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Some remarks on multiplier ideals and vector bundles

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In this paper, we give two applications of the theory of multiplier ideals to vector bundles over complex projective manifolds, generalizing to higher rank results already established for line bundles. The first addresses the existence of sections of (suitable twists) of symmetric powers of a very ample vector bundle, vanishing on a given subvariety. The second is a vanishing theorem of Griffiths type, adjusted with an additional multiplier ideal term as in the standard Nadel vanishing theorem.

To begin with, we recall that, starting with work of Bombieri and Skoda, there has been considerable interest in going from hypersurfaces in $\mathbb{P}^n$ highly singular at a given set of points S to hypersurfaces through S of relatively low degree; there is now a very direct and terse approach based on multiplier ideals [3], [4], [5], [6].

Here, the same method is shown to yield a statement in the same spirit about sections of very ample vector bundles on a projective manifold. One says that a rank r holomorphic vector bundle on a complex projective manifold is very ample if the relative hyperplane line bundle $\mathcal{O}_{\mathbb{P}\mathscr{E}^*}(1)$ on the projectivised dual $\mathbb{P}\mathscr{E}^*$ is very ample.

If X is a projective manifold and $Z \subseteq X$ an irreducible subvariety, for every integer $p \geqslant 1$ the p-th symbolic power $\mathscr{I}_Z^{\langle p \rangle} \subseteq \mathcal{O}_Z$ of the ideal sheaf of Z is the ideal sheaf of the holomorphic functions vanishing with multiplicity $\geqslant p$ along Z (that is, at a generic point of Z).

Theorem 1. *Let X be an n-dimensional complex projective manifold. Let $\mathscr{E}$ be a rank r very ample vector bundle over X. Let $Z \subseteq X$ be a codimension e irreducible subvariety. If $H^0(X, \operatorname{Sym}^d \mathscr{E} \otimes \mathscr{I}_Z^{\langle t \rangle}) \neq 0$, then*

$$H^0(X, K_X \otimes \det(\mathscr{E}) \otimes \operatorname{Sym}^{n+\ell} \mathscr{E} \otimes \mathscr{I}_Z) \neq 0$$

as soon as $\ell \geqslant \left[\frac{de}{t}\right]$.

Set $Y = \mathbb{P}\mathscr{E}^* \xrightarrow{\pi} X$, the projectivised dual, and let $\mathcal{O}_Y(1)$ be the relative hyperplane bundle on Y. Let A be any divisor in Y, not necessarily effective, such that $\mathcal{O}_Y(A) = \mathcal{O}_Y(1)$. If $D = \sum_i a_i D_i \in \operatorname{Div}_{\mathbb{Q}}(X)$, with the $D_i \subset X$ irreducible divisors, the pull-back of D to Y is the $\mathbb{Q}$-divisor $\pi^*(D) = \sum_i a_i \pi^*(D)$. We shall say that $\mathscr{E}(-D)$ is nef and big if the rational divisor $A - \pi^*(D) \in \mathrm{Div}_{\mathbb{Q}}(Y)$ is nef and big. The following statement generalizes the Griffiths vanishing theorem for ample vector bundles [7]:

Theorem 2. *Suppose that* $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, $\mathscr{E}$ *is a holomorphic vector bundle on* X, *and* $\mathscr{E}(-D)$ *is nef and big. Then*

$$H^i(X, K_X \otimes \det(\mathscr{E}) \otimes \mathrm{Sym}^m \mathscr{E} \otimes \mathscr{J}(D)) = 0 \quad \textit{if } i > 0, m \geqslant 0.$$

Vanishing theorems for vector bundles involving multiplier ideals (or multiplier subsheaves) were obtained also in [1], based on the differential-theoretic notions of t-nefness and singular hermitian metrics (see also [2]).

Proof of Theorem 1. Let $Y = \mathbb{P}\mathscr{E}^* \xrightarrow{\pi} X$ be the projectivized dual, so that

$$\mathrm{Sym}^k \mathscr{E} = \pi_* \mathscr{O}_Y(k)$$

for every $k \geqslant 0$. We shall denote the natural isomorphism $H^0(X, \mathrm{Sym}^k(\mathscr{E})) \cong H^0(Y, \mathscr{O}_Y(k))$ by $\sigma \mapsto \tilde{\sigma}$.

Suppose that $0 \neq \sigma \in H^0(X, \mathrm{Sym}^d \mathscr{E} \otimes \mathscr{I}_Z^{\langle t \rangle})$: for every $x \in Z$ and in any trivialization of $\mathscr{E}$ in an open neighbourhood U of x, σ is represented by an $\binom{r+d}{d}$-tuple of functions $f_I \in \mathscr{I}_Z^{\langle t \rangle}(U)$. Here I runs over the set $\mathscr{A}_d$ of all multiindices $I = (i_1, \dots, i_r) \geqslant 0$ with $|I| = d$.

The trivialization of $\mathscr{E}$ on U induces isomorphisms

$$\pi^{-1}(U) \cong U \times \mathbb{P}^{r-1} \quad \text{and} \quad \mathscr{O}_Y(k)|_{\pi^{-1}(U)} \cong p_2^* \mathscr{O}_{\mathbb{P}^{r-1}}(k)$$

for every k, where $p_2 : U \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the projection onto the second factor. In terms of these isomorphisms, $\tilde{\sigma}$ is then given in homogeneous coordinates on $U \times \mathbb{P}^{r-1}$ by

$$\tilde{\sigma}(y', X) = \sum_{I \in \mathscr{A}_d} f_I(y') X^I.$$

Thus, if $\tilde{Z} = \pi^{-1}(Z) \subseteq Y$ and

$$D_{\tilde{\sigma}} := \mathrm{div}(\tilde{\sigma}) \in |H^0(Y, \mathscr{O}_Y(d))|,$$

we have $\mathrm{mult}_{\tilde{Z}}(D_{\tilde{\sigma}}) \geqslant t$. In other words, $\tilde{\sigma} \in H^0(Y, \mathscr{I}_{\tilde{Z}}^{\langle t \rangle} \otimes \mathscr{O}_Y(d))$.

If $D = (e/t) \cdot D_{\tilde{\sigma}}$, then $\mathrm{mult}_{\tilde{Z}}(D) \geqslant e = \mathrm{codim}(\tilde{Z}, Y)$, and therefore the multiplier ideal satisfies $\mathscr{J}(D) \subseteq \mathscr{I}_{\tilde{Z}}$.

Set $\delta = \left[\frac{de}{t}\right]$. Then if $A \in |\mathscr{O}_Y(1)|$ the $\mathbb{Q}$-divisor $\ell A - D$ is ample for $\ell \geqslant \delta + 1$. Therefore, $K_Y \otimes \mathscr{O}_Y(\ell + n + r - 1) \otimes \mathscr{J}(D)$ is globally generated. Since the canonical line bundle of Y is $K_Y = \pi^*(K_X \otimes \det(\mathscr{E})) \otimes \mathscr{O}_Y(-r)$,

$$\pi^*(K_X \otimes \det(\mathscr{E})) \otimes \mathscr{O}_Y(\ell - 1 + n) \otimes \mathscr{J}(D)$$

is globally generated. In particular,

$$H^0(Y, \pi^*(K_X \otimes \det(\mathscr{E})) \otimes \mathcal{O}_Y(\ell - 1 + n) \otimes \mathscr{J}(D)) \neq 0,$$

and therefore $H^0(Y, \pi^*(K_X \otimes \det(\mathscr{E})) \otimes \mathcal{O}_Y(\ell - 1 + n) \otimes \mathscr{I}_{\tilde{Z}}) \neq 0$. By pushing forward, this implies

$$H^0(X, K_X \otimes \det(\mathscr{E}) \otimes \mathrm{Sym}^{\ell-1+n} \mathscr{E} \otimes \mathscr{I}_Z) \neq 0$$

for $\ell - 1 \geqslant \delta$. □

Proof of Theorem 2. By the Nadel vanishing theorem,

$$H^i(Y, K_Y \otimes \mathcal{O}_Y(m) \otimes \mathscr{J}(\pi^* D)) = 0, \quad \text{for all } i, m > 0.$$

By the projection formula and and the arguments in Chapter V of [7],

$$H^i(X, K_X \otimes \det(E) \otimes \pi_*(\mathcal{O}_Y(m - r) \otimes \mathscr{J}(\pi^*(D))) = 0, \quad \text{if } i, m > 0.$$

Lemma 1. $\mathscr{J}(\pi^*(D)) = \pi^*(\mathscr{J}(D))$.

Assuming the lemma,

$$\begin{aligned} &H^i(X, K_X \otimes \det(E) \otimes \pi_*(\mathcal{O}_Y(m - r) \otimes \mathscr{J}(\pi^*(D))) \\ &\quad \cong H^i(X, K_X \otimes \det(\mathscr{E}) \otimes \mathrm{Sym}^{m-r}(\mathscr{E}) \otimes \mathscr{J}(D)), \end{aligned}$$

again invoking the projection formula, and the theorem follows. □

Proof of Lemma 1. Let $\mu : X' \to X$ be a log-resolution of D; recall that this means that X' is non-singular, μ is a proper birational map, and $\mu^*(D) + \mathrm{Exc}(\mu)$ has simple normal crossing support, where $\mathrm{Exc}(\mu)$ denotes the sum of the exceptional divisors of μ [6].

Set $\mathscr{E}' = \mu^*(\mathscr{E})$, $Y' = \mathbb{P}\mathscr{E}'^* = Y \times_X X'$. We have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\mu} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\mu} & X \end{array}$$

Since π' is smooth, $\pi'^*(\mathrm{Exc}(\mu)) = \mathrm{Exc}(\mu')$, $\mu' : Y' \to Y$ is a log-resolution of $\pi^*(D)$, $K_{Y'/Y} = \pi'^*(K_{X'/X})$, $[\pi'^* \mu^*(D)] = \pi'^*[\mu^*(D)]$. Therefore, since $\mu'^*(\pi^*(D)) = \pi'^*(\mu^*(D))$,

$$\mathscr{J}(\pi^*(D)) = \mu'_* \mathcal{O}_{Y'}(K_{Y'/Y} - [\mu'^* \pi^*(D)]) = \mu'_* \pi'^* \mathcal{O}_{X'}(K_{X'/X} - [\mu^*(D)]).$$

Let Coh(Z) denote the category of coherent sheaves of $\mathcal{O}_X$-modules on a projective manifold Z. Lemma 1 is now an immediate consequence of the following:

Lemma 2. $\mu'_* \circ \pi'^* = \pi^* \circ \mu_* : \mathrm{Coh}(X') \to \mathrm{Coh}(Y)$.

Proof of Lemma 2. Let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_{X'}$-modules. A basis for the Zariski topology of Y is given by affine open subsets $U = \mathrm{Spec}(B) \subseteq Y$ such that $V = \pi(U) = \mathrm{Spec}(A) \subseteq X$ is also affine. The restricted projection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponds to a morphism of rings $A \to B$. Let $V' = \mu^{-1}(V) \subseteq X'$. Then $\mu_* \mathscr{F}(V) = \mathscr{F}(V')$, viewed as an A-module by the morphism $A = \mathcal{O}_X(V) \to \mathcal{O}_{X'}(V')$. Thus, $\pi^* \mu_* \mathscr{F}(U) = \mathscr{F}(V') \otimes_A B$.

Next, $\mu'_* \circ \pi'^* \mathscr{F}(U) = \pi'^* \mathscr{F}(U')$, where $U' = \mu^{-1}(U)$ and $\pi'^* \mathscr{F}(U')$ is regarded as a B-module via the homomorphism $B \to \mathcal{O}_{Y'}(U')$. On the other hand, $\pi'^* \mathscr{F}$ is the sheafification of the presheaf on Y'

$$\tilde{\mathscr{F}}(S) = \mathscr{F}(\pi'(S)) \otimes_{\mathcal{O}_{X'}(\pi'(S))} \mathcal{O}_{Y'}(S) \quad (S \subseteq Y' \text{ open}).$$

As $\pi'(\mu'^{-1}(U)) = \mu^{-1}(\pi(U)) = V' \subseteq X'$, we have

$$\tilde{\mathscr{F}}(U') = \mathscr{F}(V') \otimes_{\mathcal{O}_{X'}(V')} \mathcal{O}_{Y'}(U').$$

Since μ and μ' are projective morphisms, $\mathcal{O}_{X'}(V') \cong A$ and $\mathcal{O}_{Y'}(U') = A$. Thus, $\mathscr{F}(V') \otimes_A B \cong \mathscr{F}(V') \otimes_{\mathcal{O}_{X'}(V')} \mathcal{O}_{Y'}(U')$. □

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