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Subgroups of the Chevalley groups of type F_4 arising from a polar space

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Abstract. For any field K, we determine the quasi-simple subgroups G of the Chevalley group $F_4(K)$ which are generated by a class Σ of so-called abstract transvection subgroups of G such that any member of Σ is contained in a long root subgroup of $F_4(K)$. First, we construct a polar space with point set Σ, which is embedded in the F_4-geometry. This yields that a conjugate of G is contained in a classical standard subsystem subgroup of $F_4(K)$ or G arises from a Moufang quadrangle in characteristic 2. For the second possibility, the so-called F_4-quadrangles are worth mentioning.

Introduction

For an arbitrary commutative field K, we denote by $F_4(K)$ the universal Chevalley group of type F_4 over K. This is the group generated by symbols $x_r(t)$, $t \in K$, $r \in \Phi$, with respect to the Steinberg relations; we refer to Carter [3, 12.1.1]. Here Φ is the root system of type F_4, a subset of the Euclidean space $\mathbb{R}^4$ with orthonormal basis $\{e_1, e_2, e_3, e_4\}$. In the notation of Bourbaki [1], the extended Dynkin diagram of type F_4 is

$$\begin{array}{ccccccccc} \circ & \text{---} & \circ & \text{---} & \circ & \Rightarrow & \circ & \text{---} & \circ \\ -\alpha_* & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \end{array}$$

where $\quad \alpha_* = e_1 + e_2, \quad \alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4,$

$$\alpha_3 = e_4, \qquad \alpha_4 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

A long root subgroup of $F_4(K)$ is a conjugate of $X_{\alpha_*} = \{x_{\alpha_*}(t) \mid t \in K\} \simeq (K, +)$. The group $F_4(K)$ is generated by its class of long root subgroups and is simple.

In the following, we study which groups arising from a polar space occur as subgroups of $F_4(K)$. The groups arising from a polar space are closely related to the groups generated by a class of so-called abstract transvection subgroups, see Timmesfeld [20], [21].

Here a conjugacy class Σ of abelian subgroups of a group G is called a class of abstract transvection subgroups of G, if $G = \langle \Sigma \rangle$ and for $A, B \in \Sigma$, either $[A, B] = 1$

or $\langle A, B\rangle$ is a rank 1 group. (The latter means that for $a \in A^{\#}$, there exists $b \in B^{\#}$ such that $A^b = B^a$ and vice versa.)

This paper is devoted to the study of the following problem:

(P) Let Σ be a class of abstract transvection subgroups of G such that

(a) there are distinct commuting elements in Σ,

(b) $C_\Sigma(A) = C_\Sigma(C)$ implies $A = C$,

(c) $|A| \geqslant 3$, for $A \in \Sigma$.

We assume that G is a subgroup of $Y = F_4(K)$ such that any element $A \in \Sigma$ is contained in a long root subgroup $\hat{A}$ of Y. The problem is to determine the possible G and the embedding of G in Y. It turns out that any such G is quasi-simple.

Problem (P) contributes to the determination of subgroups of groups of Lie type generated by long root elements. For a root system Ψ which is contained in Φ and has fundamental root system $\{p_1, \dots, p_r\}$, we set $M(p_1, \dots, p_r) := \langle X_r \mid r \in \Psi\rangle$. These subgroups of $F_4(K)$ are called standard subsystem subgroups. In $F_4(K)$ there are the classical standard subsystem subgroups

$$\begin{aligned} M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) &\simeq B_4(K), \\ M(\alpha_2, \alpha_3, \alpha_4) &\simeq C_3(K), \\ M(\alpha_2, \alpha_3, \alpha_4, -e_1) &\simeq C_4(K) \quad \text{for } \mathrm{char}(K) = 2 \end{aligned}$$

and the standard subsystem subgroups of these.

When G as in (P) already embeds in a (proper) standard subsystem subgroup of $F_4(K)$, the problem to determine the possible G is reduced to the study of subgroups of classical groups. In this case, we may apply Steinbach [16] and Cuypers and Steinbach [7].

We prove the following:

Theorem 1. *For any subgroup G of $F_4(K)$ as in* (P) *above, passing to a conjugate, one of the following holds*:

(1) *G is contained in one of the classical standard subsystem subgroups $M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) \simeq B_4(K)$, $M(\alpha_2, \alpha_3, \alpha_4) \simeq C_3(K)$ or $M(\alpha_2, \alpha_3, \alpha_4, -e_1) \simeq C_4(K)$ for* $\mathrm{char}(K) = 2$.

(2) $\mathrm{char}(K) = 2$ *and G arises from a Moufang quadrangle.*

We remark that there is overlap between Cases (1) and (2). Below in Theorem 2 and Theorem 3 (which yield Theorem 1) we give more detailed information on the possible subgroups G and their embeddings in $F_4(K)$. For unexplained terminology, we refer to Section 1.

In addition to the Steinberg generators and relations for $F_4(K)$ mentioned above, we use the associated building. In this building, there are four types of objects, called points, lines, planes and symplecta, and the long root subgroups of $F_4(K)$ may be identified with the points.

For G as in (P) we prove first that Σ is the point set of a polar space $\wp(\Sigma)$, where lines embed in symplecta of the building associated to $F_4(K)$. We say that G arises from a polar space. Throughout we use the result of Buekenhout and Shult [2], which characterizes polar spaces as point-line-spaces satisfying the one-or-all axiom.

When the rank of $\wp(\Sigma)$ is at least 3, we apply the classification of polar spaces due to Tits [22]. In the rank 2 case, $\wp(\Sigma)$ is a Moufang quadrangle. We use the classification of Moufang quadrangles by Tits and Weiss [24] (as stated in Van Maldeghem [25]). Another important tool is the determination of weakly embedded polar spaces (including generalized quadrangles) in Steinbach and Van Maldeghem [14], [15]. For the definition of weak embeddings of polar spaces, we refer to (1.11).

For subgroups of $F_4(K)$ as in (P) of rank $\geqslant 3$ and rank 2, respectively, we prove:

Theorem 2. *Let G be a subgroup of $F_4(K)$ as in* (P), *such that there exist three distinct pairwise commuting elements $A, B, C \in \Sigma$ with $C \notin C_\Sigma(C_\Sigma(A, B))$.*

Then a conjugate of G is contained in $M(\alpha_2, \alpha_3, \alpha_4) = C_3(K)$, when $\mathrm{char}(K) \neq 2$, *and in $M(\alpha_2, \alpha_3, \alpha_4, -e_1) = C_4(K)$, when* $\mathrm{char}(K) = 2$ (*with underlying symplectic space denoted by V in both cases*). *Moreover, Σ is the point set of a symplectic polar space of rank* 3 *and of some orthogonal polar space* (*with degenerate associated symplectic form*) *of rank* 3 *or* 4, *respectively, which is weakly embedded in $P([V, G])$.*

Theorem 3. *Let G be a subgroup of $F_4(K)$ as in* (P), *such that $C \in C_\Sigma(C_\Sigma(A, B))$, whenever $A, B, C \in \Sigma$ are distinct and pairwise commuting.*

Then Σ is the point set of a Moufang quadrangle $\wp(\Sigma)$. When $\mathrm{char}(K) = 2$, *we assume furthermore that Σ is the class of full central elation subgroups of $\wp(\Sigma)$. Then one of the following holds*:

(a) *A conjugate of G is contained in $M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$* (*with underlying orthogonal space V*). *Moreover, the dual quadrangle $\wp(\Sigma)^D$ is an orthogonal quadrangle or a mixed quadrangle weakly embedded in $P([V, G])$.*

(b) $\mathrm{char}(K) = 2$ *and a conjugate of G is contained in $M(\alpha_2, \alpha_3, \alpha_4, -e_1) = C_4(K)$* (*with underlying symplectic space V*). *Moreover, $\wp(\Sigma)$ is some orthogonal quadrangle weakly embedded in $P([V, G])$.*

(c) $\mathrm{char}(K) = 2$ *and $\wp(\Sigma)$ is a so-called F_4-quadrangle.*

We refer to Section 5 for a description of the F_4-quadrangles and to Theorem 6.2 for their embeddings in $F_4(K)$.

In this paper we work with arbitrary fields, including non-perfect fields of characteristic 2 (as for example the field of rational functions over GF(2)). The latter are involved in many interesting phenomena, in particular in the F_4-quadrangles.

For finite groups and also for algebraic groups over an algebraically closed field, results on groups of Lie type embedded in $F_4(K)$ are in the literature. Stensholt [19] constructs embeddings among finite groups of Lie type such that long root subgroups are long root subgroups. For the exceptional types in the finite case, the embedded groups of Lie type have been determined by Cooperstein [5], [6].

Subgroups of simple algebraic groups over an algebraically closed field, which are generated by full long root subgroups have been determined by Liebeck and Seitz [10] for all classical and exceptional types. In their setting the subgroups in question arise from one of the spherical root systems.

Groups generated by abstract transvection subgroups have been classified by Timmesfeld; we refer to [20, Thm. 3] or [21, III §1]. The quasi-simple ones arise from a polar space, provided that $A \in \Sigma$ is not too small. This result applies for G as in (P) with $|A| \geqslant 4$. In this paper I preferred to exploit the fact that G is a subgroup of $F_4(K)$.

For $|A| \geqslant 3$, we obtain a polar space associated to G, which is embedded in the F_4-geometry by construction. But for $|A| = 2$, G is a 3-transposition group and does not necessarily arise from a polar space; we refer to Cooperstein [5, Part II]. We remark that when $|A| = 2$, we can also handle the (classical) groups which arise from a polar space with the methods of this paper.

The paper is organized as follows: In the preliminary Section 1, we collect properties of $F_4(K)$, classical groups and polar spaces for later use. In Section 2 we construct the polar space with point set Σ which is embedded in the F_4-geometry. In Section 3 we deal with the subgroups of rank $\geqslant 3$ and we prove Theorem 2. Next, Theorem 3 on subgroups arising from a Moufang quadrangle is proved in Section 4. Finally, Sections 5 and 6 are devoted to the F_4-quadrangles and their embeddings.

The remaining subgroups of Lie type in $F_4(K)$ are dealt with in Steinbach [13]. There the subgroups in question are generated by a non-degenerate class Σ of abstract root subgroups in the sense of Timmesfeld [20], [21]. In particular, there are $A, B \in \Sigma$ with $[A, B] \in \Sigma$.

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1 Preliminaries

For the definition and properties of Chevalley groups and the associated root systems, we refer to Carter [3], Steinberg [18] and Bourbaki [1].

1.1 Chevalley commutator relations in $F_4(K)$. Let $t, u \in K$ and $r, s \in \Phi$. When $0 \neq r + s \notin \Phi$, then $[x_r(t), x_s(u)] = 1$. When $r + s \in \Phi$, then the following holds (with signs depending on r, s, but not on t, u):

(a) If r, s are long or if $r, s, r + s$ are short, then $[x_r(t), x_s(u)] = x_{r+s}(\pm tu)$.

(b) If r, s are short and $r + s$ is long, then $[x_r(t), x_s(u)] = x_{r+s}(\pm 2tu)$.

(c) If r is long and s is short, then $[x_r(t), x_s(u)] = x_{r+s}(\pm tu)x_{r+2s}(\pm tu^2)$.

Furthermore, $\langle X_r, X_{-r} \rangle \simeq \mathrm{SL}_2(K)$.

1.2 Standard subsystem subgroups. In the root system Φ of type F_4 we consider the following root systems with a fundamental system of the indicated type:

$$\Phi(C_3) : \{\alpha_2, \alpha_3, \alpha_4\}, \quad \Phi(B_4) : \{-\alpha_*, \alpha_1, \alpha_2, \alpha_3\}, \quad \Phi(C_4) : \{\alpha_2, \alpha_3, \alpha_4, -e_1\}$$

The roots of $\Phi(C_3)$ are $\pm(e_1 - e_2)$, $\pm e_3 \pm e_4$, $\pm e_3$, $\pm e_4$, $\pm\frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4)$; the ones of $\Phi(B_4)$ are $\pm e_i \pm e_j$, $\pm e_i$ (all long roots); and in $\Phi(C_4)$ there are $\pm e_1 \pm e_2$, $\pm e_3 \pm e_4$, $\pm e_i$, $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ (all short roots).

We use the definition of a standard subsystem subgroup $M(p_1, \ldots, p_r)$ as given in the introduction. The centralizer in $F_4(K)$ of $\langle X_{\pm(e_1+e_2)}\rangle$ is $M_1 := M(\alpha_2, \alpha_3, \alpha_4)$, the one of $\langle X_{\pm(e_1-e_2)}\rangle$ is $M_2 := M\big(\alpha_2, \alpha_3, \frac{1}{2}(e_1 + e_2 - e_3 - e_4)\big)$ (both 6-dimensional symplectic groups). Furthermore, $\langle M_1, M_2\rangle$ is $M(\alpha_2, \alpha_3, \alpha_4, -e_1)$, which is $F_4(K)$ for $\mathrm{char}(K) \neq 2$ and $C_4(K)$ for $\mathrm{char}(K) = 2$.

1.3 The F_4-geometry. We consider the building associated to $F_4(K)$ (in the sense of Tits [22]) as a point-line geometry, the F_4-geometry. (In Tits [22] and in subsequent papers this point-line geometry is called a metasymplectic space.) There are four types of objects: points, lines, planes and symplecta. For properties of symplecta, we refer to Timmesfeld [21, III Sec. 7], Van Maldeghem [25, p. 80] and Cooperstein [5, p. 333].

A point is a long root subgroup, the standard point being $X_{e_1+e_2}$. Two long root subgroups A, C define a line, a so-called F_4-line, precisely when any element in AC is a long root element. The standard line is $X_{e_1+e_2}X_{e_1+e_3}$ (identified with the set of long root subgroups contained in it). Similarly, three long root subgroups (not on a line) define a plane, when any two define a line.

As follows from the Dynkin diagram of type F_4, all points, lines and planes of the F_4-geometry contained in a symplecton (seen as point-line geometry) yield a polar space of type B_3. Any two commuting long root subgroups A, B of $F_4(K)$, which do not define an F_4-line, are contained in a unique symplecton $S(A, B)$ of the F_4-geometry. The standard symplecton on $X_{e_1+e_2}$ and $X_{e_1-e_2}$ is (the set of long root subgroups contained in)

$$S := S(X_{e_1+e_2}, X_{e_1-e_2}) = \langle X_{e_1\pm e_2}, X_{e_1\pm e_3}, X_{e_1\pm e_4}, X_{e_1}\rangle.$$

All other symplecta are conjugate. Note that $S \leqslant M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$ and that $S = \mathrm{Z}(U_J)$ in the parabolic subgroup $P_J = U_J L_J$ with Levi complement associated to the diagram $(\alpha_1, \alpha_2, \alpha_3)$ of type B_3. We may consider S as a 7-dimensional natural module for $B_3(K)$.

Let S be the symplecton on $X_{e_1+e_2}$ and $X_{e_1-e_2}$ as above. When A, B are non-collinear points in S (i.e., A, B are not on an F_4-line), then $S = S(A, B)$. Furthermore, S is (the set of points contained in) the subgroup generated by A, B and all T which are collinear with both A and B. For a long root subgroup E generating $\mathrm{SL}_2(K)$ with $X_{e_1+e_2}$, there is a unique long root subgroup T contained in S which commutes with E. Any point in S, which is not on an F_4-line with T, generates $\mathrm{SL}_2(K)$ with E.

1.4 Properties of $F_4(K)$. The permutation rank of $F_4(K)$ on the class of long root subgroups is five. The class of long root subgroups is a class of abstract root subgroups of $F_4(K)$ in the sense of Timmesfeld [20], [21].

The center of $F_4(K)$ is trivial. Any diagonal automorphism of $F_4(K)$ is an inner automorphism. For any long root subgroup T in $Y = F_4(K)$ and $1 \neq t \in T$, we have $C_Y(t) = C_Y(T)$. Let A_i, B_i $(i = 1, \dots, 4)$ be long root subgroups of $Y = F_4(K)$ such that $X_i := \langle A_i, B_i \rangle \simeq \mathrm{SL}_2(K)$ and $[X_i, X_j] = 1$ for $i, j = 1, \dots, 4$, $i \neq j$. Passing to a conjugate, we may assume that $A_1, B_1, \dots, A_4, B_4$ are $X_{e_1+e_2}$, $X_{-e_1-e_2}$, $X_{e_1-e_2}$, $X_{-e_1+e_2}$, $X_{e_3-e_4}$, $X_{-e_3+e_4}$, $X_{e_3+e_4}$, $X_{-e_3-e_4}$.

For any long root subgroup E in $F_4(K)$, we denote by M_E the unipotent radical in the parabolic subgroup $N(E)$ (see Carter [3, 8.5]). For $E = X_{e_1+e_2}$, we have $M_E = \langle X_r \mid r \in \Psi \rangle$, where $\Psi := \{e_1 + e_2, e_1, e_2, \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4), e_1 \pm e_3, e_1 \pm e_4, e_2 \pm e_3, e_2 \pm e_4\}$. Furthermore, A^m is contained in $S(E, A)$ for $m \in M_E$ whenever E and A define a symplecton.

1.5 Classical groups. First, we fix notation for a vector space endowed with a form. For pseudo-quadratic forms, which are a generalization of quadratic forms and of (anti-)hermitian forms, we refer to Tits [22, 8.2].

Let L be a skew field and W a (left-)vector space over L endowed with one of the following non-degenerate forms of Witt index $\geqslant 1$:

(a) a symplectic form $f : W \times W \to L$, $\mathrm{char}(L) \neq 2$,

(b) an ordinary quadratic form $q : W \to L$ (with associated symmetric bilinear form $f : W \times W \to L$),

(c) a $(\sigma, -1)$-quadratic form $q : W \to L/\Lambda$ (with associated anti-hermitian form $f : W \times W \to L$) such that $1 \in \Lambda := \{c + c^\sigma \mid c \in L\}$.

By the form on W we always mean f in (a) and q in (b) and (c). A vector $w \in W$ where the form vanishes is called isotropic (or also singular; in particular when q is an ordinary quadratic form). The form is non-degenerate if there are no non-zero isotropic vectors in $\mathrm{Rad}(W, f) := \{w \in W \mid f(w, v) = 0 \text{ for all } v \in W\}$. For isotropic $x, y \in W$ with $f(x, y) = 1$, we call (x, y) a hyperbolic pair, spanning the hyperbolic line H. Throughout (x_i, y_i) is a hyperbolic pair spanning H_i. We say that the form has Witt index n, when the maximal (totally) isotropic subspaces are n-dimensional.

We consider the classical groups which are isometry groups $\mathrm{Aut}(W, q)$ with q as in (b) or (c). From Cuypers and Steinbach [7, §2] we use the following on isotropic transvection subgroups. An isotropic transvection associated to the isotropic point p is a (non-trivial) element t in $\mathrm{Aut}(W, q)$ with $t|_{p^\perp} = \mathrm{id}$. Note that $[W, t] \subseteq p^{\perp\perp} = p \oplus \mathrm{Rad}(W, f)$. By T_p we denote the isotropic transvection subgroup associated to p. Then $T_p \neq 1$ provided that q is not an ordinary quadratic form with $\mathrm{Rad}(W, f) = 0$.

For orthogonal groups $\Omega(W, q)$ with Siegel transvection subgroups T_ℓ (which correspond to singular lines ℓ; i.e. they are long root subgroups), we refer to Timmesfeld [21, II (1.5)].

Next, we investigate whether a classical group is generated by two subgroups with Witt index of the associated form decreased by 1. In the notation of (1.5), we have:

1.6. *Let L be a skew field and W a vector space over L endowed with a non-degenerate (pseudo-)quadratic form (or a symplectic form in characteristic* $\neq 2$*) of Witt index* $\geqslant 3$*. We denote by G the subgroup of* $\mathrm{GL}(W)$ *generated by the isotropic transvection subgroups.*

Let $W = H_1 \perp H_2 \perp H_3 \perp U$ *and set* $G_1 := \langle T_p \,|\, p \subseteq H_1^\perp \rangle$ *and similarly for* G_3*. Then* $G = \langle G_1, G_3 \rangle$*.*

Proof. For $A := T_{x_1}$, $B := T_{y_1}$, we have $A, B \leqslant G_3$ and $G = \langle C_\Sigma(A), B \rangle$, where Σ is the class of isotropic transvection subgroups. Any $A \neq T \in C_\Sigma(A)$ is of the form $T = T_{cx_1+s}$ with $0 \neq s \in H_1^\perp$, $q(s) = 0$. Since G_1 is transitive on its isotropic points, there is $g \in G_1$ with $\langle s \rangle g = \langle x_2 \rangle$. Thus $T^g \leqslant G_3$ and $G \leqslant \langle G_1, G_3 \rangle$. □

1.7. *Let L be a field of characteristic* 2 *and* $W = H_1 \perp H_2 \perp U \perp \mathrm{Rad}(W, f)$ *a vector space over L endowed with a non-degenerate quadratic form q of Witt index* 2*. We suppose that both U and* $\mathrm{Rad}(W, f)$ *are non-zero.*

Then the group G generated by the isotropic transvection subgroups with respect to q is generated by G_1, G_2*, where* G_i *is the subgroup of G which leaves* $H_i^\perp$ *invariant and centralizes* H_i $(i = 1, 2)$*.*

Proof. With respect to the basis $\{x_1, \ldots, y_1\}$, the unipotent radical M of the stabilizer in G of $\langle x_1 \rangle$ consists of the elements

$$\rho_z := \left(\begin{array}{c|c|c} 1 & & \\ \hline Jz^T & I & \\ \hline q(z) & z & 1 \end{array}\right),$$

where J is the fundamental matrix of $f|_{H_1^\perp}$. Since $G = \langle M, T_p, T_{y_1} \,|\, p$ isotropic point in $H_1^\perp \rangle$, it suffices to show that $M \leqslant \langle G_1, G_2 \rangle$. Since $U \neq 0$, any ρ_z is as a product of conjugates under G_1 of elements ρ_w, $w \in U \perp \mathrm{Rad}(W, f)$ (with $\rho_w \in G_2$). □

There exist quadratic forms as in (1.7), we refer to Dieudonné [8, n^o 26].

1.8 Polar spaces, central and axial elations. For polar spaces, we refer to Tits [22] and Cohen [4]. A point-line geometry Γ is called a polar space, if the one-or-all axiom (due to Buekenhout and Shult [2]) is satisfied: For each point p and line l of Γ, the point p is collinear with either one or all points of l.

When a point p is collinear with a point x of ℓ, then x is called a neighbour of p on ℓ. We only consider non-degenerate polar spaces, where there is no point collinear with all points. A polar space in which a point p is either on a line l or collinear with a unique point of l is also called a generalized quadrangle. In this case the rank of Γ is 2. A Moufang quadrangle is a generalized quadrangle where the automorphism group satisfies a certain transitivity condition, the so-called Moufang condition. The Moufang quadrangles have been classified by Tits and Weiss [24], we refer to Van Maldeghem [25].

A central elation of a polar space Γ with center p is an automorphism of Γ which fixes all points of Γ collinear with the point p. The central elation subgroup of Γ with center p is the subgroup of $\mathrm{Aut}(\Gamma)$ consisting of all central elations with center p. Similarly, the axial elation subgroup associated to a line ℓ fixes all lines concurrent with ℓ.

The geometry of 1- and 2-dimensional subspaces of a vector space where a form as in (1.5) vanishes yields a so-called classical polar space. Here every central elation with center p is induced by an isotropic transvection associated to p, we refer to Cuypers and Steinbach [7, (2.5), (3.4)]. Similarly, in an orthogonal polar space (arising from an ordinary quadratic form) any axial elation is induced by a Siegel transvection. We remark that any symplectic polar space with underlying $2n$-dimensional vector space over L, $\mathrm{char}(L) = 2$, is isomorphic to an orthogonal polar space in dimension $2n + \dim_{L^2} L$; compare Cohen [5, (3.27)].

1.9 Classical Moufang quadrangles. These are the classical polar spaces of rank 2 (up to duality). We use the notation of (1.5). In Case (a), we say Γ is a symplectic quadrangle in characteristic $\neq 2$. In Case (b), Γ is called an orthogonal quadrangle. We write $W = H_1 \perp H_2 \perp W_0$ where (x_i, y_i) is a hyperbolic pair spanning H_i and W_0 is anisotropic. For $f_0 := f|_{W_0}$ and $\mathscr{B}_0$ a basis of W_0, we denote by J the associated fundamental matrix of f_0. We may consider (x_1, x_2, y_1, y_2) as an apartment of Γ and (x_2, x_1, y_2) as a half apartment with associated root group U_2. Note that the matrices of the elements in U_2 are (with respect to the basis $\{x_1, x_2, \mathscr{B}_0, y_2, y_1\}$)

$$\left(\begin{array}{c|ccc|c} 1 & & & & \\ \hline 0 & 1 & & & \\ s & & I & & \\ 0 & & & 1 & \\ \hline c & 0 & z & 0 & 1 \end{array}\right),$$

where $s = -J(z^\sigma)^T$ and $q(z) + (c + \Lambda) = 0$ in (b), (c) (with obvious block decomposition, empty entries are 0). Whence U_2 is abelian exactly in Cases (a) and (b) and in Case (c) with $f_0 = 0$. Clearly, Siegel transvections are axial elations for orthogonal quadrangles. Hence dual orthogonal quadrangles admit central elations.

1.10. *When a classical polar space* Γ *as in* (1.8) *admits* (*non-trivial*) *axial elations, then necessarily* Γ *is orthogonal.*

Proof. Let t be a non-trivial axial elation of Γ with respect to the line $\ell = \langle x_1, x_2\rangle$. When p is a point on ℓ and s is a line of Γ through p, then $st = s$.

We assume that Γ is not orthogonal. Then $\varepsilon = -1$. We have $\langle x_i\rangle t = \langle x_i\rangle$ $(i = 1, 2)$. Since $\langle y_1\rangle$ is on $\langle x_2, y_1\rangle$, but not perpendicular to $\langle x_1\rangle$, we get that $\langle y_1\rangle t = \langle a x_2 + y_1\rangle$ with $a \in L$. Similarly, $\langle y_2\rangle t = \langle \lambda x_1 + y_2\rangle$ with $\lambda \in K$. The points $\langle y_1\rangle t$ and $\langle y_2\rangle t$ are perpendicular, hence $a = \lambda^\sigma$. Necessarily, $\lambda \neq 0$. (Otherwise t is an axial elation for $\langle x_1, x_2\rangle$ fixing $\langle y_1\rangle$ and $\langle y_2\rangle$ and thus is the identity.)

For any $\mu \in L$, the line $s := \langle x_1 - \mu^\sigma x_2, \mu y_1 + y_2 \rangle$ of Γ meets ℓ and is hence fixed by t. Therefore we may calculate $\langle \mu y_1 + y_2 \rangle t$ as the intersection of $\langle y_1, y_2 \rangle t$ and s. By comparing coefficients we get $\mu \lambda^\sigma = -\lambda \mu^\sigma$ for $\mu \in L$. With $x := \lambda \mu^\sigma$ this means $x^\sigma = -x$ for $x \in L$. Thus $\operatorname{char} L = 2$ (and Γ is not symplectic in $\operatorname{char} L \neq 2$), $\sigma = \mathrm{id}$ and $\Lambda = \{c + c^\sigma \mid c \in L\} = 0$. This means that Γ cannot be of type (c) (where $1 \in \Lambda$), a contradiction. □

1.11 Weak embeddings of polar spaces. Let V be a vector space over some skew field K and Γ a polar space. We say that Γ is weakly embedded in the projective space $P(V)$, if there exists an injective map π from the set of points of Γ to the set of points of $P(V)$ such that

(a) the set $\{\pi(x) \mid x \text{ point of } \Gamma\}$ generates $P(V)$,

(b) for each line l of Γ, the subspace of $P(V)$ spanned by $\{\pi(x) \mid x \in l\}$ is a line,

(c) if x, y are points of Γ such that $\pi(y)$ is contained in the subspace of $P(V)$ generated by the set $\{\pi(z) \mid z \text{ collinear with } x\}$, then y is collinear with x.

The map π is called the weak embedding and (c) is the weak embedding axiom. We say that Γ is weakly embedded of degree > 2 in $P(V)$, if each line of $P(V)$ which is spanned by the images of two non-collinear points of Γ contains the image of a third point of Γ. Similarly, we define when the weak embedding has degree 2.

Weak embeddings of classical polar spaces and of generalized quadrangles have been classified by Steinbach and Van Maldeghem [14, 15]. The main result is that with known exceptions they are induced by semi-linear mappings.

We close this section with a construction of a weak embedding to be used later. For a group G generated by the class Σ of transvection groups, we consider the point-line geometry $\wp(\Sigma)$ with point set Σ and lines $C_\Sigma(C_\Sigma(A, C))$ for $[A, C] = 1$; compare (2.4).

1.12. *Let K be a field and V a vector space over K endowed with a non-degenerate quadratic form Q of Witt index $\geqslant 2$. Let G be a quasi-simple subgroup of $\Omega(V, Q)$ generated by the class Σ of abstract transvection groups such that any $A \in \Sigma$ is contained in a Siegel transvection group $\hat{A}$ of $\Omega(V, Q)$.*

We assume that $\wp(\Sigma)$ is a Moufang generalized quadrangle and that there exists an apartment (E, B, F, D) of $\wp(\Sigma)$ such that $[V, E] + [V, F] = [V, B] + [V, D]$.

Then the dual generalized quadrangle $\wp(\Sigma)^D$ is weakly embedded of degree 2 in $P([V, G])$.

Proof. We use that G acts transitively on the set of ordered ordinary 4-gons by the Moufang condition. The 4-dimensional subspace $[V, E] + [V, F]$ is the orthogonal sum of two hyperbolic lines. We consider the map $\pi : \wp(\Sigma)^D \to P(V)$ which maps each line $A \in \Sigma$ of $\wp(\Sigma)^D$ to the singular line $[V, A]$ of V and each point $p = \bigcap\{T \mid T$ a line of $\wp(\Sigma)^D$ on $p\}$ to $\bigcap\{[V, T] \mid T$ a line of $\wp(\Sigma)^D$ on $p\}$.

We prove that π is a weak embedding. Clearly, π maps lines to lines and is injective on lines. Furthermore, π maps points to points. (Indeed, for any two lines A and C of

$\wp(\Sigma)^D$ on the point p, necessarily $[V, A] + [V, C]$ is 3-dimensional and non-singular, whence $\pi(p) = [V, A] \cap [V, C]$.) We deduce that for non-collinear points x, y of $\wp(\Sigma)^D$ and different points z, t of $\wp(\Sigma)^D$ collinear with both x and y, the same relations hold for the images of these four points under π in the polar space associated to (V, Q). In particular, π is injective on points. Thus the weak embedding axiom holds and π is a weak embedding (of degree 2). □

2 The construction of a polar space

Let G be a subgroup of $F_4(K)$ as in Problem (P) of the introduction. We show that Σ is the point set of a polar space $\wp(\Sigma)$, where $A, C \in \Sigma$ are on a line if and only if they commute, and lines in $\wp(\Sigma)$ embed in symplecta of the F_4-geometry. For symplecta in the F_4-geometry, we refer to (1.3). Proposition 2.3 below, compare Cuypers and Steinbach [7, (5.4)], is crucial for the construction of $\wp(\Sigma)$. The proof needs that $|A| \geqslant 3$, for $A \in \Sigma$. First, we deduce from (P):

2.1. For $A \in \Sigma$, we denote by $\hat{A}$ the unique long root subgroup of $F_4(K)$ which contains A. An arbitrary long root subgroup of $F_4(K)$ is denoted by a letter, like T, without a 'hat'.

Let $A, B \in \Sigma$. If $[A, B] = 1$, then $[\hat{A}, \hat{B}] = 1$. If $\langle A, B \rangle$ is a rank 1 group, then $\langle \hat{A}, \hat{B} \rangle \simeq \mathrm{SL}_2(K)$. Furthermore, A is the unique element in Σ contained in $\hat{A}$, since we assume that $C_\Sigma(A) = C_\Sigma(B)$ implies $A = B$.

When $[A, B] = 1$ and $A \neq B$, then $\hat{A}$ and $\hat{B}$ are not collinear in the F_4-geometry. Otherwise (1.3) implies that for $a \in A^\#$, $b \in B^\#$ there is a long root subgroup T of $F_4(K)$ which contains $t := ab$. The role of A and B is symmetric and by (P) we may choose $C \in \Sigma$ with $[A, C] = 1$ and $\langle B, C \rangle$ a rank 1 group. We obtain $\mathrm{SL}_2(K) \simeq X := \langle \hat{B}, \hat{C} \rangle = \langle \hat{C}, \hat{C}^b \rangle = \langle \hat{C}, \hat{C}^t \rangle \leqslant \langle \hat{C}, T \rangle$. Thus $\langle \hat{C}, T \rangle$ is a rank 1 group and there is $c \in \hat{C}$ such that $T^c = \hat{C}^t = \hat{C}^b$. We obtain $t = ab \in X$. This yields $a \in Z(X)$ for $a \in A^\#$, a contradiction. (Compare also Timmesfeld [20, (3.6)] or [21, II (2.3)].)

We have shown that two different commuting elements in Σ define a symplecton. Next, we investigate a certain subgroup of $F_4(K)$ generated by three long root subgroups of $F_4(K)$.

2.2. *Let A, B, C be different long root subgroups in $F_4(K)$, $|K| \geqslant 3$, such that $\langle A, B \rangle \simeq \mathrm{SL}_2(K)$, $\langle B, C \rangle \simeq \mathrm{SL}_2(K)$ and A, C define a symplecton. Then the unique long root subgroup E contained in the symplecton $S(A, C)$ which commutes with B is the center of $\langle A, B, C \rangle$. Furthermore, E and C are conjugate in the unipotent radical of $N(A)$.*

Proof. We set $Y := \langle A, B, C \rangle$. Without loss $A = X_{e_1+e_2}$, $B = X_{-e_1-e_2}$, $E = X_{e_1-e_2}$. Since C is contained in $S(A, E)$, we obtain that $A, B, E, C \leqslant M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$ with associated orthogonal space (V, Q). We choose notation such that A, B, E are the Siegel transvection subgroups (see at the end of (1.5)) associated to the singular lines $\langle x_1, x_2 \rangle$, $\langle y_1, y_2 \rangle$ and $\langle x_1, y_2 \rangle$, respectively, where (x_1, x_2) and

(x_2, y_2) are orthogonal hyperbolic pairs. Then the singular line associated to C is $\langle x_1, y_2 - Q(s)x_2 + s\rangle$, where $s \in \langle x_1, y_1, x_2, y_2\rangle^{\perp}$ with $Q(s) \neq 0$.

Thus $V_5 := [V, \langle Y, E\rangle] = \langle x_1, y_1\rangle \perp \langle x_2, y_2\rangle \perp \langle s\rangle$ is a 5-dimensional non-degenerate subspace of V. In $\Omega(V_5, Q)$, the structure of Y is $K^{1+2}\mathrm{SL}_2(K)$, $|K| \geqslant 3$, and the center of Y is a Siegel transvection group, T say. Necessarily, $[V, T] = \langle x_1, y_2\rangle = [V, E]$ and $T = E$. This proves the first claim. The second one holds in the orthogonal group. □

2.3 Proposition. *Let $A, B, C \in \Sigma$ with $[A, B] \neq 1$, $[B, C] \neq 1$ and $[A, C] = 1$, $A \neq C$. Then there exists $F \in \Sigma$ with $[B, F] = 1$ and $\hat{F} \in S(\hat{A}, \hat{C})$.*

Proof. Let $a \in A^{\#}$, $b \in B^{\#}$ with $A^b = B^a$. Since $|C| \geqslant 3$, there exists $c \in C^{\#}$ such that $C^{bc} \neq B$. Let $b' \in B^{\#}$ such that $(C^{bc})^{b'} = C$ and set $D := A^{bcb'} = B^{acb'}$. Then $[A^{bc}, B] \neq 1$, whence $[D, B] \neq 1$, and $D \neq A, C$.

For the unique point T of $S(\hat{A}, \hat{C})$ with $[T, \hat{B}] = 1$, necessarily $S(\hat{A}, \hat{C}) = S(T, \hat{C})$. Conjugation with bcb' yields that $S(\hat{D}, \hat{C}) = S(T, \hat{C})$.

Since $\langle B, D\rangle$ is a rank 1 group, there exists $d \in D^{\#}$ such that $(B^{ac})^d = B$. Hence $z := acd$ centralizes A, C and normalizes B. Thus $z \in \mathrm{Z}(\langle A, B, C\rangle) \leqslant \mathrm{Z}(\langle \hat{A}, \hat{B}, \hat{C}\rangle)$. The latter is T by (2.2).

By (P) there is $E \in \Sigma$ with $[E, A] = 1$, $[E, C] \neq 1$. Then $\hat{A}$ is the unique point in $S(\hat{A}, \hat{C})$ which commutes with $\hat{E}$, whence $\langle E, D\rangle$ and $\langle \hat{E}, T\rangle$ are rank 1 groups.

Let $e \in E^{\#}$ with $E^c = C^e$. With C, E, D in the roles of A, B, C we see that there exists $e' \in E^{\#}$ such that $S(\hat{C}, \hat{D}) = S(\hat{F}, \hat{D})$, for $F := E^{cde'}$. We have $F = E^{ze'}$ and $\hat{F} \leqslant S(\hat{C}, \hat{D}) = S(\hat{A}, \hat{C})$. Hence $\hat{F}$ and T are commuting long root subgroups in the rank 1 group $\langle \hat{E}, T\rangle$. This yields that $\hat{F} = T$, whence $F \in C_{\Sigma}(B)$ and $\hat{F} \in S(\hat{A}, \hat{C})$, as desired. □

2.4 Theorem. *We define the line on two different commuting elements A, C of Σ as $\ell_{A,C} := \{T \in \Sigma \mid \hat{T} \in S(\hat{A}, \hat{C})\}$. Then the point-line space $\wp(\Sigma)$, with point set Σ together with the set of all these lines, is a non-degenerate polar space. Any line has at least three points and through any point there are at least three lines.*

Proof. There is a unique line on two distinct collinear points by (1.3). The 2-then-all axiom holds in $\wp(\Sigma)$. Otherwise there exist $T \in \Sigma$ and distinct points A, B, D on a line ℓ with $[T, A] = 1 = [T, B]$ and $\langle T, D\rangle$ a rank 1 group. But then for suitable $d \in D^{\#}$, $t \in T^{\#}$, we obtain $T = D^{td^{-1}} \in \ell^{td^{-1}} = \ell$, a contradiction.

Hence $\wp(\Sigma)$ is a (non-degenerate) polar space by (2.3). For different commuting $A, C \in \Sigma$, the conjugacy class Σ is not contained in $C_{\Sigma}(A) \cup C_{\Sigma}(C)$. Hence there is B not collinear with A or C. Now (2.3) implies that B is collinear with a third point F on the line on A and C. Furthermore F is collinear with B and C^b for $b \in B$. This proves the theorem. □

2.5. We give another description of the lines of the polar space $\wp(\Sigma)$ of (2.4). In a non-degenerate polar space, $\ell^{\perp\perp} = \ell$, for any line ℓ, see Cohen [4, (3.1)]. Thus for distinct commuting elements A, C of Σ, the line on A and C in $\wp(\Sigma)$ is $\ell_{A,C} = C_\Sigma(C_\Sigma(A,C))$.

The group G acts on the polar space $\wp(\Sigma)$ with kernel $Z(G)$. For $A \in \Sigma$, any $a \in A^\#$ fixes all points collinear with A, whence is a central elation with center A.

Next, we deduce some properties of G and $\wp(\Sigma)$ for later use. For $M_{\hat{E}}$, see (1.4).

2.6. *For three distinct collinear points E, T, C of $\wp(\Sigma)$, there exists $n \in N := \langle C_\Sigma(E)\rangle \cap M_{\hat{E}}$ such that $C = T^n$. Furthermore, for non-collinear points E, F of $\wp(\Sigma)$, we have $\langle C_\Sigma(E)\rangle = \langle C_\Sigma(E) \cap C_\Sigma(F)\rangle N$.*

Proof. By (2.2) there exists $m \in M_{\hat{E}}$ such that $\hat{C} = \hat{T}^m$. We may choose $S \in C_\Sigma(E)$, $S \notin C_\Sigma(T)$. Let $1 \neq s \in S$ and $t \in T$, $c \in C$ such that $C^s = S^c$ and $T^s = S^t$. For $n := st^{-1}cs^{-1} = sct^{-1}s^{-1}$, we obtain $T^n = C$. Since $S(\hat{E}, \hat{C}) = S(\hat{E}, \hat{T})$, also $S(\hat{E}, \hat{S})^n = S(\hat{E}, \hat{S})$. Furthermore, $n \in (CT)^{s^{-1}} \leqslant (\hat{T}^m\hat{T})^{s^{-1}} \leqslant \hat{T}^{s^{-1}} M_{\hat{E}}$. Let $\hat{t} \in \hat{T}$ and $m_0 \in M_{\hat{E}}$ with $n = \hat{t}^{s^{-1}} m_0$. Then $\hat{S}^{\hat{t}^{s^{-1}}} = \hat{S}^{nm_0^{-1}} \in S(\hat{E}, \hat{S})^{nm_0^{-1}} = S(\hat{E}, \hat{S})$ (with (1.4)). Thus $\hat{S}$ and $\hat{S}^{\hat{t}^{s^{-1}}}$ commute and necessarily $\hat{t} = 1$. This proves the first claim. For $E \neq C \in C_\Sigma(E)$, $E \notin C_\Sigma(F)$, we denote by T the unique neighbour of F on $\ell_{E,C}$. Then $C = T^n$ with $n \in \langle C_\Sigma(E)\rangle \cap M_{\hat{E}}$ and $C \leqslant T \cdot N$, as desired. $\square$

2.7. *Let E, F be non-collinear points of $\wp(\Sigma)$. Any point not collinear to E is conjugate to F in $\langle C_\Sigma(E)\rangle$. In particular $G = \langle C_\Sigma(E), F\rangle$. Furthermore, G is quasi-simple. When $\wp(\Sigma)$ has rank* 2, *then $\wp(\Sigma)$ is a Moufang quadrangle.*

Proof. The first claim follows from (2.4) and the proof of Steinbach [17, (3.1)]. For the quasi-simplicity of G, one can proceed as in Cuypers and Steinbach [7, (7.3)]. With (2.6) the proof of the Moufang condition is as in Steinbach [17, (3.2), (3.3)]. $\square$

3 Subgroups of $F_4(K)$ arising from a polar space of rank $\geqslant 3$

Let G be a subgroup of $F_4(K)$ as in Problem (P) of the introduction. In this section, we suppose that the polar space $\wp(\Sigma)$ of (2.4) has rank at least 3.

Our aim is to show that a conjugate of G is contained in the standard subsystem subgroup $M(C_3) := M(\alpha_2, \alpha_3, \alpha_4) = C_3(K)$, when $\mathrm{char}(K) \neq 2$, and in $M(C_4) := M(\alpha_2, \alpha_3, \alpha_4, -e_1) = C_4(K)$, when $\mathrm{char}(K) = 2$.

3.1. We fix $E, F \in \Sigma$ with $\langle E, F\rangle$ a rank 1 group. Passing to a conjugate of G we may assume that $\hat{E} = X_{e_1+e_2}$, $\hat{F} = X_{-e_1-e_2}$. Let V_6 be the underlying 6-dimensional symplectic space of $M(C_3)$. We consider the point set $\Delta := C_\Sigma(E) \cap C_\Sigma(F)$ as a polar space of rank at least 2. Each $A \in \Delta$ is contained in a symplectic transvection subgroup of $M(C_3)$. Therefore the rank of Δ is at most 3 and $\wp(\Sigma)$ has rank 3 or 4. For the definition of a weak embedding of a polar space, we refer to (1.11).

3.2. *In the notation of* (3.1), *we assume that A and C are different commuting points of Δ. Then $[V_6, T] \subseteq [V_6, A] + [V_6, C]$ for all points T on $\ell_{A,C}$. Furthermore, Δ is weakly embedded of degree > 2 in $P(V_0)$, $V_0 = [V, G]$.*

Proof. Passing to a conjugate of G under $M(C_3)$, we may assume that $\hat{A} = X_{e_3-e_4}$, $\hat{C} = X_{e_3+e_4}$. Let T be a third point on $\ell_{A,C}$. Then $\hat{T}$ is contained in the symplecton $S(\hat{A}, \hat{C}) = \langle X_{e_3\pm e_4}, X_{e_3\pm e_1}, X_{e_3\pm e_2}, X_{e_3}\rangle$ and $\hat{T}$ commutes with both $X_{e_1+e_2}$ and $X_{-e_1-e_2}$. The Chevalley commutator formula, see (1.1), implies that $\hat{T} \leqslant \langle \hat{A}, \hat{C}, X_{e_3}\rangle$. Whence $[V_6, T] \subseteq [V_6, \hat{A}] + [V_6, \hat{C}]$. From this the lemma follows. □

For the non-embeddable polar space $E_7^{\mathscr{C}}$ of rank 3 whose planes are not Desarguesian, we refer to Tits [22, (9.1)]. Here $\mathscr{C}$ is a Cayley division algebra (with anisotropic norm form) over a commutative field, L say.

3.3. *The polar space $\wp(\Sigma)$ is not isomorphic to the polar space $E_7^{\mathscr{C}}$.*

Proof. Otherwise the polar space Δ of (3.1) is isomorphic to the dual of the orthogonal quadrangle associated to the orthogonal space $\mathscr{C} \times L^4$ of vector space dimension 12. This is a contradiction to Steinbach and Van Maldeghem [14, (7.2.4)] which asserts that a weakly embedded dual orthogonal quadrangle which is not mixed has a standard embedding in a vector space of dimension at most 8. □

We say a symplectic form f has rank n, if the underlying vector space is $W = H_1 \perp \cdots \perp H_n \perp \mathrm{Rad}(W, f)$ with $H_1, \ldots, H_n$ hyperbolic lines for f. When q is a non-degenerate quadratic form in characteristic 2 with associated symplectic form f, then the Witt index of q and the rank of f may differ, see after (1.7).

Using the classification of non-degenerate polar spaces of (finite) rank at least 3 due to Tits (see Tits [22], Cohen [4, (3.34)]) we see that $\wp(\Sigma)$ is one of the following:

3.4 Proposition. *We assume that $\wp(\Sigma)$ is a polar space of rank $\geqslant 3$. Then there exists a commutative field L and a vector space W over L such that $\wp(\Sigma)$ is isomorphic to the polar space of* 1- *and* 2-*dimensional subspaces of W, where one of the following non-degenerate forms vanishes*:

(a) *a symplectic form $f : W \times W \to L$ of rank* 3 *or* 4 *in* $\mathrm{char}(L) \neq 2$,

(b) *an ordinary quadratic form $q : W \to L$ of Witt index* 3 *or* 4 (*with degenerate associated symplectic form $f : W \times W \to L$ of rank* 3 *or* 4) *in* $\mathrm{char}(L) = 2$.

Furthermore, there is an embedding $\alpha : L \to K$. In particular $\mathrm{char}(K) = \mathrm{char}(L)$.

Proof. By (3.2) we know that $\wp(\Sigma)$ has rank 3 or 4. From Tits [22] and (3.3) we deduce that $\wp(\Sigma)$ arises from a vector space W endowed with a form as in (1.5).

Let H be a hyperbolic line in W such that the underlying vector space of Δ is $W_0 := H^{\perp}$. By Steinbach and Van Maldeghem [14], the weak embedding of Δ in $P(V_0)$ of (3.2) is induced by a semi-linear mapping $\varphi : W_0 \to V_0$ (with respect to $\alpha : L \to K$). In particular, L is commutative.

By (2.5) the polar space $\wp(\Sigma)$ admits central elations. These are induced by isotropic transvections by (1.8). Thus when $\wp(\Sigma)$ arises from an ordinary quadratic form q, necessarily $\mathrm{char}(L) = 2$ and the associated symplectic form f is degenerate (see (1.5)). The rank of f is 3 or 4 by Steinbach and Van Maldeghem [15, (5.4)].

By Tits [22, (8.2.2)] we are left with the case where $\wp(\Sigma)$ arises from a $(\sigma, -1)$-quadratic from q with $1 \in \Lambda = \{c + c^\sigma \mid c \in L\}$. But this situation cannot occur. Indeed, since $\sigma \neq \mathrm{id}$ (by definition of q) and L is commutative, $\Lambda = \{c \in L \mid c^\sigma = c\}$. Hence $\wp(\Sigma)$ and the unitary space arising from f (the anti-hermitian form associated to q) coincide, see Tits [22, (8.2.4)]. Let (x_1, y_1), (x_2, y_2) be orthogonal hyperbolic pairs in W_0. For $c \in L$ with $c^\sigma \neq c$, the vector $a := x_2 + cy_2$ is anisotropic, but $p := x_1 - cy_1 + a$ is isotropic. In the symplectic space the vectors $a\varphi$ and $p\varphi$ are perpendicular (since $a\varphi$ is isotropic). By Steinbach and Van Maldeghem [15, (5.3)] this yields $f(a, a) = f(a, p) = 0$, a contradiction. □

The group $G/Z(G)$ is isomorphic to the normal subgroup of $\mathrm{Aut}(\wp(\Sigma))$ generated by the central elation subgroups, see Cuypers and Steinbach [7, (8.2)].

In the following we identify $\wp(\Sigma)$ with the classical polar space of (3.4), considering $A \in \Sigma$ as an isotropic point p of W. We say that A corresponds to the isotropic transvection group T_p (see (1.5)). Throughout (x_i, y_i) is a hyperbolic pair spanning H_i.

3.5. *If* $\mathrm{char}(L) \neq 2$, *then the rank of the symplectic form* f *is* 3. *In particular, G is not isomorphic to* $\mathrm{Sp}_8(L)$.

Proof. We assume that the rank of f is 4. Then W is spanned by four orthogonal hyperbolic pairs (v_i, w_i). Passing to a conjugate of G, we may choose notation such that the following correspondence holds (see (1.4)):

T_{v_1}	T_{w_1}	T_{v_2}	T_{w_2}	T_{v_3}	T_{w_3}	T_{v_4}	T_{w_4}
$X_{e_1+e_2}$	$X_{-e_1-e_2}$	$X_{e_1-e_2}$	$X_{-e_1+e_2}$	$X_{e_3+e_4}$	$X_{-e_3-e_4}$	$X_{e_3-e_4}$	$X_{-e_3+e_4}$
$T_{\langle x_1, x_2\rangle}$	$T_{\langle y_1, y_2\rangle}$	$T_{\langle x_1, y_2\rangle}$	$T_{\langle y_1, x_2\rangle}$	$T_{\langle x_3, x_4\rangle}$	$T_{\langle y_3, y_4\rangle}$	$T_{\langle x_3, y_4\rangle}$	$T_{\langle y_3, x_4\rangle}$

The first row lists symplectic transvection groups on W, the second row lists the long root subgroups of $F_4(K)$, which contain the corresponding element of Σ. We denote the first four of these elements in Σ by E, F, A, B. An entry in the last row writes the long root subgroup of $F_4(K)$ in the entry above as Siegel transvection subgroup of $M(B_4)/\langle -1\rangle$, where $M(B_4) := M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$. The underlying 9-dimensional orthogonal space (with quadratic form Q) is $H_1 \perp H_2 \perp H_3 \perp H_4 \perp \langle a\rangle$.

We define $G_0 := \langle T \in \Sigma \mid T$ corresponds to T_p with $p \in \langle v_3, w_3\rangle \perp \langle v_4, w_4\rangle\rangle$. Then G_0 centralizes $X_{\pm(e_1+e_2)}$ and $X_{\pm(e_1-e_2)}$ and hence $G_0 \leqslant M(\alpha_2, \alpha_3) = B_2(K)$ with underlying 5-dimensional orthogonal space $V_5 := H_3 \perp H_4 \perp \langle a\rangle$.

Let T be a third point on $\ell_{E,A}$. Then $\hat{T} \in S(\hat{E}, \hat{A})$, see (2.4), and there exists $0 \neq s \in V_5$ such that $\hat{T}$ is the Siegel transvection group corresponding to the singular line $\ell_T := \langle x_1, x_2 - Q(s)y_2 + s\rangle$ intersecting V_5 trivially.

There is $T_0 \in \Sigma \cap G_0$ such that $[V_5, T_0] \nsubseteq H_3 \perp H_4$ (since $\mathrm{Sp}_4(L)$ does not embed in $\Omega_4^+(K) = \mathrm{SL}_2(K) * \mathrm{SL}_2(K)$). For an arbitrary element in $\Sigma \cap G_0$, let ℓ be the associated line in V_5. Since $[G_0, \hat{T}] = 1$ and $\ell \cap \ell_T = 0$, we see that $\ell_T + \ell$ is singular.

This implies that $x_2 - Q(s)y_2 + s$ is contained in $V_5^{\perp}$, which is $H_1 \perp H_2$ in characteristic $\neq 2$. Hence $s \in V_5 \cap (H_1 \perp H_2) = 0$, a contradiction. $\square$

3.6 Theorem. *Let G be a subgroup of $F_4(K)$ as in* (P) *with associated polar space $\wp(\Sigma)$ of rank at least* 3. *Then a conjugate of G is contained in the standard subsystem subgroup $M(C_3) := M(\alpha_2, \alpha_3, \alpha_4) = C_3(K)$, when* $\mathrm{char}(K) \neq 2$, *and in $M(C_4) := M(\alpha_2, \alpha_3, \alpha_4, -e_1) = C_4(K)$, when* $\mathrm{char}(K) = 2$.

Proof. First we assume that $\mathrm{char}(K) \neq 2$. By (3.5) the symplectic form f associated to G has rank 3 with underlying vector space $W = H_1 \perp H_2 \perp H_3$. Passing to a conjugate of G, we may choose notation as follows (compare (3.5)):

T_{x_1}	T_{y_1}	T_{x_2}	T_{y_2}	T_{x_3}	T_{y_3}
$X_{e_1-e_2}$	$X_{-e_1+e_2}$	$X_{e_3-e_4}$	$X_{-e_3+e_4}$	$X_{e_3+e_4}$	$X_{-e_3-e_4}$

For the second row, we denote the corresponding elements Σ by E, F, A_1, B_1, A_2 and B_2 respectively.

We set $G_1 := \langle A \in \Sigma \mid A$ corresponds to T_p with $p \subseteq H_1^{\perp} \rangle$ and similarly G_3 for H_3. Then G_1 centralizes $X_{\pm(e_1-e_2)}$. For $\beta := \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$, we obtain $G_1 \leqslant M(\alpha_2, \alpha_3, \beta) \simeq C_3(K)$, with underlying symplectic space V_6. As in (3.2) the polar space $\Delta := C_\Sigma(E) \cap C_\Sigma(F)$ is weakly embedded in $P(V_0)$, where $V_0 := \langle [V_6, A_1], [V_6, B_1], [V_6, A_2], [V_6, B_2] \rangle$, the 4-dimensional subspace of V_6 underlying $M(\alpha_2, \alpha_3) \simeq C_2(K)$. This yields that $G_1 \leqslant M(\alpha_2, \alpha_3) \simeq C_2(K)$.

Similarly, $G_3 \leqslant M\big(\alpha_2, \frac{1}{2}(e_1 - e_2 - e_3 + e_4)\big) \simeq C_2(K)$. Since $G = \langle G_1, G_3 \rangle$ by (1.6), the claim follows.

Next, we deal with the case where $\mathrm{char}(K) = 2$. The orthogonal space associated to G contains $H_1 \perp H_2 \perp H_3$. Passing to a conjugate of G we may assume that the rank 1 groups corresponding to H_1, H_3 are contained in $\langle X_{\pm(e_1+e_2)} \rangle$, $\langle X_{\pm(e_1-e_2)} \rangle$, respectively. As above we define the subgroups G_1, G_3 of G associated to $H_1^{\perp}$ and $H_3^{\perp}$, respectively. Then $G_1 \leqslant M(\alpha_2, \alpha_3, \alpha_4) \simeq C_3(K)$ and $G_3 \leqslant M(\alpha_2, \alpha_3, \beta) \simeq C_3(K)$, where $\beta := \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$. Since $G = \langle G_1, G_3 \rangle$ by (1.6), the claim follows with (1.2). $\square$

Theorem 3.6 reduces the determination of the groups G in question to the study of subgroups of symplectic groups generated by parts of symplectic transvection groups. We refer to Cuypers and Steinbach [7, (1.5)] for the latter problem. The results obtained in this section, together with a construction of a weak embedding as in the proof of (3.2), yield Theorem 2.

4 Embedding classical and mixed Moufang quadrangles in $F_4(K)$

Because of the results in Section 3 we are left with the case where the polar space $\wp(\Sigma)$ of (2.4) has rank 2. By (2.7) $\wp(\Sigma)$ is a Moufang quadrangle, which admits central elations.

4.1. We fix $E, F, A, B \in \Sigma$ such that (E, A, F, B) is an apartment in $\wp(\Sigma)$. Passing to a conjugate of G, we may assume (see (1.4))

$$\hat{E} = X_{e_1+e_2}, \quad \hat{F} = X_{-e_1-e_2}, \quad \hat{A} = X_{e_1-e_2}, \quad \hat{B} = X_{-e_1+e_2}.$$

We call (E, A, F, B) the standard apartment in G. By U_2 we denote the root group of G associated with the half apartment (B, E, A) which has E in the middle.

4.2 Theorem. *Let (E, A, B, F) be the standard apartment in G. If $\hat{T} \leqslant \langle \hat{A}, \hat{B} \rangle$, for any $T \in C_\Sigma(E) \cap C_\Sigma(F)$, then $G \leqslant M := M(-\alpha_*, \alpha_1, \alpha_2, \alpha_3) = B_4(K)$ (with associated 9-dimensional orthogonal space (V, q)). Furthermore, the dual generalized quadrangle $\wp(\Sigma)^D$ is weakly embedded of degree 2 in $P(V_0)$, where $V_0 := [V, G]$.*

Proof. By (2.7) $G = \langle C_\Sigma(E), F \rangle$. Fix $E \neq C \in C_\Sigma(E)$ and denote by T the neighbour of F on the line $\ell_{E,C}$. Then $T \in C_\Sigma(E) \cap C_\Sigma(F)$ and by assumption $\hat{T}$ is a long root subgroup of $F_4(K)$ contained in $X := \langle \hat{A}, \hat{B} \rangle$. We obtain $\hat{T} = \hat{A}^x$ with $x \in X$ and $S(\hat{E}, \hat{T}) = S(\hat{E}, \hat{A})^x \leqslant M^x = M$. But $C \in \ell_{E,T}$, hence $\hat{C}$ is in $S(\hat{E}, \hat{T})$ by (2.4). In particular $C \leqslant M$, whence $G \leqslant M$.

We identify M with $\Omega(V, q)$ (neglecting that $B_4(K)/\langle -1 \rangle = \Omega_9(K)$) and obtain the claim with (1.12). □

4.3. In the situation of (4.2) $\wp(\Sigma)^D$ is an orthogonal quadrangle (arising from an ordinary quadratic form) or a so-called mixed quadrangle (in characteristic 2 only, see (4.7)) by Steinbach and Van Maldeghem [15]. In particular, $\wp(\Sigma)$ has abelian root groups, see (1.9). Furthermore, the weak embedding is induced by a semi-linear mapping.

Next we show that (4.2) applies when $\operatorname{char} K \neq 2$.

4.4 Theorem. *Let (E, A, B, F) be the standard apartment in G. If* $\operatorname{char}(K) \neq 2$, *then $\hat{T} \leqslant \langle \hat{A}, \hat{B} \rangle$, for any $T \in C_\Sigma(E) \cap C_\Sigma(F)$.*

Proof. We assume that there exists $C \in C_\Sigma(E) \cap C_\Sigma(F)$ with $\hat{C} \not\leqslant \langle \hat{A}, \hat{B} \rangle$. Let $Y_C := \langle A, B, C \rangle$ and $\hat{Y}_C := \langle \hat{A}, \hat{B}, \hat{C} \rangle$. Note that $\hat{Y}_C \leqslant M(\alpha_2, \alpha_3, \alpha_4) = C_3(K) \simeq \mathrm{Sp}_6(K)$ with $\hat{A}, \hat{B}$ and $\hat{C}$ pairwise non-commuting symplectic transvection groups. Hence we may write $\hat{A} = T_{x_1}$, $\hat{B} = T_{y_1}$, $\hat{C} = T_{x_1+\mu y_1+s}$, where (x_1, y_1) is a hyperbolic pair in the underlying symplectic space, $0 \neq \mu \in K$ and $0 \neq s \in \langle x_1, y_1 \rangle^\perp$. Now $\hat{Y}_C = \langle T_{x_1+\mu y_1}, \hat{B}, \hat{C} \rangle$ has the following structure: $\hat{Y}_C = NX \simeq K^{1+2}\mathrm{SL}_2(K)$ (semidirect product), where $X := \langle \hat{A}, \hat{B} \rangle$, $N/\mathrm{Z}(\hat{Y}_C)$ is a natural module for X and $\mathrm{Z}(\hat{Y}_C) = T_s$. Moreover, $\langle \hat{A}^N \rangle$ is abelian.

Let $n \in Y_C \cap N$. Then C and C^n commute, but $C_\Sigma(E) \cap C_\Sigma(F)$ does not contain different commuting elements. Hence $C^n = C$ and $[C, n] \leqslant C \cap N = 1$. Similarly, $[A, n] = 1$ and $[B, n] = 1$, thus $n \in \mathrm{Z}(Y_C)$. Since $\operatorname{char}(K) \neq 2$, there exists a central involution z in $\langle A, B \rangle$. For $y = xn \in Y_C$ with $n \in N$, $x \in X$, we have $[y, z] = [n, z] \in Y_C \cap N \leqslant \mathrm{Z}(Y_C)$. Hence $[Y_C, [Y_C, z]] = 1$. The three-subgroup lemma yields

$[Y'_C, z] = 1$, whence $[Y_C, z] = 1$ (also when $|A| = 3$). We obtain $Y_C \leqslant C_{\hat{Y}_C}(z) \leqslant XZ(\hat{Y}_C)$ and $Y_C = Y'_C \leqslant X$. But this is a contradiction to the choice of C. □

For char $K = 2$, we need a different approach. In any Moufang quadrangle one kind of root groups is abelian, but not necessarily both. Our next aim is to show that in characteristic 2 all root groups of $\wp(\Sigma)$ are abelian. From Tits [23] we deduce that for any Moufang quadrangle admitting central elations, the root groups associated to half-apartments with a line in the middle are abelian. Next, we construct 'root subgroups' (associated to points) in G.

4.5. *Let the group G act on the polar space $\wp(\Sigma)$ via $\rho : G \to \mathrm{Aut}(\wp(\Sigma))$. The map $\rho : (\langle C_\Sigma(E)\rangle \cap M_{\hat{E}}) \cap N_G(A) \cap N_G(B) \to U_2$ is an isomorphism.*

Proof. We set $N := (\langle C_\Sigma(E)\rangle \cap M_{\hat{E}}) \cap N_G(A) \cap N_G(B)$. Since $Z(G) \cap M_{\hat{E}} = 1$, the restriction of ρ to N is injective. For any $n \in N$, the image $n\rho$ is in U_2. (Indeed, $G \cap M_{\hat{E}}$ fixes any line on E by (1.4). For any point C on $\ell_{E,A}$, we have $\hat{C} = \hat{A}^m$ with $m \in M_{\hat{E}}$ by (2.2). Since the commutator subgroup of $M_{\hat{E}}$ is contained in $\hat{E}$, we obtain that $C^n = C$ for $n \in N_G(A) \cap M_{\hat{E}}$.)

Next, let $u \in U_2$. Then F and F^u are conjugate in $\langle C_\Sigma(E)\rangle$ by (2.7). Thus there exists $n \in \langle C_\Sigma(E)\rangle \cap M_{\hat{E}}$ with $F^u = F^n$ by (2.6). Let $x := (n\rho)u^{-1}$. As before x fixes all lines on E, whence also E, F, A, B. Therefore x fixes all points on $\ell_{E,A}$ and all points on $\ell_{E,B}$. We obtain $x \in U_2$ with $F^x = F$; i.e., x is the identity in $\mathrm{Aut}(\wp(\Sigma))$ and $u = n\rho$. □

4.6. *Let (E, A, B, F) be the standard apartment in G. Then the root group U_2 (in G) is contained in $U^* := \langle X_{e_1+e_2}, X_{(1/2)(e_1+e_2 \pm e_3 \pm e_4)}\rangle$. In particular, all root groups of $\wp(\Sigma)$ are abelian.*

Proof. Since U_2 stabilizes the point $p := X_{e_1+e_2}$ and the symplecton $S := S(X_{e_1+e_2}, X_{e_1-e_2})$ of the F_4-geometry, U_2 is contained in P_J, the intersection of the two parabolic subgroups $N(p)$ and $N(S)$ of Y (where the Levi complement has the diagram $(\alpha_2, \alpha_3, \alpha_4)$ and $(\alpha_1, \alpha_2, \alpha_3)$, respectively).

Now U_2 commutes with the central elation group with center A or B (see Steinbach [17, (3.5)]), whence centralizes $X_{e_1 \pm e_2}$. Thus U_2 is contained in the standard subsystem subgroup $M := M\big(\alpha_2, \alpha_3, \frac{1}{2}(e_1 + e_2 - e_3 - e_4)\big) \simeq C_3(K)$. But $P_J \cap M$ is a parabolic subgroup of M with unipotent radical U^* and Levi complement L^* associated to the diagram (α_2, α_3). By (4.5) we have $U_2 \leqslant M_{\hat{E}}$, whence $U_2 \leqslant M_{\hat{E}} \cap U^*L^* = U^*$.

If $\mathrm{char}(K) = 2$, U^* is abelian. If $\mathrm{char}(K) \neq 2$, (4.2) and (4.3) apply by (4.4). □

We use the classification of Moufang quadrangles due to Tits and Weiss [24], as stated in Van Maldeghem [25, 5.5].

4.7. From the classification of Moufang quadrangles we use that (up to duality) any Moufang quadrangle arises from a vector space with a form, is a mixed quadrangle or is an exceptional quadrangle of type F_4 or E_n, $n = 6, 7, 8$.

We do not need an explicit description of the exceptional Moufang quadrangles of type E_n, only that they have non-abelian root groups. The so-called mixed quadrangles, which are by definition subquadrangles of a symplectic quadrangle in characteristic 2, were introduced by Tits; see Van Maldeghem [25, (3.4.2)], Steinbach and Van Maldeghem [14, (6.1.1)], Cuypers and Steinbach [7, (4.2)]. For mixed quadrangles, the standard root subgroups U_i satisfy $[U_1, U_3] = 1 = [U_2, U_4]$. The dual of a mixed quadrangle is also a mixed quadrangle, see Van Maldeghem [25, (3.2.9)]. For the exceptional Moufang quadrangles of type F_4, we refer to (5.2). Their duals are also of type F_4, see Van Maldeghem [25, (7.4.2)].

4.8 Proposition. *The Moufang quadrangle $\wp(\Sigma)$ is in the following list. Furthermore, Σ is as stated, provided that Σ is the class of full central elation subgroups.*

(i) *$\wp(\Sigma)$ is a dual orthogonal quadrangle and Σ is the class of Siegel transvection subgroups on the orthogonal space associated to $\wp(\Sigma)^D$.*

(ii) *$\wp(\Sigma)$ arises from the vector space $W = H_1 \perp H_2 \perp \mathrm{Rad}(W, f)$ over L endowed with the non-degenerate $(\sigma, -1)$-quadratic form $q : W \to L/\Lambda$ of Witt index* 2 *(with associated anti-hermitian form $f : W \times W \to L$) such that $1 \in \Lambda$. Here Σ is the class of isotropic transvection subgroups.*

(iii) *$\wp(\Sigma)$ is an orthogonal quadrangle in characteristic* 2, *arising from a non-degenerate quadratic form of Witt index* 2, *with degenerate associated symplectic form. Here Σ is the class of isotropic transvection subgroups.*

(iv) *$\wp(\Sigma)$ is a mixed quadrangle and Σ is the class of central elation subgroups.*

(v) *$\wp(\Sigma)$ is an exceptional Moufang quadrangle of type F_4 and Σ is the class of central elation subgroups.*

Proof. We use (4.7). By (4.6) $\wp(\Sigma)$ cannot be an exceptional Moufang quadrangle of type E_n or a dual of it. We may assume that $\wp(\Sigma)$ arises (up to duality) from a vector space with a form.

We know that $\wp(\Sigma)$ admits central elations and has abelian root groups. Hence by (1.9) and (1.10) the list of candidates for $\wp(\Sigma)$ is as stated in (4.8). (Note that the symplectic quadrangle in characteristic $\neq 2$ is included in the first case.) For a dual orthogonal quadrangle, any central elation of $\wp(\Sigma)$ is an axial elation of the orthogonal quadrangle $\wp(\Sigma)^D$ and hence induced by a Siegel transvection, see (1.8). In Cases (ii) and (iii) any central elation of $\wp(\Sigma)$ is induced by an isotropic transvection, see (1.8). $\square$

4.9 Theorem. *Let (E, A, B, F) be the standard apartment in G. We assume that $\wp(\Sigma)$ is a dual orthogonal quadrangle or a mixed quadrangle or that $\wp(\Sigma)$ arises from a pseudo-quadratic form as in* (4.8)(ii) *with Σ the class of full central elation subgroups.*

Then $T \leqslant \langle A, B \rangle$, for $T \in C_\Sigma(E) \cap C_\Sigma(F)$. Hence (4.2) *applies. Moreover, any $\wp(\Sigma)$ as in* (4.8)(ii) *is necessarily a dual orthogonal quadrangle.*

Proof. Let $A \neq T \in C_\Sigma(E) \cap C_\Sigma(F)$. We deal with the three cases separately. First assume that $\wp(\Sigma)$ is dual orthogonal, see (1.5) for Siegel transvections. In the orthogonal space (W, q) associated to $\wp(\Sigma)^D$ we choose notation such that $E = T_{\langle x_1, x_2 \rangle}$, $F = T_{\langle y_1, y_2 \rangle}$, $A = T_{\langle x_1, y_2 \rangle}$, $B = T_{\langle x_2, y_1 \rangle}$, where (x_1, y_1) and (x_2, y_2) are orthogonal hyperbolic pairs. Then both $[W, T] + \langle x_1, x_2 \rangle$ and $[W, T] + \langle y_1, y_2 \rangle$ are 3-dimensional non-singular (note that q has Witt index 2) and $[W, T]$ is singular. Hence $[W, T] \in \{\langle x_2 - cx_1, cy_2 + y_1 \rangle, \langle x_1, y_2 \rangle \mid c \in L\}$. Since A is the full (projective) Siegel transvection subgroup associated to $\langle x_1, y_2 \rangle$, we obtain $T \in B^A$.

The case of a mixed quadrangle is similar, using the information of Cuypers and Steinbach [7, (4.3)]. Finally, when $\wp(\Sigma)$ arises from a pseudo-quadratic form as in (4.8)(ii), again $T \in B^A$. Thus $\wp(\Sigma)^D$ is weakly embedded of degree 2 by (4.2). Whence Steinbach and Van Maldeghem [15, (6.4)] yields that $\wp(\Sigma)^D$ is orthogonal. □

Under the assumptions of (4.9), unitary groups of Witt index 2 arise only when the underlying vector space is 4-dimensional over a commutative field or over a quaternion division ring (as follows from Steinbach and Van Maldeghem [15, (6.4)]).

An orthogonal quadrangle, with f trivial on $(H_1 \perp H_2)^\perp$, is mixed, see Van Maldeghem [25, p. 220]. Thus in (4.8)(iii), we are left with the case where f has rank $\geqslant 3$; i.e., W contains three orthogonal hyperbolic lines with respect to f.

4.10. *Let (E, A, B, F) be the standard apartment in G. We assume that* $\mathrm{char}(K) = 2$ *and that $\wp(\Sigma)$ arises from an orthogonal space (W, q) with Σ the class of isotropic transvection subgroups. Here q is a non-degenerate quadratic form of Witt index* 2, *with degenerate associated symplectic form of rank* $\geqslant 3$.

Then $G \leqslant M(C_4) := M(\alpha_2, \alpha_3, \alpha_4, -e_1) = C_4(K) \simeq \mathrm{Sp}_8(K)$, *with underlying symplectic space V_8. Moreover, $\wp(\Sigma)$ is weakly embedded in $P(V_0)$, $V_0 := [V_8, G]$.*

Proof. We write $W = H_1 \perp H_2 \perp U \perp \mathrm{Rad}(W, f)$ with U and $\mathrm{Rad}(W, f)$ nonzero. In W we choose notation such that $E = T_{x_1}$, $F = T_{y_1}$, $A = T_{x_2}$, $B = T_{y_2}$, where (x_i, y_i) is a hyperbolic pair. By (1.7) $G = \langle G_1, G_2 \rangle$ with G_i the group associated to $H_i^\perp$ $(i = 1, 2)$. Now $G_1 \leqslant M(\alpha_2, \alpha_3, \alpha_4) \simeq C_3(K)$ and $G_2 \leqslant M(\alpha_2, \alpha_3, \beta) \simeq C_3(K)$, where $\beta := \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$. With (1.2) we obtain that $G \leqslant M(C_4)$.

Next, we prove that $[V_8, C] \subseteq [V_8, E] + [V_8, A]$, for C on $\ell_{E,A}$. By (2.4), $\hat{C}$ is contained in the symplecton on $\hat{E}$ and $\hat{A}$, whence $\hat{C} \leqslant \langle X_{e_1 \pm e_2}, X_{e_1 \pm e_3}, X_{e_1 \pm e_4}, X_{e_1} \rangle$. Thus $\hat{C}$ centralizes the long root subgroups $\hat{E}, \hat{A}$ and the short root subgroups $X_{(1/2)(e_1 \pm e_2 \pm e_3 \pm e_4)}$ of $M(C_4)$. A calculation in the 8-dimensional symplectic group shows that $[V_8, C] \subseteq [V_8, E] + [V_8, A]$, as desired. This yields a weak embedding of $\wp(\Sigma)$ in $P(V_0)$. □

We remark that in (4.10) the rank of f is at most 4 by Steinbach and Van Maldeghem [15, (5.4)]. The results obtained in Section 4 yield Theorem 3.

We close this section with an example that in characteristic 2 we cannot expect that for G as in (P) the central elation subgroup associated to the point A in the polar space $\wp(\Sigma)$ of (2.4) is contained in $\hat{A}$. (This phenomenon already occurs for subgroups of $C_3(K)$, $\mathrm{char}(K) = 2$.) This is why we added in Theorem 3 the assumption that Σ is the class of full central elation subgroups.

4.11 Example. Let K be the field of rational functions over GF(2). Then $K = \mu K^2 \oplus K^2$ for a suitable $\mu \in K$. We consider the Chevalley group of type C_3 over K with associated fundamental root system $\{f_1 - f_2, f_2 - f_3, 2f_3\}$. For $x = \mu v^2 + a^2$ and b in K, we define

$$u_1(b) := x_{f_1-f_2}(b), \quad u_2(x) := x_{f_1-f_3}(v) \cdot x_{f_1+f_3}(\mu v) \cdot x_{2f_1}(a^2),$$
$$u_3(b) := x_{f_1+f_2}(b), \quad u_4(x) := x_{f_2-f_3}(v) \cdot x_{f_2+f_3}(\mu v) \cdot x_{2f_2}(a^2).$$

By U_i we denote the group consisting of the u_i $(i = 1, \dots, 4)$. Then $U_1, U_3 \simeq (K, +)$ and the same holds for U_2, U_4. Furthermore the only non-trivial commutator relation among the U_i is $[u_1(b), u_4(x)] = u_2(b^2x)u_3(bx)$. Thus the group G generated by $U_1, \dots, U_4$ and the corresponding 'negative root groups' $U_5, \dots U_8$ is a Chevalley group of type C_2 over K.

Let $V_2 := \{u_2(a^2) \mid a \in K\}$. Then $\Sigma := V_2^G$ is a class of abstract transvection groups of G. Any $A \in \Sigma$ is contained in a long root subgroup $\hat{A}$ of $C_3(K)$. But this does not hold for the full symplectic transvection group U_2 in G.

5 The exceptional Moufang quadrangles of type F_4

The exceptional Moufang quadrangles of type F_4 were discovered by Richard Weiss in February 1997, defined in terms of commutator relations. Their central elation subgroups are contained in long root subgroups of $F_4(L)$, where L is a suitable non-perfect field of characteristic 2.

We describe the Moufang quadrangles of type F_4 in (5.2), following Mühlherr and Van Maldeghem [11]. Then we show that the group generated by the associated central elation subgroups is generated by two classical subgroups S_1, S_2 of Witt index 2. In Section 6 these two subgroups are crucial for the determination of the embeddings of the F_4-quadrangles in Chevalley groups of type F_4. In particular, we will apply the results on the embeddings of S_1 and S_2 obtained in Section 4.

5.1 Commutator relations. In Chevalley groups $F_4(L)$, $\mathrm{char}(L) = 2$, we define

$$\begin{aligned}
u_2 := \bar{u}_2(p_1, p_2, p_3, p_4, c) &:= x_{(1/2)(e_1+e_2-e_3-e_4)}(p_1)x_{(1/2)(e_1+e_2+e_3+e_4)}(p_2)\\
&\quad \cdot x_{(1/2)(e_1+e_2-e_3+e_4)}(p_3)x_{(1/2)(e_1+e_2+e_3-e_4)}(p_4) \cdot x_{e_1+e_2}(c),\\
u_4 := \bar{u}_4(t_1, t_2, t_3, t_4, a) &:= x_{(1/2)(e_1-e_2-e_3-e_4)}(t_1)x_{(1/2)(e_1-e_2+e_3+e_4)}(t_2)\\
&\quad \cdot x_{(1/2)(e_1-e_2-e_3+e_4)}(t_3)x_{(1/2)(e_1-e_2+e_3-e_4)}(t_4) \cdot x_{e_1-e_2}(a),\\
u_1 := \bar{u}_1(s_1, s_2, s_3, s_4, b) &:= x_{e_2-e_3}(s_1)x_{e_2+e_3}(s_2) \cdot x_{e_2-e_4}(s_3)x_{e_2+e_4}(s_4) \cdot x_{e_2}(b),\\
u_3 := \bar{u}_3(q_1, q_2, q_3, q_4, d) &:= x_{e_1-e_3}(q_1)x_{e_1+e_3}(q_2) \cdot x_{e_1-e_4}(q_3)x_{e_1+e_4}(q_4) \cdot x_{e_1}(d).
\end{aligned}$$

Then the following commutator relations hold:

$$[u_1, u_3] = x_{e_1+e_2}(s_1 q_2 + s_2 q_1 + s_3 q_4 + s_4 q_3),$$
$$[u_2, u_4] = x_{e_1}(t_1 p_2 + t_2 p_1 + t_3 p_4 + t_4 p_3),$$
$$\begin{aligned}[u_1, u_4] = u_2(&bt_1 + s_1 t_4 + s_3 t_3, bt_2 + s_2 t_3 + s_4 t_4, bt_3 + s_1 t_2 + s_4 t_1, bt_4 + s_2 t_1 + s_3 t_2,\\ &ab^2 + as_1 s_2 + as_3 s_4 + s_1 s_3 t_2^2 + s_1 s_4 t_4^2 + s_2 s_3 t_3^2 + s_2 s_4 t_1^2)\\ &\cdot u_3(as_1 + s_3 t_3^2 + s_4 t_1^2, as_2 + s_3 t_2^2 + s_4 t_4^2, as_3 + s_1 t_4^2 + s_2 t_1^2, as_4 + s_1 t_2^2 + s_2 t_3^2,\\ &ab + bt_1 t_2 + bt_3 t_4 + s_1 t_2 t_4 + s_2 t_1 t_3 + s_3 t_2 t_3 + s_4 t_1 t_4).\end{aligned}$$

These relations follow from Chevalley's commutator formula for $F_4(L)$, see (1.1). Since the characteristic is 2, we do not have to take care of signs. Furthermore, some commutators vanish; e.g., $[X_r, X_s] = 1$, when r, s are short roots such that $r + s$ is a long root. (The check of the above relations, with an implementation of the Chevalley commutator relations in the unipotent subgroup of a Chevalley group, is part of the diploma thesis of Haller [9].)

5.2 Description. For the Moufang quadrangles $\mathcal{Q} := Q(K, L, K'; \alpha, \beta)$ of type F_4, see Mühlherr and Van Maldeghem [11, 2.2] or Van Maldeghem [25, p. 218]. We say $\mathcal{Q}$ is an F_4-quadrangle. Here L is a field of characteristic 2 with an automorphism σ of order 2 and L' is a subfield of L containing L^2. For $t \in L$, we write $\bar{t} := t^\sigma$. The fixed field of σ is $K := \{t \in L \mid \bar{t} = t\}$ and $K' := L' \cap K$. Furthermore, $\alpha \in K'$ and $\beta \in K$ satisfy the following:

$$\text{when } u, v \in L,\ a \in K' \text{ and } u\bar{u} + \alpha v\bar{v} + \beta a = 0, \text{ then } u = v = a = 0,$$
$$\text{when } x, y \in L',\ b \in K \text{ and } x\bar{x} + \beta^2 y\bar{y} + \alpha b^2 = 0, \text{ then } x = y = b = 0.$$

(Because of the above assertions, there exist certain anisotropic quadratic forms.) In the universal Chevalley group $F_4(L)$ we consider the subgroup $F_4(L', L)$ of mixed type F_4; i.e., $F_4(L', L) = \langle x_r(t'), x_s(t) \mid r \text{ long}, t' \in L', s \text{ short}, t \in L\rangle \leqslant F_4(L)$ (with long and short root subgroups isomorphic to $(L', +)$ and $(L, +)$), respectively, see Tits [22]. The root groups $U_1, \dots, U_8$ of $\mathcal{Q}$ are the following subgroups of $F_4(L', L)$:

$$U_2 = \{u_2(u, v, a) \mid u, v \in L, a \in K'\}, \quad U_1 = \{u_1(x, y, b) \mid x, y \in L', b \in K\},$$
$$U_4 = \{u_4(u, v, a) \mid u, v \in L, a \in K'\}, \quad U_3 = \{u_3(x, y, b) \mid x, y \in L', b \in K\},$$

where (with the notation of (5.1))

$$u_2(u, v, a) := \bar{u}_2(\beta^{-1}, \alpha\bar{v}, \beta^{-1}u, \alpha\bar{u}, a), \quad u_1(x, y, b) := \bar{u}_1(y, \alpha\beta^2\bar{y}, x, \alpha\bar{x}, b),$$
$$u_4(u, v, a) := \bar{u}_4(\beta^{-1}, \alpha\bar{v}, \beta^{-1}u, \alpha\bar{u}, a), \quad u_3(x, y, b) := \bar{u}_3(y, \alpha\beta^2\bar{y}, x, \alpha\bar{x}, b).$$

We get U_5, U_6, U_7, U_8 by replacing all roots r in the root system of type F_4, which occur in U_1, U_2, U_3, U_4, by the negative root $-r$. The commutator relations between the U_i are the ones obtained from $F_4(L)$ in (5.1). Among U_1, U_2, U_3, U_4 there are non-trivial commutator relations only for $[U_1, U_3] \leqslant X_{e_1+e_2}$, $[U_2, U_4] \leqslant X_{e_1}$ and $[U_1, U_4] \leqslant U_2 U_3$. Note that $U_2 \leqslant U^*$ in the notation of (4.6).

We consider the root groups U_{2i} as root groups belonging to a half apartment with a point in the middle. All root groups U_{2i-1} and U_{2i} are abelian; the latter ones are abelian, since L has characteristic 2.

5.3. By Mühlherr and Van Maldeghem [11] there is an automorphism τ of the building associated to $F_4(L, L')$ such that the F_4-quadrangle $\mathcal{Q}$ arises as a set of fixed points of τ. This automorphism fixes points and symplecta of the building, but no line or plane. For the convenience of the reader we extract the action of a suitable τ on $F_4(L', L)$ from [11].

The map $e_1 \mapsto e_1$, $e_2 \mapsto e_2$, $e_3 \mapsto -e_3$ and $e_4 \mapsto -e_4$ extends to an isometry w of the 4-dimensional Euclidean space spanned by the root system Φ of type F_4 (with fundamental system $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$) which permutes Φ. We define $c_1 := \alpha\beta^2$, $c_2 := \beta^{-2}$, $c_3 := \alpha^{-1}$, $c_4 := \alpha\beta$. For $r = \lambda_1\alpha_1 + \cdots + \lambda_4\alpha_4 \in \Phi$, the image of $x_r(t)$ under τ is $x_{rw}(c_1^{\lambda_1} c_2^{\lambda_2} c_3^{\lambda_3} c_4^{\lambda_4} \bar{t})$. All elements in the U_i defined in (5.2) are fixed under the map τ. This gives an impression why in U_2, say, the scalars in $X_{(1/2)(e_1+e_2-e_3-e_4)}$ and in $X_{(1/2)(e_1+e_2+e_3+e_4)}$ are not independent.

5.4. *For any F_4-quadrangle as in* (5.2), *the group of central elations in the root group U_2 is* $\{x_{e_1+e_2}(a) \mid a \in K'\}$.

Proof. An element $u_2 \in U_2$ is a central elation if and only if $[u_2, U_4] = 1$, see Steinbach [17, (3.5)] for example. With the commutator relations in the F_4-quadrangle $\mathcal{Q}$ the lemma follows. $\square$

5.5 Proposition. *For the F_4-quadrangle $\mathcal{Q}$ described in* (5.2), *we denote by Σ the class of central elation subgroups and we set $S := \langle U_1, \ldots, U_8 \rangle \leqslant F_4(L)$. Then Σ is a class of abstract transvection groups of S and any $A \in \Sigma$ is contained in a long root subgroup of $F_4(L)$.*

Proof. By Steinbach [17, (3.6)], the class Σ of central elation subgroups is a class of abstract transvection subgroups of $\langle\Sigma\rangle \leqslant \mathrm{Aut}(\mathcal{Q})$. Both $\langle\Sigma\rangle$ and S are simple subgroups of $\mathrm{Aut}(\mathcal{Q})$, see Van Maldeghem [25, 5.8]. Hence they coincide. The claim follows with (5.4). $\square$

5.6. We recall the root systems $\Phi(C_4)$ and $\Phi(B_4)$ of (1.2). Let

$$V_1 := \{u_1(0,0,b) \mid b \in K\} \leqslant X_{e_2}, \quad V_2 := \{u_2(0,0,a) \mid a \in K'\} \leqslant X_{e_1+e_2},$$

and similarly for $V_3, \ldots, V_8$. Then $S_1 := \langle V_1, U_2, \ldots, V_7, U_8 \rangle \leqslant C_4(L)$, $S_2 := \langle U_1, V_2, \ldots, U_7, V_8 \rangle \leqslant B_4(L)$ and S_1 and S_2 generate $S = \langle U_1, \ldots, U_8 \rangle$.

Next we show that S_1 and S_2 are isomorphic to classical groups of Witt index 2 over K and K', respectively. We refer to (1.5) for the notation for classical groups.

The quadrangles associated with S_1 and S_2, respectively, have been identified as classical quadrangles also in Mühlherr and Van Maldeghem [11]. But for later use (and since there seems to be a notational error in [11, Sec. 8]) we give some details.

We fix $E_1 \in L'$ with $\overline{E_1} = E_1 + 1$. Then $L = K(E_1)$ and $L' = K'(E_1)$. We use the notation for vector spaces and forms from (1.5). By (x_i, y_i) we denote a hyperbolic pair spanning H_i.

5.7. *The group* $S_1 = \langle V_1, U_2, \dots, V_7, U_8 \rangle$ *defined in* (5.6) *is isomorphic to the classical group of Witt index 2 generated by the isotropic transvection subgroups in the isometry group of the orthogonal space* $W_1 = H_1 \perp H_2 \perp \mathbf{H} \perp (K')^{1/2}$ *over* K, *where* $\mathbf{H}$ *is endowed with the quadratic from* $\beta^{-1}n$. *Here* n *is the norm on the quaternion division ring* $\mathbf{H} := L \oplus LE_2$ *over* K *such that* $E_2E_1 = E_1E_2 + E_2$, $E_2\overline{E_2} = \alpha$ *and* $\overline{E_2} = E_2$.

Proof. Recall from (5.6) that $S_1 \leqslant C_4(L)$. Using the standard matrices for root elements in $C_4(L)$ as given in Carter [3, p. 186], we write the root elements $u_2(u, v, a)$ as symplectic 8×8-matrices over L. The underlying symplectic space is $H_1 \perp H_2 \perp H_3 \perp H_4$. Let $\mathscr{B}_0$ be a basis of $\langle x_3, y_3, x_4, y_4 \rangle$ and denote by J_0 the corresponding fundamental matrix. For any 4-tuple z with entries in L and any $c \in L$, we define

$$M(z,c) := \left(\begin{array}{c|ccc|c} 1 & 0 & z & 0 & c \\ \hline & 1 & & & 0 \\ & & I & & J_0 z^T \\ & & & 1 & 0 \\ \hline & & & & 1 \end{array}\right), \quad N(c) := \left(\begin{array}{cc|c|cc} 1 & & & c & \\ & 1 & & & c \\ \hline & & I & & \\ \hline & & & 1 & \\ & & & & 1 \end{array}\right)$$

compare (1.9). Set $\mathscr{B}_1 := \{x_3, x_4, y_4, y_3\}$.

The matrices of $u_2(u, v, a)$ and of $u_1(0, 0, b)$ with respect to the basis $\{x_1, x_2, \mathscr{B}_1, y_2, y_1\}$ are $M((\alpha\bar{v}, \bar{u}, \beta^{-1}u, \beta^{-1}v), \beta^{-1}(u\bar{u} + \alpha v\bar{v}) + a)$ and $N(b)$, respectively. The matrices of elements in V_3 and U_4 are of a similar form. We define a new basis $\mathscr{B}_2 := \{v_1, v_2, v_3, v_4\}$ of $\langle x_3, y_3, x_4, y_4 \rangle$ by

$$w_1 := x_4 + \beta^{-1}y_4, \qquad w_2 := \overline{E_1}x_4 + \beta^{-1}E_1y_4,$$
$$w_3 := \alpha x_3 + \beta^{-1}y_3, \quad w_4 := \alpha\overline{E_1}x_3 + \beta^{-1}E_1y_3.$$

We write $u, v \in L$ as $u = u_0 + u_1E_1$ and $v = v_0 + v_1E_1$ with $u_0, u_1, v_0, v_1 \in K$. The matrix of $u_2(u, v, a)$ with respect to the basis $\{x_1, x_2, w_1, w_2, w_3, w_4, y_2, y_1\}$ is

$$M((u_0, u_1, v_0, v_1), \beta^{-1}(u\bar{u} + \alpha v\bar{v}) + a)$$

and has only entries from K. We define an ordinary quadratic form Q on $\langle w_1, w_2, w_3, w_4 \rangle_K$ by $Q(u_0w_1 + u_1w_2 + v_0w_3 + v_1w_4) := \beta^{-1}(u\bar{u} + \alpha v\bar{v}) \in K$, where $u := u_0 + u_1E_1$ and $v := v_0 + v_1E_1$. Because of the properties of α and β in (5.2) Q

is anisotropic. The mapping $w_1 \mapsto 1$, $w_2 \mapsto E_1$, $w_3 \mapsto E_2$, $w_4 \mapsto E_1E_2$ extends to an isometry from $(\langle v_1, v_2, v_3, v_4\rangle_K, Q)$ to $(\mathbf{H}, \beta^{-1}n)$, where $\mathbf{H}$ is as in the statement of (5.7). We denote also the 4-dimensional vector space $\langle v_1, v_2, v_3, v_4\rangle_K$ by $\mathbf{H}$.

We define a quadratic form q (of Witt index 2) on $W_1 := \langle x_1, y_1, x_2, y_2\rangle_K \perp \mathbf{H} \perp (K')^{1/2}$ over K such that (x_1, y_1), (x_2, y_2) are orthogonal hyperbolic pairs with respect to q, $q|_{\mathbf{H}} = \beta^{-1}n$, $q(\hat{a}) = \hat{a}^2$ for $\hat{a} \in (K')^{1/2}$. Then $S_1 = \langle V_1, U_2, \ldots, V_7, U_8\rangle$ is isomorphic to the group generated by the isotropic transvection groups on W_1. □

5.8. *The group $S_2 = \langle U_1, V_2, \ldots, U_7, V_8\rangle$ defined in* (5.6) *is isomorphic to the classical group of Witt index* 2 *generated by the Siegel transvection subgroups in the isometry group of the orthogonal space $W_2 = H_1 \perp H_2 \perp \mathbf{H}' \perp K$ over K', where $\mathbf{H}'$ is endowed with the quadratic from $\alpha n'$. Here n' is the norm on the quaternion division ring $\mathbf{H}' := L' \oplus L'E_2'$ over K' such that $E_2'E_1 = E_1E_2' + E_2'$, $E_2'\overline{E_2'} = \beta^2$ and $\overline{E_2'} = E_2'$.*

Proof. The proof is similar to the proof of (5.7). The new basis of $H_3 \perp H_4$ is

$$v_1 := \alpha x_4 + y_4, \qquad v_2 := \alpha\overline{E_1}x_4 + E_1y_4,$$
$$v_3 := \alpha\beta^2 x_3 + y_3, \quad v_4 := \alpha\beta^2\overline{E_1}x_3 + \beta^{-1}E_1y_3. \qquad \square$$

5.9. *For $S = \langle U_1, \ldots, U_8\rangle \leqslant F_4(L)$, we have $\mathrm{Z}(S) = 1$.*

Proof. The center $\mathrm{Z}(S)$ commutes with long root elements in $X_{\pm(e_1+e_2)}$ and in $X_{\pm(e_1-e_2)}$ and is thus contained in the standard subsystem subgroup $M(\alpha_2, \alpha_3) = C_2(L)$. Let $u_2 \in U_2$ and $z \in \mathrm{Z}(S)$. Using the notation of the proof of (5.7), we write both elements with respect to the basis $\mathscr{E}' = \{x_1, x_2, v_1, v_2, v_3, v_4, y_2, y_1\}$ of the 8-dimensional symplectic space underlying the standard subsystem subgroup $M(\alpha_2, \alpha_3, \alpha_4, -e_1) = C_4(L)$. The matrix of u_2 is $M((u_0, u_1, v_0, v_1), \beta^{-1}(u\bar{u} + \alpha v\bar{v}) + a)$, as was shown in (5.7). The matrix of z is of the form $\mathrm{diag}(1, 1, z, 1, 1)$ with z considered as a 4×4-matrix. Because of $u_2^z = u_2$, we obtain $(u_0, u_1, v_0, v_1)z = (u_0, u_1, v_0, v_1)$, for arbitrary $u_2 \in U_2$. This shows $z = 1$. □

6 Embedding the F_4-quadrangles in Chevalley groups of type F_4

6.1. For the definition of the F_4-quadrangles and their parameters L, L', K, K', α, β and root groups U_i, see (5.2). We set

$$F_4Q(K, L, K', \alpha, \beta) := \langle U_1, \ldots, U_8\rangle \leqslant F_4(L', L).$$

(The abbreviation F_4Q indicates F_4-**Q**uadrangle.) For any embedding $\gamma : L \to O$, where O is a field, we define the embedding $\varepsilon_\gamma : F_4(L) \to F_4(O)$, $x_r(t) \mapsto x_r(t^\gamma)$, where $r \in \Phi$, $t \in L$. As K is a parameter of the F_4-quadrangle, we study subgroups of $F_4(O)$. Let G be a subgroup of $F_4(O)$ as in Problem (P) of the introduction. For the polar space $\wp(\Sigma)$, we refer to (2.4). With this notation the following holds:

6.2 Theorem. *We assume that $\wp(\Sigma)$ is an F_4-quadrangle with Σ the class of full central elation subgroups, whence $G/Z(G) \simeq S := F_4(K, L, K', \alpha, \beta)$. Then after extending scalars from O to $\hat{O}$ for a suitable extension $\hat{O}$ of O of degree $\leqslant 2$, there is an embedding $\gamma : L \to \hat{O}$ such that a conjugate of G in $F_4(\hat{O})$ is $F_4Q(K^\gamma, L^\gamma, (K')^\gamma, \alpha^\gamma, \beta^\gamma)$.*

Proof. We consider the subset $\Psi := \{e_1 + e_2, e_1 - e_2\}$ of Φ^+. Passing to a conjugate of G in $Y = F_4(O)$, we achieve the following: if $r \in \Psi \,\dot\cup\, (-\Psi)$ and $T \in \Sigma$ corresponds to X_r (in S), then $\hat{T} = X_r$ (in Y), see (1.4).

By assumption there is a central extension $\rho : G \to S$. By (5.7), (5.8) the groups $S_1 := \langle V_1, U_2, \dots, V_7, U_8\rangle \leqslant C_4(L)$, $S_2 := \langle U_1, V_2, \dots, U_7, V_8\rangle \leqslant B_4(L)$ are isomorphic to classical groups of Witt index 2. Denote by M_i the subgroup of G generated by all elements in Σ which correspond to an isotropic transvection subgroup or a Siegel transvection subgroup, respectively, in S_i $(i = 1, 2)$. Then $G = \langle M_1, M_2\rangle$. By previous results, $M_1 \leqslant C_4(O)$, see (4.10), and $M_2 \leqslant B_4(O)$, see (4.9). Thus M_1 and M_2 embed in classical subgroups of $F_4(O)$.

First, we consider the embedding $M_1 \leqslant C_4(O)$. Denote by $\mathscr{E} = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$ a basis of O^8, the underlying 8-dimensional symplectic space over O, such that the fundamental matrix of the symplectic form is $J = \begin{pmatrix} & I \\ I & \end{pmatrix}$. We write each element of $C_4(O) = \mathrm{Sp}_8(K)$ as an 8×8-matrix with respect to $\mathscr{E}$ as in Carter [3, p. 186]. For $C_4(L) = \mathrm{Sp}_8(L)$, we also introduce such a basis $\mathscr{E}$. (It will be clear from the context, whether $\mathscr{E}$ denotes the basis of L^8 or of O^8.) For S_1, we also use $\mathscr{E}' = \{x_1, x_2, v_1, v_2, v_3, v_4, y_2, y_1\}$, the basis of L^8 which was used in the proof of (5.7) to identify S_1 as a classical group (with underlying vector space W_1 over K). Recall that $W_1 = \langle\mathscr{E}'\rangle_K \perp \mathrm{Rad}(W_1)$. We consider the elements in S_1 as 8×8-matrices over K. By J' we denote the fundamental matrix of the symplectic form on $\langle\mathscr{E}'\rangle_K$ with respect to the basis $\mathscr{E}'$.

By (4.10) the polar space associated to M_1 is weakly embedded in $P(V_0)$, where $V_0 := [O^8, M_1] = O^8$. (See Steinbach and Van Maldeghem [15, (5.4)], for the last assertion.) By Cuypers and Steinbach [7] there exist an embedding $\gamma : K \to O$ and a semi-linear mapping $\varphi : W_1 \to O^8$ (with $\ker\varphi = \mathrm{Rad}(W_1)$) such that

$$(w\varphi)m = (w(m\rho))\varphi, \quad w \in W, m \in M_1.$$

Denote (again) by E_1 a root of the quadratic polynomial $x^2 + x + (E_1\overline{E_1})^\gamma$ over O. Then $\hat{O} := O(E_1)$ is an extension of degree $\leqslant 2$ of O. We extend scalars from O to $\hat{O}$ and in the following we consider the embedding $G \leqslant F_4(\hat{O}) =: \hat{Y}$. Via $E_1 \mapsto E_1$ we obtain an embedding $\gamma : L \to \hat{O}$. Let $\mathscr{B}$ denote the image of $\mathscr{E}'$ under $\varphi : W_1 \to \hat{O}^8$.

We define a basis $\mathscr{E}'$ of $\hat{O}^8$ such that $M^{\mathscr{E}}_{\mathscr{E}'}(\mathrm{id}) = T^\gamma$, where T is the matrix of the base change from $\mathscr{E}$ to $\mathscr{E}'$ over L. The fundamental matrix with respect to $\mathscr{E}'$ over $\hat{O}$ is hence $(J')^\gamma$. Above we have shown that $M^{\mathscr{E}'}_{\mathscr{E}'}(m) = D^{-1}M^{\mathscr{E}'}_{\mathscr{E}'}(m\rho)^\gamma D$ with $D := M^{\mathscr{B}}_{\mathscr{E}'}(\mathrm{id})$, for $m \in M_1$. We identify D with $M^{\mathscr{E}'}_{\mathscr{E}'}(D)$. The matrix D is of the form $D = \mathrm{diag}(c, d, D_0, d', c')$ with D_0 a 4×4-matrix. Furthermore, $D(J')^\gamma D^T = \mu(J')^\gamma$, for a scalar $\mu \in \hat{O}$. The base change to $\mathscr{E}$ yields that $m = m\rho\varepsilon_\gamma\delta$ with an automorphism δ of $C_4(\hat{O})$ which is a product of a diagonal automorphism and of an inner automorphism of $C_2(\hat{O})$. Since in $e_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ the root α_1 occurs with coefficient 1, the diagonal automorphism of $C_4(\hat{O})$ may be extended to a diagonal (and hence inner) automorphism of $F_4(\hat{O}) = \hat{Y}$. Hence passing to a conjugate in $\hat{Y}$, we may assume that $m = m\rho\varepsilon_\gamma$ for $m \in M_1$.

For the embedding $M_2 \leqslant B_4(\hat{O})$, we proceed similarly (with bases $\mathscr{E}, \mathscr{E}'$ and $\mathscr{B}$) and obtain that $M_{\mathscr{E}'}^{\mathscr{E}'}(m) = D^{-1}M_{\hat{\mathscr{E}}'}^{\mathscr{E}'}(m\rho)^\gamma D$ for $m \in M_2$, with $D = \operatorname{diag}(c, d, D_0, d', c')$, D_0 a 5×5-matrix, and $\gamma : L \to \hat{O}$ as constructed above. This yields that

$$m \in \langle X_{e_2 \pm e_3}, X_{e_2 \pm e_4}, X_{e_2} \rangle, \quad \text{for } m \in M_2 \text{ with } m\rho \in U_1,$$
$$m \in \langle X_{e_1 \pm e_3}, X_{e_1 \pm e_4}, X_{e_1} \rangle, \quad \text{for } m \in M_2 \text{ with } m\rho \in U_3.$$

Similarly, as in (5.9), we see that $Z(G) = 1$. Hence $\rho : G \to S$ is an isomorphism. Fix $m_1 \in G$ such that $m_1\rho = u_1(x, 0, 0) =: u_1$, $x \in L'$. Let $m_4 \in G$ with $m_4\rho = u_4(u, v, 0) =: u_4$, $u, v \in L$ arbitrary.

We use the commutator relations in (5.1). On one hand $[m_1, m_4] = m_2 m_3$ with $m_2\rho = u_2(\alpha\bar{x}v, xu, \cdot) \in U_2$ (we do not need the value of the third parameter) and $m_3\rho \in U_3$. On the other hand $m_1 \in \langle X_{e_2 \pm e_3}, X_{e_2 \pm e_4}, X_{e_2} \rangle$ and $m_3 \in \langle X_{e_1 \pm e_3}, X_{e_1 \pm e_4}, X_{e_1} \rangle$ by the above. Hence there are scalars $s_1, s_2, s_3, s_4, b \in \hat{O}$ such that $m_1 = x_{e_2 - e_3}(s_1) x_{e_2 + e_3}(s_2) x_{e_2 - e_4}(s_3) x_{e_2 + e_4}(s_4) x_{e_2}(b)$. Thus

$$\begin{aligned}[m_1, m_4] &= x_{(1/2)(e_1+e_2-e_3-e_4)}(p_1) x_{(1/2)(e_1+e_2+e_3+e_4)}(p_2) \\ &\quad \cdot x_{(1/2)(e_1+e_2-e_3+e_4)}(p_3) x_{(1/2)(e_1+e_2+e_3-e_4)}(p_4) \cdot x_{e_1+e_2}(c) \cdot y_3\end{aligned}$$

with $y_3 \in \langle X_{e_1 \pm e_3}, X_{e_1 \pm e_4}, X_{e_1} \rangle$ and

$$(1) \quad p_1 = b\beta^{-1}v + s_1\bar{u} + s_3\beta^{-1}u, \qquad (2) \quad p_2 = b\alpha\bar{v} + s_2\beta^{-1}u + s_4\bar{u},$$
$$(3) \quad p_3 = b\beta^{-1}u + s_1\alpha\bar{v} + s_4\beta^{-1}v, \quad (4) \quad p_4 = b\bar{u} + s_2\beta^{-1}v + s_3\alpha\bar{v}.$$

(Here we omit the application of γ on the right hand side to simplify notation.) Each element in $U := \langle X_r \mid r \in \Phi^+ \rangle$ has a unique factorization as a product of root elements in increasing order. Since all root elements involved above commute, comparing the coefficients of $x_{(1/2)(e_1+e_2 \pm e_3 \pm e_4)}$ yields

$$p_1 = \beta^{-1}xu, \quad p_2 = \alpha\overline{xu}, \quad p_3 = \alpha\beta^{-1}\bar{x}v, \quad p_4 = \alpha x\bar{v},$$

for all $u, v \in L$. With $u = 0$, $v = 1$, we see $b = 0$. Next, (1) with $u = 1$, (4) with $v = 1$ and (2) with $u = 1$ yield

$$s_1 = \beta^{-1}s_3 + \beta^{-1}x, \quad s_2 = \alpha\beta s_3 + \alpha\beta x, \quad s_4 = \beta^{-1}s_2 + \alpha\bar{x} = s_3\alpha + \alpha x + \alpha\bar{x}.$$

With (1) we obtain $\beta^{-1}s_3\bar{u} + s_3\beta^{-1}u = \beta^{-1}xu + \beta^{-1}x\bar{u}$, for all $u \in L$. Setting $u = E_1$, we get $s_3 = x$ and $s_1 = s_2 = 0$, $s_4 = \alpha\bar{x}$. We have shown that $m = m\rho\varepsilon_\gamma$, for $m\rho = u_1(x, 0, 0)$ with $x \in L'$.

A similar calculation yields $m = m\rho\varepsilon_\gamma$, for $m\rho = u_1(0, y, 0)$ with $y \in L'$. Thus $m = m\rho\varepsilon_\gamma$ for $m\rho \in U_1$. Now U_3 is conjugate to U_1 in $\langle U_4, U_8 \rangle$, see Van Maldeghem [25, (5.2.6)] for example. Hence we get the same result for U_3 and also for U_5, U_7. Together this proves $g = g\rho\varepsilon_\gamma$ for $g \in G$, thus the theorem. □

Theorems 2 and 3 proved in Sections 3 and 4, respectively, imply Theorem 1. Detailed information on the embeddings of the F_4-quadrangles is given in Theorem 6.2.

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