ define distinct planes π_y. Let $M \in \mathcal{T}$. Then the $q + 1$ planes π_m, with $m \in M$, are distinct. Hence $P \notin \{Q^+(5, q), H(5, q)\}$, and so $P = Q(6, q)$. Let N be a line of $Q(6, q)$ which is not contained in $\mathcal{T}$. If N is contained in π_z and $\pi_{z'}$, with z, z' distinct points of $Q(6, q)$, then $\langle \pi_z, \pi_{z'} \rangle$ is a subspace on $Q(6, q)$, clearly a contradiction. Hence the planes π_z, $z \in Q(6, q)$, contain exactly $(q^3 + 1)(q^2 + q + 1)q^2$ lines of $Q(6, q)$ not in $\mathcal{T}$. It follows that each line of $Q(6, q)$ not in $\mathcal{T}$ belongs to exactly one plane π_z, $z \in Q(6, q)$. Now consider the incidence structure $\mathcal{H} = (Q(6, q), \mathcal{T}, \mathrm{I})$, with I the natural incidence. Let $M \in \mathcal{T}$ and $z \in Q(6, q)$, $z \notin M$. We will show that in the point graph of $\mathcal{H}$ there is exactly one point on M nearest to z and either at distance 1 or 2 from z. If $z \in \pi_y$ for some $y \in M$, then y is the unique point of M nearest to z and $d(y, z) = 1$. Next, assume that for no point $y \in M$ we have $z \in \pi_y$. Then $\langle z, M \rangle$ does not belong to $Q(6, q)$ and on M there is exactly one point y for which yz belongs to $Q(6, q)$. By a previous argument there is a unique point $u \in Q(6, q)$ with $y, z \in \pi_u$. Then the point y is the unique point of M nearest to z and $d(y, z) = 2$. Consequently $\mathcal{H}$ is a generalized hexagon of order q. Now by Cameron and Kantor [5] (see also Thas and Van Maldeghem [14]) $\mathcal{H}$ is the classical generalized hexagon $H(q)$ embedded in $Q(6, q)$.

The theorem is completely proved.

Case $n \geqslant 3$. Let $\mathcal{R}$ be a spread of the nonsingular polar space P, with P one of $Q(2n + 2, q)$, with q even and $n \geqslant 3$, $Q(4m + 2, q)$, with q odd and $m \geqslant 2$, $Q^+(4m + 3, q)$, with $m \geqslant 1$. Notice that $Q(2n + 2, q)$, with q even and $n \geqslant 3$, $Q^+(4m + 3, q)$ with $m \geqslant 1$ and q even, and $Q^+(7, q)$ with either q an odd prime, or q odd with $q \not\equiv 1 \pmod 3$, admit a spread, that for $Q^+(7, q)$ with q odd, $q \equiv 1 \pmod 3$ and not a prime, for $Q^+(4m + 3, q)$ with q odd and $m > 1$, and for $Q(4m + 2, q)$ with $m > 1$ and q odd, it is an open problem whether or not the polar space admits a spread, and that $Q^+(4m + 1, q)$ with $m > 1$, $Q(4m, q)$ with $m > 1$ and q odd, and $H(2n + 1, q)$ with $n > 2$, do not admit a spread; see Hirschfeld and Thas [11]. Then the set of all hyperplanes of all elements of $\mathcal{R}$ is an SPG system $\mathcal{T}$ of P.

Let P be a singular polar space with projective index $n \geqslant 3$ having as radical the point x, with $\tilde{P} = P/\{x\}$. If $\mathcal{R}$ is a spread of $\tilde{P}$, then the set of all $(n - 1)$-dimensional subspaces of the ambient space of P contained in the elements of $\mathcal{R}$, but not containing x, constitutes an SPG system of P.

Remark 3.8. A more general way to introduce the sets $\mathcal{T}$ goes as follows. Let P be a polar space with projective index n, $n \geqslant 1$, which is either nonsingular or singular with a point x as radical. A *partial* SPG *system* of P is any set $\mathcal{T}$ of $(n - 1)$-dimensional totally singular subspaces of P satisfying the following condition: if π,

$\pi' \in \mathcal{T}$ with $\pi \neq \pi'$ and $\pi \cap \pi' \neq \emptyset$, then $\langle \pi, \pi' \rangle$ contains a generator of P. If $\mathcal{T}$ is a partial SPG system of P and if either $|\mathcal{T}| = |P|$ in the nonsingular case or $|\mathcal{T}| = o(P/\{x\})q^n$ in the singular case, then $\mathcal{T}$ is called an SPG *system* of P. For our purposes, that is, the construction of semipartial geometries described in Section 4, Definition 3.1 suffices and is more natural.

4 Semipartial geometries arising from SPG systems

The Construction. Let P be either a nonsingular polar space, with $P \in \{Q(2n+2,q), Q^+(2n+1,q), H(2n+1,q)\}$, $n \geqslant 1$, or a singular polar space with projective index $n, n \geqslant 1$, and having as radical the point x. Let $\mathrm{PG}(d,q)$ be the ambient space of P. Further, let $\mathcal{T}$ be an SPG system of P and let P be embedded in a nonsingular polar space $\bar{P}$ with ambient space $\mathrm{PG}(d+1,q)$, of the same type as P and with projective index n (so, P and $\bar{P}$ are both orthogonal, Hermitian, or symplectic). Hence for $P = Q(2n+2,q)$ we have $\bar{P} = Q^-(2n+3,q)$, for $P = Q^+(2n+1,q)$ we have $\bar{P} = Q(2n+2,q)$, and for $P = H(2n+1,q)$ we have $\bar{P} = H(2n+2,q)$.

If $\bar{P}$ is not symplectic and $y \in \bar{P}$, then let τ_y be the tangent hyperplane of $\bar{P}$ at y; if $\bar{P}$ is symplectic and θ is the corresponding symplectic polarity of $\mathrm{PG}(d+1,q)$, then let $\tau_y = y^\theta$ for any $y \in \mathrm{PG}(d+1,q)$.

For $y \in \bar{P} - P$, let $\bar{y}$ be the set of all points z of $\bar{P} - P$ for which $\tau_z \cap P = \tau_y \cap P$. As $\tau_u \neq \tau_z$ for distinct points u, z of $\bar{P}$, no two distinct points of $\bar{y}$ are collinear in $\bar{P}$. For P nonsingular and orthogonal we have $|\bar{y}| = 2$, except when $P = Q^+(2n+1,q)$ and q even where $|\bar{y}| = 1$, for P nonsingular and Hermitian we have $|\bar{y}| = \sqrt{q}+1$, for P singular and orthogonal we have $|\bar{y}| = 1$, except when P is parabolic and q is even where $|\bar{y}| = q$, for P singular and Hermitian we have $|\bar{y}| = \sqrt{q}$, and for P singular and symplectic we have $|\bar{y}| = q$. Notice also that all points of $\bar{y}$ belong to some common line of $\mathrm{PG}(d+1,q)$ except when P is singular parabolic and q is even where they belong to a common nonsingular conic. Then $\{\bar{y} \mid y \in \bar{P} - P\}$ is the point set $\mathcal{P}$ of the incidence structure $\mathcal{S}$.

Let ξ be any maximal totally singular subspace of $\bar{P}$, not contained in P for which $\xi \cap P \in \mathcal{T}$ and let $y \in (\bar{P} - P) \cap \xi$. Further, let $\bar{\xi}$ be the set of all maximal totally singular subspaces η of $\bar{P}$, not contained in P for which $\xi \cap P = \eta \cap P$ and $\eta \cap \bar{y} \neq \emptyset$. From $\xi \cap P = \eta \cap P$ follows that no point of $\xi - P$ is collinear on $\bar{P}$ with a point of $\eta - P$. Let z be a point of $\xi - P$, $z \neq y$, and let $yz \cap P = \{u\}$. Further let y' be the element of $\bar{y}$ in η. The hyperplanes $\tau_l, l \in uy$, form a bundle of hyperplanes which coincides with the bundle of hyperplanes with elements $\tau_{l'}, l' \in uy'$. Hence on uy' there is a point z' with $\tau_z = \tau_{z'}$. So $\bar{z}$ has also a point in common with η. The lines of $\mathcal{S}$ are all sets $\bar{\xi}$, and the line set is denoted by $\mathcal{L}$.

Incidence in $\mathcal{S}$ is defined as follows. If $\bar{y} \in \mathcal{P}$ and $\bar{\xi} \in \mathcal{L}$, then $\bar{y}\,\mathrm{I}\,\bar{\xi}$ if and only if for some $z \in \bar{y}$ and some $\eta \in \bar{\xi}$, we have $z \in \eta$.

Theorem 4.1. *The incidence structure* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *satisfies*

(i) *each line is incident with* $s+1$ *points, with* $s = q^n - 1$,

(ii) *each point is incident with* $t+1 = o(P)$ *lines in the nonsingular case and* $t+1 = o(P/\{x\})$ *lines in the singular case,*

(iii) *any two distinct points are incident with at most one common line,*

(iv) *if a point $\bar{y}$ and a line $\bar{\xi}$ are not incident, then there are either* 0 *or* α ($\alpha \geqslant 1$) *points which are collinear with $\bar{y}$ and incident with $\bar{\xi}$; here $\alpha = \beta q^{n-1}$, with β the number of points of $\bar{P}$ in any set $\bar{y} \in \mathscr{P}$.*

Proof. Let $\bar{\xi} \in \mathscr{L}$. Then the number of elements $\bar{y} \in \mathscr{P}$ incident with $\bar{\xi}$ is the number of points $\bar{P} - P$ in any element of $\bar{\xi}$, that is, q^n.

Let $\bar{y} \in \mathscr{P}$. Then the number of elements $\bar{\xi} \in \mathscr{L}$ incident with $\bar{y}$ is the number of n-dimensional subspaces on $\bar{P}$ containing an element of $\mathscr{T}$ and any given element of $\bar{y}$. By the definition of $\mathscr{T}$ that is respectively $o(Q^-(2n+1,q))$, $o(Q(2n,q))$, $o(H(2n,q))$, $o(P/\{x\})$. In the nonsingular case we have $o(Q^-(2n+1,q)) = o(P)$, $o(Q(2n,q)) = o(P)$, and $o(H(2n,q)) = o(P)$. Hence in the nonsingular case $t+1 = o(P)$, and in the singular case $t+1 = o(P/\{x\})$.

Assume, by way of contradiction, that the distinct points $\bar{y}$, $\bar{z}$ of $\mathscr{P}$ are both incident with the distinct lines $\bar{\xi}$, $\bar{\eta}$ of $\mathscr{L}$. Let $y \in \bar{y}$, $z \in \bar{z}$, $\xi \in \bar{\xi}$, $\eta \in \bar{\eta}$, with $y, z \in \xi$ and $y \in \eta$. As $\bar{z}\,\mathrm{I}\,\bar{\eta}$ and zy is a line of $\bar{P}$, also z belongs to η. Hence $yz \cap P$ belongs to $\xi \cap P$ and to $\eta \cap P$. As either $(\xi \cap P) \cap (\eta \cap P) = \emptyset$, or $\xi \cap P = \eta \cap P$, we have $\xi \cap P = \eta \cap P$. It follows that $\bar{\xi} = \bar{\eta}$, a contradiction. Consequently (iii) is satisfied.

Let $\bar{y} \in \mathscr{P}$ and $\bar{\xi} \in \mathscr{L}$, with $\bar{y} \not\mathrm{I}\, \bar{\xi}$. Further, let $\bar{y} = \{y_1, y_2, \dots, y_\beta\}$ and $\xi \in \bar{\xi}$. If τ_{y_i}, $i \in \{1, 2, \dots, \beta\}$, contains $\xi \cap P$, then there are no points of $\mathscr{S}$ collinear with $\bar{y}$ and incident with $\bar{\xi}$. Now assume that $\mathscr{T}_{y_i}$ does not contain $\xi \cap P$, $i = 1, 2, \dots, \beta$. If $\tau_{y_i} \cap \xi = \tau_{y_j} \cap \xi$, $i \neq j$ and $i, j \in \{1, 2, \dots, \beta\}$, then $\tau_{y_i} = \tau_{y_j}$, a contradiction. Hence $(\tau_{y_i} \cap \xi) - P$ has no point in common with $(\tau_{y_j} \cap \xi) - P, i \neq j$. Now assume that $\eta \in \bar{\xi}$, with $\xi \neq \eta$. Let $u \in (\tau_{y_i} \cap \xi) - P$. If $\bar{u} \cap \eta = \{u'\}$, then by a foregoing argument, the line $u'v$, with $\{v\} = uy_i \cap P$, contains a point y_j of $\bar{y}$. Hence η and ξ define the same points collinear with $\bar{y}$ and incident with $\bar{\xi}$. It follows that in $\mathscr{S}$ there are exactly βq^{n-1} points which are collinear with $\bar{y}$ and incident with $\bar{\xi}$.

Now the theorem is completely proved.

Theorem 4.2. *If the polar space P is either nonsingular with $P \in \{Q(2n+2,q)$, $Q^+(2n+1,q)$ with $q \in \{2,3\}$, $H(2n+1,q)\}$ or singular with*

$$P/\{x\} \in \{Q(2n,q) \text{ with } q \text{ even},\ Q^+(2n-1,q),\ Q^-(2n+1,q), H(2n-1,q), H(2n,q), W_{2n-1}(q)\},$$

then $\mathscr{S} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ is a semipartial geometry.

Proof. Let $\bar{y}$ and $\bar{z}$ be noncollinear points of $\mathscr{S}$. We must show that in $\mathscr{S}$ the number of points collinear with both $\bar{u}$ and $\bar{z}$ is a constant μ ($\mu > 0$).

First, let $P = Q(2n+2,q)$. If $y \in \bar{y}$ and $\bar{z} = \{z_1, z_2\}$, then the number of points of $\mathscr{S}$ collinear with both $\bar{y}$ and $\bar{z}$ is equal to

$$\sum_{i=1}^{2} |(\tau_y \cap \tau_{z_i} \cap \bar{P}) - P| = 2(|Q^-(2n+1,q)| - |Q(2n,q)|) = 2q^n(q^n - 1).$$

A similar argument applies to $P = H(2n+1, q)$, and $P/\{x\} \in \{Q^+(2n-1, q), Q^-(2n+1, q), H(2n-1, q), H(2n, q), W_{2n-1}(q)\}$.

Now let P be the nonsingular polar space $Q^+(2n+1, q)$, with $q \in \{2, 3\}$. First, let $q = 2$. Then $|\bar{y}| = |\bar{z}| = 1$, say $\bar{y} = \{y\}$ and $\bar{z} = \{z\}$. The number of points of $\mathscr{S}$ collinear with both $\bar{y}$ and $\bar{z}$ is equal to

$$|(\tau_y \cap \tau_z \cap \bar{P}) - P| = |Q(2n, 2)| - |Q^-(2n-1, 2)| = 2^{2n-1} + 2^{n-1}.$$

Next, let $q = 3$. Then $|\bar{y}| = |\bar{z}| = 2$, say $y \in \bar{y}$ and $\bar{z} = \{z_1, z_2\}$. The number of points of $\mathscr{S}$ collinear with both $\bar{y}$ and $\bar{z}$ is equal to

$$\sum_{i=1}^{2} |(\tau_y \cap \tau_{z_i} \cap \bar{P}) - P| = 2(|Q(2n, 3)| - |Q^-(2n-1, 3)|)$$
$$= 2(3^{2n-1} + 3^{n-1}).$$

Finally, let P be singular, with $P/\{x\} = Q(2n, q)$ and q even. Then $|\bar{y}| = |\bar{z}| = q$. Since $\alpha = s + 1 = q^n$ and $|\mathscr{P}| = q^{2n}$, the geometry $\mathscr{S}$ is a $2-(q^{2n}, q^n, 1)$ design, so an affine plane of order q^n. The theorem is proved.

Now we will consider in more detail these cases which yield an SPG. Known classes of SPG and PG will appear, but also new classes will arise.

Classes of semipartial and partial geometries. (a) $P = Q(2n+2, q)$. In such a case the SPG has parameters

$$s = q^n - 1, \quad t = q^{n+1}, \quad \alpha = 2q^{n-1}, \quad \mu = 2q^n(q^n - 1), \quad n \geqslant 1.$$

For $n = 1$ the SPG system is the point set of $Q(4, q)$.

For $n = 2$ the SPG system arises either from the classical generalized hexagon of order q, or from a spread of $Q(6, q)$. So the SPG exists for any prime power q.

For $n \geqslant 3$ any spread of $Q(2n+2, q)$ defines an SPG system. Such a spread is known to exist for q even.

For $n = 1$ the SPG was known and there was a construction by R. Metz (private communication) which is essentially the foregoing one; see De Clerck and Van Maldeghem [8]. For $n = 2$ and $q = 3$, the SPG arising from a spread of $Q(6, 3)$ was known (but not the construction given here). It was discovered by Delanote [10]; Delanote [10] also described the SPG for $q = 3$ and $\mathscr{T}$ a hypothetical spread of $Q(2n, 3)$, $n > 3$. However the description of Delanote is the one given in the next section. The other classes are new.

Now let r be the pole of $PG(2n+2, q) \supset Q(2n+2, q)$ with respect to the polar space $Q^-(2n+3, q) \subset PG(2n+3, q)$; the polarity defined by $Q^-(2n+3, q)$ is orthogonal for q odd and symplectic for q even. Now we project $PG(2n+3, q) - \{r\}$ from r onto a hyperplane Π not through r of $PG(2n+3, q)$. For q odd $Q(2n+2, q)$ is projected onto a quadric $\tilde{Q}(2n+2, q)$; for q even the orthogonal polar space $Q(2n+2, q)$

is projected onto a symplectic polar space in a hyperplane Δ of Π. Further, $Q^-(2n+3,q) - Q(2n+2,q)$ is projected onto the set Γ. For q even $\Gamma \cup \Delta$ is a set of type $(1,(q/2)+1,q+1)$ of Π, see Hirschfeld and Thas [11]; for q odd Γ is one of the imprimitivity classes defined by $\mathrm{PGO}(2n+3,q)$. With any point $\bar{y} \in \mathscr{P}$ there corresponds, by projection, a point of Γ, and with any line $\bar{\xi} \in \mathscr{L}$ there corresponds, by projection, a n-dimensional space $\tilde{\xi}$ of Π. Also, $\tilde{\xi} \cap \tilde{Q}(2n+2,q)$ for q odd and $\tilde{\xi} \cap \Delta$ for q even, is an $(n-1)$-dimensional space which belongs to the projection $\tilde{\mathscr{T}}$ of $\mathscr{T}$. In this way there arises an SPG $\tilde{\mathscr{S}} \cong \mathscr{S}$, where incidence is inherited from Π. For $n = 1$ this description of $\tilde{\mathscr{S}}$ is due to Hirschfeld and Thas [11].

For $q = 2$ the geometry $\tilde{\mathscr{S}}$ is a $2-(2^n(2^{n+1}-1), 2^n, 1)$ design, were the points are points of $\mathrm{AG}(2n+2,2)$ and the blocks are n-dimensional affine subspaces of $\mathrm{AG}(2n+2,2)$.

(b) $P = Q^+(2n+1,q)$, $q \in \{2,3\}$. First, let $q = 2$. Then the SPG has parameters

$$s = 2^n - 1, \quad t = 2^n, \quad \alpha = 2^{n-1}, \quad \mu = 2^{2n-1} + 2^{n-1}, \quad n \geqslant 1.$$

As $\mu = \alpha(t+1)$, the SPG is a PG.

For n odd the polar space $Q^+(2n+1,2)$ admits a spread, hence an SPG system $\mathscr{T}$.

Let w be the nucleus of $Q(2n+2,2) \supset Q^+(2n+1,2)$. Let Π be a hyperplane of $\mathrm{PG}(2n+2,2) \supset Q(2n+2,2)$, with $w \notin \Pi$. Further, let $\mathscr{T}$ be the SPG system arising from a spread of $Q^+(2n+1,2)$, n odd. By projecting the SPG $\mathscr{S}$ from w onto Π, we now obtain, for $n \geqslant 2$, the PG of De Clerck, Dye and Thas [7]. For $n = 1$ the PG is the complete bipartite graph $K_{3,3}$.

Next, let $q = 3$. Then the SPG has parameters

$$s = 3^n - 1, \quad t = 3^n, \quad \alpha = 2.3^{n-1}, \quad \mu = 2(3^{2n-1} + 3^{n-1}).$$

As $\mu = \alpha(t+1)$, the SPG is a PG.

Let w be the pole of $\mathrm{PG}(2n+1,3) \supset Q^+(2n+1,3)$ with respect to the polar space $Q(2n+2,3) \subset \mathrm{PG}(2n+2,3)$, with $Q^+(2n+1,3) \subset Q(2n+2,3)$. Further, let $\mathscr{T}$ be the SPG system arising from a spread of $Q^+(2n+1,3)$, n odd. By projecting the SPG $\mathscr{S}$ from w onto $\mathrm{PG}(2n+1,3)$, we now obtain, for $n \geqslant 2$, the PG of Thas [12]. Notice that just for $n \in \{1,3\}$ a spread of $Q^+(2n+1,3)$ is known. For $n = 1$ the PG is the dual of a net of order 4 and degree 3.

(c) $P = H(2n+1,q)$, *q a square*. In such a case the SPG has parameters

$$s = q^n - 1, \quad t = q^n\sqrt{q}, \quad \alpha = (\sqrt{q}+1)q^{n-1}, \quad \mu = q^{n-1}(q^n-1)\sqrt{q}(\sqrt{q}+1), \quad n \geqslant 1.$$

For $n = 1$, $\mathscr{T} = H(3,q)$ and $\mathscr{S}$ has parameters

$$s = q - 1, \quad t = q\sqrt{q}, \quad \alpha = \sqrt{q}+1, \quad \mu = (q-1)\sqrt{q}(\sqrt{q}+1).$$

For $n \geqslant 2$ no SPG system of $H(2n+1,q)$ is known.

Now let w be the pole of $\mathrm{PG}(2n+1,q) \supset H(2n+1,q)$ with respect to the polar space $H(2n+2,q) \subset \mathrm{PG}(2n+2,q)$, with $H(2n+1,q) \subset PG(2n+2,q)$. Now we project $\mathrm{PG}(2n+2,q) - \{w\}$ from w onto $\mathrm{PG}(2n+1,q)$. With any point $\bar{y} \in \mathscr{P}$ there corresponds, by projection, a point of $\mathrm{PG}(2n+1,q) - H(2n+1,q)$; in such a way there arises a bijection from $\mathscr{P}$ onto the point set $\mathrm{PG}(2n+1,q) - H(2n+1,q)$. With the lines of $\mathscr{S}$ there correspond the n-dimensional subspaces γ of $\mathrm{PG}(2n+1,q)$ for which $H(2n+1,q) \cap \gamma \in \mathscr{T}$. In this way there arises an SPG $\tilde{\mathscr{S}} \cong \mathscr{S}$, where incidence is inherited from $\mathrm{PG}(2n+1,q)$. For $n=1$ this description of $\tilde{\mathscr{S}}$ was already found by Thas; see De Clerck and Van Maldeghem [8].

(d) $P/\{x\} = Q(2n,q)$ *with q even*. In such a case we have

$$s = q^n - 1, \quad t = q^n, \quad \alpha = q^n.$$

Hence $\mathscr{S}$ is a $2-(q^{2n}, q^n, 1)$ design, so an affine plane of order q^n. Let w be the nucleus of $Q(2n+2,q) \supset P$. Now we project $\mathrm{PG}(2n+2,q) - \langle x, w\rangle$, with $Q(2n+2,q) \subset \mathrm{PG}(2n+2,q)$ from $\langle x, w\rangle$ onto a subspace $\mathrm{PG}(2n,q) \subset \mathrm{PG}(2n+2,q)$ which is skew to the line $\langle x, w\rangle$. The projection of $P - \{x\}$ is a $\mathrm{PG}(2n-1,q)$, the projection of a point of $\mathscr{S}$ is a point of $\mathrm{AG}(2n,q) = \mathrm{PG}(2n,q) - \mathrm{PG}(2n-1,q)$ (in this way there arises a bijection of $\mathscr{P}$ onto $\mathrm{AG}(2n,q)$), and the projection of a line of $\mathscr{S}$ is an n-dimensional affine subspace of $\mathrm{AG}(2n,q)$. With the elements of $\mathscr{T}$ correspond maximal totally isotropic subspaces of some symplectic polarity θ of $\mathrm{PG}(2n-1,q)$. In this way there arises an affine plane $\tilde{\mathscr{S}} \cong \mathscr{S}$, where the incidence is inherited from $\mathrm{AG}(2n,q)$.

For $n=1$ we have $\tilde{\mathscr{S}} = \mathrm{AG}(2,q)$. For $n>1$ and $\mathscr{T}$ defined by a spread of $Q(2n,q)$, $\tilde{\mathscr{S}}$ is the André–Bose–Bruck representation of the translation plane $\mathscr{S}$; in this case the projection of $\mathscr{T}$ is an SPG regulus $\tilde{\mathscr{T}}$, and $\tilde{\mathscr{S}}$ is constructed from $\tilde{\mathscr{T}}$ in the usual way.

(e) $P/\{x\} = Q^+(2n-1,q)$. In such a case we have

$$s = q^n - 1, \quad t = q^{n-1}, \quad \alpha = q^{n-1}, \quad \mu = q^{n-1}(q^{n-1}+1).$$

Hence $\mathscr{S}$ is the dual of a net of order $q^{n-1}+1$ and degree q^n. Now we project $\mathrm{PG}(2n+1,q)$, with $Q^+(2n+1,q) \subset \mathrm{PG}(2n+1,q)$ and $Q^+(2n-1,q) \subset Q^+(2n+1,q)$ from x onto a subspace $\mathrm{PG}(2n,q) \subset \mathrm{PG}(2n+1,q)$ which does not contain x. Then there arises a dual net with point set $\mathrm{AG}(2n,q) = \mathrm{PG}(2n,q) - \mathrm{PG}(2n-1,q)$, with n-dimensional affine subspaces of $\mathrm{AG}(2n,q)$ as lines, and where incidence is inherited from $\mathrm{AG}(2n,q)$.

For $\mathscr{T}$ defined by a spread of $Q^+(2n-1,q)$ (then either $n=1$ or n is even), the projection of $\mathscr{T}$ is an SPG regulus $\tilde{\mathscr{T}}$ and $\tilde{\mathscr{S}}$ is constructed from $\tilde{\mathscr{T}}$ in the usual way.

(f) $P/\{x\} = Q^-(2n+1,q)$. In such a case we have

$$s = q^n - 1, \quad t = q^{n+1}, \quad \alpha = q^{n-1}, \quad \mu = q^n(q^n - 1).$$

By projection from x we obtain again a representation $\tilde{\mathscr{S}}$ of $\mathscr{S}$ in an affine space $\mathrm{AG}(2n+2,q)$. For $\mathscr{T}$ defined by a spread of $Q^-(2n+1,q)$ the projection of $\mathscr{T}$ is an

SPG regulus $\tilde{\mathcal{T}}$ in PG$(2n+1,q)$ and $\tilde{\mathcal{S}}$ is constructed from $\tilde{\mathcal{T}}$ in the usual way; these examples were also described at the end of Section 2.

(g) $P/\{x\} = H(2n-1,q)$, *q a square*. In such a case we have

$$s = q^n - 1, \quad t = q^{(2n-1)/2}, \quad \alpha = q^{(2n-1)/2}, \quad \mu = q^{(2n-1)/2}(q^{(2n-1)/2} + 1).$$

Hence $\mathcal{S}$ is the dual of a net of order $q^{(2n-1)/2} + 1$ and degree q^n. By projection from x we obtain a representation $\tilde{\mathcal{S}}$ of $\mathcal{S}$ in an affine space AG$(2n,q)$. For $n=1$ we obtain the dual of a net of order $\sqrt{q}+1$ and degree q.

(h) $P/\{x\} = H(2n,q)$, *q a square*. Then the SPG has parameters

$$s = q^n, \quad t = q^{(2n+1)/2}, \quad \alpha = q^{(2n-1)/2}, \quad \mu = q^n(q^n - 1).$$

By projection from x we obtain a representation $\tilde{\mathcal{S}}$ of $\mathcal{S}$ in an affine space AG$(2n+1,q)$.

For $n=1$ we have $s = q-1$, $t = q^{3/2}$, $\alpha = q^{1/2}$, $\mu = q(q-1)$; see also 1.3 of Debroey and Thas [6]. For $\mathcal{T}$ defined by a spread of $H(2n,q)$ (but no such spread is known) the projection of $\mathcal{T}$ is an SPG regulus $\tilde{\mathcal{T}}$ in PG$(2n,q)$ and $\tilde{\mathcal{S}}$ is constructed from $\tilde{\mathcal{T}}$ in the usual way.

(i) $P/\{x\} = W_{2n-1}(q)$. Then we have

$$s = q^n - 1, \quad t = q^n, \quad \alpha = q^n.$$

Hence $\mathcal{S}$ is a $2-(q^{2n}, q^n, 1)$ design, so an affine plane of order q^n. By projection from x we obtain a representation $\tilde{\mathcal{S}}$ of $\mathcal{S}$ in AG$(2n,q)$. For $n=1$ we have $\tilde{\mathcal{S}} = $ AG$(2,q)$. For $n>1$ and $\mathcal{T}$ defined by a spread of $W_{2n-1}(q)$, $\tilde{\mathcal{S}}$ is the André–Bose–Bruck representation of the translation plane $\mathcal{S}$.

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