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Alexander duality in subdivisions of Lawrence polytopes

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Abstract. The class of simplicial complexes representing triangulations and subdivisions of Lawrence polytopes is closed under Alexander duality. This gives a new geometric model for oriented matroid duality.

Key words. Alexander duality, oriented matroid, Lawrence polytope, triangulation.

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1 Introduction

The aim of this note is to show that oriented matroid duality can be seen as an instance of Alexander duality of simplicial complexes (see e.g. [2]). We represent an affine oriented matroid $(\mathcal{M}, f)$ on the ground set $\{1, \dots, n, f\}$ by a simplicial complex $\Delta(\mathcal{M}, f)$ on the vertex set $\{x_1, \dots, x_n,\ y_1, \dots, y_n\}$ as follows. The facets of $\Delta(\mathcal{M}, f)$ are the complements of the sets

$$\{x_i : i \in C^+\} \cup \{y_j : j \in C^-\},$$

where $C = (C^+, C^-)$ runs over all signed cocircuits of $(\mathcal{M}, f)$ such that the distinguished element f lies in C^+. We have the following result:

Theorem 1. *The Alexander dual of $\Delta(\mathcal{M}, f)$ is the simplicial complex $\Delta({}_{-f}\mathcal{M}^*, f)$ associated with the affine oriented matroid $({}_{-f}\mathcal{M}^*, f)$. Here ${}_{-f}\mathcal{M}^*$ denotes the oriented matroid dual to $\mathcal{M}$ with the element f reoriented.*

This duality can be expressed geometrically in terms of Lawrence polytopes. Suppose that the contraction $\mathcal{M}/f$ is represented by a $d \times n$-matrix $\boldsymbol{D}$ of rank d. Then the

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associated Lawrence polytope (see e.g. [11, §6.6]) is the convex hull of the columns of the $(d+n) \times 2n$-matrix

$$\Lambda(\boldsymbol{D}) = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{I} \end{pmatrix}. \tag{1}$$

Here $\boldsymbol{I}$ is the $n \times n$-identity matrix, $\boldsymbol{0}$ is the $d \times n$-zero matrix, and the columns are indexed by $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. Recall that $\{x_i, y_i\}$ is the complement of a facet of $\Lambda(\boldsymbol{D})$, for all i. It turns out that $\Delta(\mathscr{M}, f)$ is a polyhedral subdivision of the Lawrence polytope $\Lambda(\boldsymbol{D})$, where each maximal face in the subdivision is represented by the simplex on its set of vertices. This subdivision is a triangulation if and only if the matroid $\mathscr{M} \backslash f$ is uniform. The Lawrence polytope $\Lambda(\boldsymbol{D})$ itself is called *uniform* if all $d \times d$-minors of $\boldsymbol{D}$ are nonzero, or, in the non-realizable case, if the matroid $\mathscr{M}/f$ is uniform.

The following is our main result:

Theorem 2. *The following families of simplicial complexes on the* $2n$*-element set* $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ *are closed under Alexander duality*:

(1) *Regular triangulations of uniform Lawrence polytopes*,

(2) *regular subdivisions of Lawrence polytopes*,

(3) *triangulations of uniform Lawrence matroid polytopes*,

(4) *subdivisions of Lawrence matroid polytopes*.

Moreover, Alexander duality gives a bijection between regular triangulations of Lawrence polytopes and regular subdivisions of uniform Lawrence polytopes. These two families are not closed under Alexander duality.

The families (3) and (4) in Theorem 2 refer to the case when the oriented matroid $\mathscr{M}/f$ cannot be represented by a matrix $\boldsymbol{D}$. For the relevant definitions and notations used here we refer to the books [3] and [10]. In particular, see [3, §9.3] for Lawrence (matroid) polytopes and [3, §9.6] for subdivisions of (matroid) polytopes. The first author proved in [10, Theorem 4.14] that every subdivision of a Lawrence (matroid) polytope is induced by a lifting of oriented matroids $\mathscr{M}/f \to \mathscr{M}$.

Our presentation is organized as follows. In Section 2 we prove Theorem 1 and we interpret $\Delta(\mathscr{M}, f)$ in terms of hyperplane arrangements. The proof of Theorem 2 is given in Section 4. Examples of Alexander dual pairs of subdivided Lawrence polytopes are given in Section 3. The smallest non-trivial example is the pair of triangular prisms in Figure 1.

Section 5 concerns the Alexander duals of simplicial balls and spheres in general. This section was added after we received the very helpful comments of an anonymous referee. He or she pointed us to the work of Dong [5] and proposed the extension stated in part 2 of Theorem 9.

The original motivation for this project came from commutative algebra and hyperkähler geometry. The simplicial complex $\Delta(\mathscr{M}, f)$ is represented algebraically

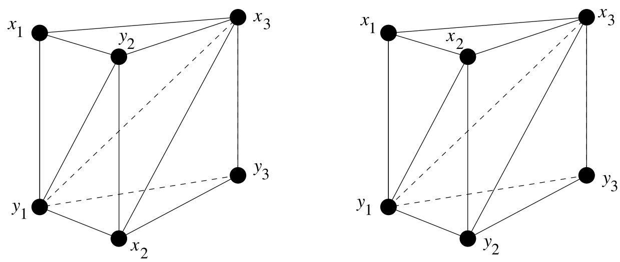

Figure 1. The triangulation of a triangular prism is Alexander self-dual, after relabeling the vertices. The non-edges on the left are the complements of the tetrahedra on the right.

as a square-free monomial ideal in $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. The minimal free resolution of this ideal constructed in [9] can be interpreted as a (suitably homogenized) coboundary complex on the Alexander dual $\Delta({}_{-f}\mathscr{M}^*, f)$. In particular, part (1) in Theorem 2 furnishes a large class of Stanley–Reisner rings which are Cohen–Macaulay and have an explicit linear resolution. The quotient of such a Stanley–Reisner ring modulo a linear system of parameters was shown in [6] to equal the cohomology ring of a toric hyperkähler variety. These varieties are complete intersections in the toric variety whose fan is a cone over $\Delta(\mathscr{M}, f)$. It would be interesting to explore the duality of toric hyperkähler varieties arising from our results.

2 Oriented matroid duality is Alexander duality

We recall the combinatorial definition of Alexander duality. Let K be a simplicial complex on the vertex set V. Then the *Alexander dual* of K is the simplicial complex

$$K^\vee := \{V \backslash \sigma : \sigma \notin K\}$$

The Alexander Duality Theorem states that the i-th reduced homology group $\tilde{H}_i(K, \boldsymbol{Z})$ of K equals the $(|V| - 3 - i)$-th reduced cohomology group $\tilde{H}^{|V|-3-i}(K^\vee, \boldsymbol{Z})$ of $K^\vee$. See, e.g., [2, Equation (2)] or [1, (9.17)]. In particular, the Alexander dual of an acyclic simplicial complex is acyclic, although the Alexander dual of a contractible simplicial complex need not be contractible. See Section 5 for a discussion of this and related topological issues.

Proof of Theorem 1. The statement can be rephrased as the following claim: given an oriented matroid $\mathscr{M}$ on the ground set $\{1, \ldots, n, f\}$, for any pair of subsets $\sigma_1, \sigma_2 \subseteq \{1, \ldots, n\}$ one and only one of the following happens:

(1) There is a cocircuit (C^+, C^-) in $\mathscr{M}$ with $C^- \subseteq \sigma_1$, and $f \in C^+ \subseteq \sigma_2 \cup \{f\}$, or

(2) There is a cocircuit (D^+, D^-) in $\mathscr{M}^*$ (that is, a circuit in $\mathscr{M}$) with $f \in D^- \subseteq \{1, \ldots, n, f\} \backslash \sigma_1$ and $D^+ \subseteq \{1, \ldots, n\} \backslash \sigma_2$.

Indeed, condition (1) above is equivalent to

$$\{x_i : i \notin \sigma_2\} \cup \{y_j : j \notin \sigma_1\} \in \Delta(\mathscr{M}, f),$$

and condition (2) is equivalent to

$$\{x_i : i \in \sigma_2\} \cup \{y_j : j \in \sigma_1\} \in \Delta({}_{-f}\mathscr{M}^*, f).$$

The claim follows from Lemma 3 below, taking $e = f$ and color classes $B = (\sigma_2 \backslash \sigma_1) \cup \{f\}$, $W = \sigma_1 \backslash \sigma_2$, $R = \sigma_1 \cap \sigma_2$, and $G = \{1, \dots, n\} \backslash (\sigma_1 \cup \sigma_2)$. We also set $(C^+, C^-) = (Y^+, Y^-)$ and $(D^+, D^-) = (X^-, X^+)$.

Lemma 3 is just a rephrasing of the 4-painting axiom of oriented matroid circuits and cocircuits. The notation in the lemma is chosen to exactly match the axiom as it appears in [3, Theorem 3.4.4]. This is the reason why we have $X = -D$ above rather than reorienting X in the lemma. □

Lemma 3. *Let B, W, G and R be a partition of the ground set of an oriented matroid $\mathscr{M}$. Let $e \in B \cup W$ be one of the elements. Then, exactly one of the following happens*:

(1) *There is a circuit (X^+, X^-) with $X^- \subseteq W \cup G$ and $e \in X^+ \subseteq B \cup G$, or*

(2) *There is a cocircuit (Y^+, Y^-) with $e \in Y^+ \subseteq B \cup R$ and $Y^- \subseteq W \cup R$.*

We now interpret $\Delta(\mathscr{M}, f)$ in terms of hyperplane arrangements. By the Topological Representation Theorem [3, §4], an affine oriented matroid $(\mathscr{M}, f)$ of rank d on $\{1, \dots, n, f\}$ represents an affine arrangement $\mathscr{H}(\mathscr{M}, f)$ of n pseudo-hyperplanes in $\mathbb{R}^{d-1}$, with the distinguished element f playing the role of the hyperplane at infinity. We can regard $\mathscr{H}(\mathscr{M}, f)$ as a cover of $\mathbb{R}^{d-1}$ by $2n$ closed half-spaces $\{x_1, \dots, x_n, y_1, \dots, y_n\}$, where x_i and y_i label respectively the positive and negative sides of the i-th oriented hyperplane. It is straightforward to check that a subset of these half-spaces has a non-empty intersection in $\mathbb{R}^{d-1}$ if and only if the corresponding subset of $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ is a simplex in $\Delta(\mathscr{M}, f)$. In other words:

Remark 4. The simplicial complex $\Delta(\mathscr{M}, f)$ is the nerve of the cover of $\mathbb{R}^{d-1}$ consisting of the $2n$ closed half-spaces in the arrangement $\mathscr{H}(M, f)$.

The facets of $\Delta(\mathscr{M}, f)$ are maximal intersecting families of closed half-spaces. They correspond to the vertices of the arrangement $\mathscr{H}(\mathscr{M}, f)$. The face poset of $\mathscr{H}(\mathscr{M}, f)$ appears as a subposet in the face poset of $\Delta(\mathscr{M}, f)$. A simplex $\sigma \in \Delta(\mathscr{M}, f)$ is called *full* if $\sigma \cap \{x_i, y_i\} \neq \emptyset$ for all i.

Remark 5. If $\mathscr{M} \backslash f$ is uniform, then the face poset of $\mathscr{H}(\mathscr{M}, f)$ is anti-isomorphic to the poset of full simplices of $\Delta(\mathscr{M}, f)$. If $\mathscr{M} \backslash f$ is not uniform, then the former is a strict subposet of the latter.

This implies that the oriented matroid $\mathcal{M}$ can be recovered from the simplicial complex $\Delta(\mathcal{M}, f)$ provided $\mathcal{M}$ is uniform. The same statement is not true for general oriented matroids. For instance, consider an arbitrary arrangement of hyperplanes which intersect in a line, and then adjoin two parallel hyperplanes transverse to that line. Here $\Delta(\mathcal{M}, f)$ consists of two simplices of the same dimension which share a common facet, regardless of which arrangement we started with.

3 Lawrence polytopes in dimension three, four and five

In Section 4 we are going to prove Theorem 2 by translating Theorem 1 into the language of subdivisions of Lawrence (matroid) polytopes. As a preparation for that we describe in this section all the Lawrence polytopes which exist in dimensions up to 5, and an example of our Alexander duality result involving two Lawrence polytopes of respective dimensions 4 and 5.

We first recall the construction of Lawrence polytopes in oriented matroid language, and then we discuss low-dimensional Lawrence polytopes. Let $\mathcal{M}$ be an oriented matroid of rank d on $\{1, \ldots, n\}$, and let $\mathcal{M}^*$ be its dual. Let $\mathcal{M}^* \cup (-\mathcal{M}^*)$ be the oriented matroid on $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ defined by labeling the i-th element of $\mathcal{M}^*$ as x_i and extending $\mathcal{M}^*$ by an element y_i opposite to each x_i. The dual of $\mathcal{M}^* \cup (-\mathcal{M}^*)$ is called the *Lawrence oriented matroid* (or *Lawrence polytope*, since it is a matroid polytope) of $\mathcal{M}$, and denoted $\Lambda(\mathcal{M})$. It has $2n$ elements and rank $d + n$. Lawrence (matroid) polytopes are studied in Section 9.3 of [3] and in Chapter 4 of [10]. For example, [3, Lemma 4.11(ii)] implies that $\Lambda(\mathcal{M})$ has $n - l + 2c$ facets, where c is the number of cocircuits of $\mathcal{M}$ and l the number of coloops.

Since all the oriented matroids with $d + n \leqslant 11$ are realizable, all Lawrence matroid polytopes of dimension at most 10 are honest polytopes, that is, they can be realized by $(d + n) \times 2n$-matrices of the form $\Lambda(\boldsymbol{D})$ as in (1). In what follows we describe all Lawrence polytopes of dimension $d + n - 1 \leqslant 5$.

Let us first discuss the degenerate cases when $\mathcal{M}$ has a loop or coloop. If x_i is a coloop in $\mathcal{M}$ (i.e. if the i-th column of $\boldsymbol{D}$ is linearly independent of all others), then it becomes a loop in $\mathcal{M}^*$. Then, x_i and y_i are loops in $\mathcal{M}^* \cup (-\mathcal{M}^*)$ and coloops in $\Lambda(\mathcal{M})$. Geometrically, $\Lambda(\boldsymbol{D})$ is an iterated pyramid over the Lawrence polytope $\Lambda(\boldsymbol{D} \backslash \{x_i\})$. If x_i is a loop in $\mathcal{M}$ (i.e. if the i-th column of $\boldsymbol{D}$ is zero), then $\Lambda(\mathcal{M})$ is obtained from $\Lambda(\mathcal{M} \backslash \{x_i\})$ by adjoining a pair of parallel elements which forms a positive cocircuit. Geometrically, $\Lambda(\boldsymbol{D})$ is a pyramid over $\Lambda(\boldsymbol{D} \backslash \{x_i\})$ with apex at a pair of identified points x_i and y_i. The right picture of Figure 2 represents this situation. The apex of the pyramid corresponds to the identified points y_3 and x_3. Note that the triangulation uses x_3 and not y_3 as a vertex. This is indicated in the diagram with a filled dot for x_3 and an empty dot for y_3.

We now consider only Lawrence polytopes that are not pyramids over other Lawrence polytopes, which is the same as allowing only oriented matroids without loops or coloops. There are eight combinatorial types of such Lawrence polytopes having dimension at most five. The corresponding parameters (n, d) are $(2, 1), (3, 1), (4, 1), (5, 1), (3, 2), (4, 2), (4, 2), (4, 2)$:

If $d = 1$, then the Lawrence polytope of $\mathcal{M}$ equals the product $\Delta^1 \times \Delta^{n-1}$ of a seg-

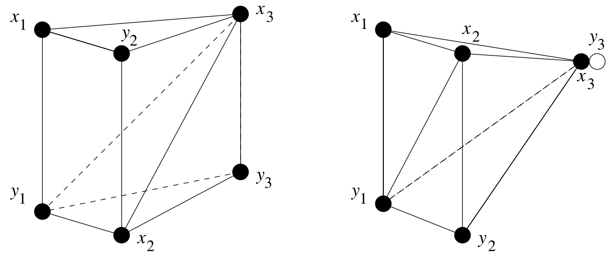

Figure 2. This subdivision of a uniform Lawrence polytope (the triangular prism) is Alexander dual to a triangulation of a non-uniform Lawrence polytope (the pyramid).

ment and a simplex of dimension $n-1$. The polytope $\Delta^1 \times \Delta^{n-1}$ has $n!$ triangulations each isomorphic to the well-known staircase triangulation. The case $n = 2$ is featured in Figure 1. The case $n = 3$ appears in (4) below.

If $n - d = 1$, then $\mathscr{M}^*$ and $\mathscr{M}^* \cup (-\mathscr{M}^*)$ have rank 1, and $\Lambda(\mathscr{M})$ has corank 1, i.e., it has a unique circuit. Assuming without loss of generality that all the elements of $\mathscr{M}$ have the same orientation, this unique circuit is $(\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\})$. The polytope $\Lambda(\mathscr{M})$ can be realized as the convex hull of the union of two $(n-1)$-simplices in $\mathbb{R}^{2n-2}$ whose relative interiors intersect in a unique point. This Lawrence polytope is the cyclic $(2n-2)$-polytope with $2n$ vertices.

Up to reorientation, there are three oriented matroids $\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_3$ of rank 2 on 4 elements. They are represented by 2×4-matrices

$$\boldsymbol{D}_1 = (v_1, v_1, v_2, v_2), \quad \boldsymbol{D}_2 = (v_1, v_1, v_2, v_3), \quad \boldsymbol{D}_3 = (v_1, v_2, v_3, v_4).$$

Here the v_i are pairwise linearly independent vectors in the plane. In each case, $\Lambda(\boldsymbol{D}_i)$ is a five-dimensional Lawrence polytope with eight vertices and with $6 + 2i$ facets. For instance, $\Lambda(\boldsymbol{D}_1)$ is the join of two squares.

We shall examine the Lawrence polytope $\Lambda(\boldsymbol{D}_3)$ by computing one of its triangulations along with its Alexander dual. We start out with the 2×5-matrix

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & f \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \end{array},$$

and we fix the following Gale dual 3×5-matrix, with last column reoriented:

$$B = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \bar{f} \end{matrix} \\ \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 \end{pmatrix} \end{array}.$$

Thus A and B represent uniform matroids. Let $A' = A/f$ and $B' = B/\bar{f}$ denote the matrices gotten from A and B by contracting the last column. Contracting f means projecting every vector $v \in A \backslash \{f\}$ along the direction of f to a linear hyperplane not containing f. In our case:

$$A' = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc} 4 & 3 & 2 & 1 \end{array} \right) \end{array}, \quad B' = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \end{array}.$$

The 2×4-matrix B' has the form of $\boldsymbol{D}_3$ in the previous paragraph and will play the role of $\boldsymbol{D}$ in the big matrix $\Lambda(\boldsymbol{D})$ of Equation (1). The polytopes $\Lambda(A')$ and $\Lambda(B')$ are 4-dimensional and 5-dimensional, both with eight vertices. As we saw above, $\Lambda(A')$ is (affinely isomorphic to) the product of a segment and a tetrahedron.

There are precisely six signed cocircuits of B (or circuits of A) in which the element $\bar{f}$ is positive:

$$\{y_1, x_2, \bar{f}\}, \quad \{y_1, x_3, \bar{f}\}, \quad \{y_1, x_4, \bar{f}\}, \quad \{y_2, x_3, \bar{f}\}, \quad \{y_2, x_4, \bar{f}\}, \quad \{y_3, x_4, \bar{f}\}. \tag{2}$$

There are precisely four signed cocircuits of A (or circuits of B) in which the element f is positive:

$$\{x_2, x_3, x_4, f\}, \quad \{y_1, x_3, y_4, f\}, \quad \{y_1, y_2, x_4, f\}, \quad \{y_1, y_2, y_3, f\}. \tag{3}$$

Taking complements in (2) we obtain the maximal simplices in a regular triangulation of the 5-dimensional Lawrence polytope $\Lambda(B')$:

$$\begin{gathered} \{x_1, x_2, x_3, y_1, y_2, y_4\}, \quad \{x_1, x_2, x_3, y_1, y_3, y_4\}, \quad \{x_1, x_2, x_4, y_1, y_3, y_4\}, \\ \{x_1, x_2, x_3, y_2, y_3, y_4\}, \quad \{x_1, x_2, x_4, y_2, y_3, y_4\}, \quad \{x_1, x_3, x_4, y_2, y_3, y_4\}. \end{gathered}$$

Taking complements in (3) we obtain the maximal simplices in a staircase triangulation of the 4-dimensional Lawrence polytope $\Lambda(A') = \Delta^1 \times \Delta^3$:

$$\{x_1, y_1, y_2, y_3, y_4\}, \quad \{x_1, x_2, y_2, y_3, y_4\}, \quad \{x_1, x_2, x_3, y_3, y_4\}, \quad \{x_1, x_2, x_3, x_4, y_4\}. \tag{4}$$

These two simplicial complexes are Alexander dual to each other. The Stanley–Reisner ideals of the two triangulations are gotten from (2) and (3) by deleting f and $\bar{f}$ and regarding each set as square-free monomial. Namely, the Stanley–Reisner ideal of our triangulation of $\Delta(B')$ is

$$\langle y_1x_2, y_1x_3, y_1x_4, y_2x_3, y_2x_4, y_3x_4 \rangle, \tag{5}$$

and the Stanley–Reisner ideal of our triangulation of $\Delta(A')$ is

$$\langle x_2x_3x_4, y_1x_3x_4, y_1y_2x_4, y_1y_2y_3 \rangle. \tag{6}$$

4 Duality of subdivided Lawrence polytopes

The proof of Theorem 2 is based on the non-trivial fact that all subdivisions of a Lawrence matroid polytope are lifting subdivisions. This fact is one of the main results in the monograph [10].

We recall the definition of lifting subdivisions. Let $(\mathcal{M}, f)$ be an affine oriented matroid on the ground set $\{1, \dots, n, f\}$, and assume that f belongs to some positive cocircuit. Consider the sets $\{x_i : i \notin C^+\}$ where C runs over all *positive* cocircuits of $\mathcal{M}$ with $f \in C^+$. These sets form (the maximal cells of) a subdivision of the oriented matroid $\mathcal{M}/f$. Subdivisions of an oriented matroid obtained in this manner are called *lifting subdivisions*. For the general definition of subdivisions of oriented matroids see [3, §9.6] or [10].

If $\mathcal{M}/f$ is realized by a vector configuration, then subdivisions of $\mathcal{M}/f$ are the same as polyhedral subdivisions (also called polyhedral fans) of it. If not only $\mathcal{M}/f$ but also $\mathcal{M}$ is realized by a vector configuration A, then the lifting subdivision induced by $(\mathcal{M}, f)$ is the regular subdivision of A/f corresponding to the lifting $A/f \to A$. Some lifting subdivisions of vector configurations are not regular, and some polyhedral subdivisions are not lifting. See [3, Corollary 9.6.8]. By [3, Proposition 9.1.1], every lifting subdivision is either a $(d-1)$-ball or a $(d-1)$-sphere, where d is the rank of $\mathcal{M}/f$, and the latter happens exactly when $\mathcal{M}$ is acyclic and $\mathcal{M}/f$ totally cyclic. The topological type, or even the homotopy type, is not known for general subdivisions of non-realizable oriented matroids.

Proposition 6. *Let S be a lifting subdivision of a rank d oriented matroid on n elements. If S is not a triangulation we consider it as a simplicial complex whose facets are the maximal faces of S. Then, the Alexander dual $S^\vee$ of S is either contractible or homotopy equivalent to an $(n-d-2)$-sphere, depending on whether S itself is contractible or a $(d-1)$-sphere.*

Proof. A subset $\sigma \subseteq \{1, \dots, n\}$ is in $S^\vee$ if and only if $\mathcal{M}$ has no positive cocircuit with $f \in C^+ \subseteq \sigma$. By Lemma 3 (with $W = R = \emptyset$, $B = \sigma$ and $G = \{1, \dots, n, f\} \backslash \sigma$) this happens if and only if $\mathcal{M}$ has a circuit (D^+, D^-) with $f \in D^+$ and $D^- \cap \sigma = \emptyset$. Equivalently, if the closed positive half-spaces labeled by σ have non-empty intersection in the arrangement $\mathcal{H}(\mathcal{M}^*, f)$.

In other words, $S^\vee$ is the nerve of the family of closed positive half-spaces of $\mathcal{H}(\mathcal{M}^*, f)$. By the Nerve Theorem (see [1, §11]) $S^\vee$ has the homotopy type of the union of these half-spaces, which equals the complement of the (open) cell of $\mathcal{H}(\mathcal{M}^*, f)$ corresponding to the covector $(f, \{1, \dots, n\})$, or the entire affine space if that covector does not appear in $\mathcal{M}^*$. This complement is contractible unless the covector exists and the corresponding cell is bounded, in which case it is an $(n-d-2)$-sphere. The cell $(f, \{1, \dots, n\})$ exists and is bounded if and only if $\mathcal{M}^* \backslash f$ is acyclic and $\mathcal{M}^*$ totally cyclic. □

We now shift gears and replace $\mathcal{M}/f$ by $\Lambda(\mathcal{M}/f)$. It was proved in [10, Theorem 4.14] that every subdivision of a Lawrence matroid polytope $\Lambda(\mathcal{M}/f)$ is a lifting subdivision. See also [7, §4] for the realizable case. Moreover, lifts of $\Lambda(\mathcal{M}/f)$ and lifts of $\mathcal{M}/f$ are essentially the same thing. In particular, $(\mathcal{M}, f)$ represents a lift of $\Lambda(\mathcal{M}/f)$ and a lifting subdivision of it. We denote this subdivision by $S(\mathcal{M}, f)$. Its maximal faces are the sets

$$\{x_i : i \notin C^+\} \cup \{y_i : i \notin C^-\}$$

where C runs over all cocircuits of $\mathcal{M}$ with $f \in C^+$. Hence $S(\mathcal{M}, f)$ coincides with $\Delta(\mathcal{M}, f)$ if we regard $S(\mathcal{M}, f)$ as a simplicial complex as in the statement of Proposition 6. Observe that $S(\mathcal{M}, f)$ is a triangulation if and only if $\mathcal{M}\backslash f$ is uniform. Theorem 1 can be rephrased as:

Corollary 7. *Let $(\mathcal{M}, f)$ be an affine oriented matroid. Let $({}_{-f}\mathcal{M}^*, f)$ be its dual, reoriented at f. The subdivisions $S(\mathcal{M}, f)$ and $S({}_{-f}\mathcal{M}^*, f)$ of $\Lambda(\mathcal{M}/f)$ and $\Lambda(\mathcal{M}^*/f)$ are Alexander dual to one another.*

Proof of Theorem 2. Part (4) follows from Corollary 7. Part (3) corresponds to the case where both $\mathcal{M}/f$ and $\mathcal{M}\backslash f$ are uniform and part (2) is the case where both $\mathcal{M}/f$ and $\mathcal{M}\backslash f$ are realizable. Part (1) is the intersection of both cases. Observe that $\mathcal{M}\backslash f$ is uniform or realizable if and only if $\mathcal{M}^*/f = (\mathcal{M}\backslash f)^*$ has that property. □

Triangulations of Lawrence matroid polytopes and subdivisions of uniform Lawrence matroid polytopes, intermediate between Cases (3) and (4) of Theorem 2, correspond respectively to $\mathcal{M}\backslash f$ and $\mathcal{M}/f$ being uniform. Hence they are not self-dual classes of simplicial complexes $\Delta(\mathcal{M}, f)$, but classes dual to one another. Adding the attribute "regular" to both sides gives another two dual classes. Figure 2 was an example of this. Figure 3 below summarizes Theorem 2 and this remark, showing how Alexander duality acts on the following eight families of simplicial complexes on $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$:

S = {Subdivisions of matroid Lawrence polytopes}.

R = {Regular subdivisions of Lawrence polytopes}.

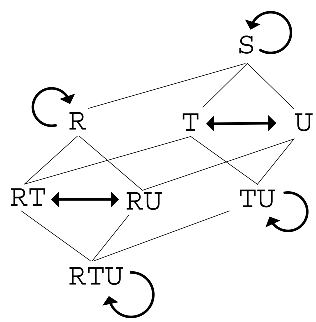

Figure 3. A diagram showing the action of Alexander duality on several families of simplicial complexes.

$\mathtt{T} = \{$Triangulations of matroid Lawrence polytopes$\}$.

$\mathtt{U} = \{$Subdivisions of uniform matroid Lawrence polytopes$\}$.

$\mathtt{RT} = \mathtt{R} \cap \mathtt{T}$, $\mathtt{RS} = \mathtt{R} \cap \mathtt{S}$, $\mathtt{TU} = \mathtt{T} \cap \mathtt{U}$, $\mathtt{RTU} = \mathtt{R} \cap \mathtt{T} \cap \mathtt{U}$.

This is a Hasse diagram: thin lines represent set-theoretic inclusions among the eight families. Thick arrows indicate the action of Alexander duality.

Remark 8. When we say "$\Delta(\mathscr{M}, f)$ is a regular triangulation of a Lawrence polytope" we mean "there is a realization $\boldsymbol{D}$ of $\mathscr{M}/f$ for which the subdivision corresponding to $\Delta(\mathscr{M}, f)$ is regular". A stronger meaning would be "in every realization $\boldsymbol{D}$ of $\mathscr{M}/f$ the subdivision corresponding to $\Delta(\mathscr{M}, f)$ is regular". Theorem 2 is not true with this stronger meaning, as the following example shows. Let $\mathscr{M}$ be the oriented matroid realized by

$$A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & f \end{array} \\ \begin{pmatrix} 1 & 2 & -\varepsilon & 0 & \varepsilon - 1 & -2 & 0 \\ \varepsilon & 0 & 1 & 2 & -1 & -2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array},$$

where ε is sufficiently small and positive. Let $A_1 = A \backslash f$ and let $A_2 = \{v_1, \ldots, v_6\}$ be a realization of $\mathscr{M} \backslash f$ in which the planes spanned by $\{v_1, v_2\}, \{v_3, v_4\}$ and $\{v_5, v_6\}$ meet in a line. Let B_1 and B_2 be Gale transforms of A_1 and A_2, respectively. Since A_2 cannot be extended to a realization of $\mathscr{M}$, $\Delta({}_{-f}\mathscr{M}^*, f)$ is a regular triangulation of $\Lambda(B_1)$ but not of $\Lambda(B_2)$, even though both represent the same matroid polytope $\Lambda(\mathscr{M}^*/f)$. On the other hand, $\Delta(\mathscr{M}, f)$ is a regular triangulation of any realization of $\Lambda(\mathscr{M}/f)$, because any realization of $\mathscr{M}/f$ is the contraction of one of $\mathscr{M}$.

In closing we relate our discussion to *zonotopal tilings*, which is the geometric model for oriented matroids featured prominently in [11]. Suppose that $\mathscr{M}/f$ can be realized as a vector configuration $\boldsymbol{D} = \{v_1, \ldots, v_n\} \subset \mathbb{R}^{d-1}$. The Bohne–Dress Theorem (see [11, §7.5]) says that the cell-complex dual to the arrangement $\mathscr{H}(\mathscr{M}, f)$ is a zonotopal tiling $\mathscr{Z}(\mathscr{M}, f)$ of the zonotope $Z(\boldsymbol{D}) = \sum_{i=1}^{n} [O, v_i]$. The exact relation between $\mathscr{Z}(\mathscr{M}, f)$ and $S(\mathscr{M}, f)$ is as follows. Let $\pi : \Lambda(\boldsymbol{D}) \to \Delta^{n-1}$ be the projection sending the pair of vertices x_i and y_i to the i-th vertex of the standard $(n-1)$-simplex Δ^{n-1}. In coordinates, this projection just forgets the first d rows in the matrix $\Lambda(\boldsymbol{D})$ given in (1). Let P be the centroid of Δ^{n-1}. Then, $\pi^{-1}(P)$ is a scaled copy of the zonotope $Z(\boldsymbol{D})$. The Cayley Trick [7] states that the zonotopal tiling $\mathscr{Z}(\mathscr{M}, f)$ is the intersection of the subdivision $S(\mathscr{M}, f)$ with that zonotope.

5 The topology of Alexander duals

We start by showing that the Alexander dual of a contractible simplicial complex need not be contractible, with the following reasoning suggested to us by Anders Björner. Let K be any acyclic but not contractible simplicial complex with at least 5 more vertices than its dimension. Small such complexes, with dimension 2 and 10 vertices, are described in [4, p. 284]. By the assumption on dimension, every three vertices form a triangle in $K^\vee$, and hence $K^\vee$ is simply connected. It is also acyclic by the Alexander Duality Theorem. By standard algebraic topology results, acyclic and simply connected simplicial complexes are contractible.

This fact contrasts the following result, pointed out to us by an anonymous referee. Part 1 is taken from [5]. The proof of the second part is due to the referee.

Theorem 9 (Dong [5]). *Let S be a simplicial complex of dimension d with n vertices. Then*:

(1) *If S is a d-sphere then $S^\vee$ has the homotopy type of the $(n-d-3)$-sphere.*

(2) *If S is a d-ball then $S^\vee$ is contractible.*

Proof. If $n \geqslant d+5$, the argument above gives that $S^\vee$ is simply connected. This, together with the fact that it has the homology groups of the $(n-d-3)$-sphere (respectively, of a contractible space) implies that it is homotopy equivalent to the $(n-d-3)$-sphere (respectively, it is contractible).

Let us now assume that $n \leqslant d+4$. In part 1, this implies that S is actually polytopal, by a classical result of Mani [8]. Corollary 22 in [5] implies that the Alexander dual of a simplicial d-polytope with n vertices is homotopy equivalent to the $(n-d-3)$-sphere.

In part 2, the case $n \leqslant d+3$ is proved by similar arguments: Coning the boundary of S to a new vertex we get a simplicial d-sphere with at most $d+4$ vertices, hence a polytopal one. This implies that S is a shellable ball, hence collapsible (see Lemma 17 in [5]). The Alexander dual of a collapsible space is contractible, by [5, Corollary 12].

We still have to deal with the case $n = d+4$ in part 2. We will prove that in this case $S^\vee$ is simply connected. Hence, the same arguments as in the case $n \geqslant d+5$ apply. The complex $S^\vee$ has a complete 1-skeleton, but not a complete 2-skeleton. The triangles missing are precisely the complements of the maximal simplices in S, and our task is to show that they all produce null-homotopic loops. To see this, let σ be a d-simplex in S, with complement $\{p,q,r\}$. If σ has a boundary facet $\sigma \backslash \{s\}$, then $\{p,q,s\}, \{p,r,s\}$, and $\{q,r,s\}$ are triangles in $S^\vee$, hence the loop $\{p,q,r\}$ is null-homotopic. If σ has no boundary facet, let σ' a d-simplex of $S^\vee$ adjacent to σ. Suppose the complement of σ' is $\{p,q,s\}$. Then the triangles $\{p,r,s\}$ and $\{q,r,s\}$ are in $S^\vee$ and prove that the loops $\{p,q,r\}$ and $\{p,q,s\}$ are homotopic. In other words, missing triangles of $S^\vee$ corresponding to adjacent d-simplices of S are homotopic. Any maximal simplex in the ball S can be connected to one incident to the boundary. This proves that every missing triangle is homotopic to a null-homotopic one. □

This result in particular implies Proposition 6 for lifting triangulations. But actually Dong's paper [5] contains the ingredients needed to generalize it to arbitrary subdivisions. Indeed, his Theorem 27 (together with his Lemma 25) states that the Alexander dual of every polyhedral decomposition of a d-sphere, considered as a simplicial complex as we did in Proposition 6, is homotopy equivalent of a $(n-d-3)$-sphere.

But the three properties of polyhedral complexes that he uses are also satisfied by subdivisions of oriented matroids. Namely: (1) they are regular cell complexes, (2) the intersection of any two closed cells is a closed cell (Dong calls this the *meet property*) and (3) they can be refined to triangulations without the addition of new vertices by the so-called pulling construction (for the pulling refinement of oriented matroid subdivisions see [3, Section 9.6] or [10, Remark 4.4]). Hence, we can generalize Proposition 6 as follows:

Theorem 10. *Let S be a subdivision of a rank d oriented matroid on n elements. If S is not a triangulation we consider it as a simplicial complex whose facets are the maximal faces of S. Then*:

(1) *If S (as a cell complex) is a $(d-1)$-sphere, then $S^\vee$ is homotopy equivalent to a $(n-d-2)$-sphere.*

(2) *If S (as a cell complex) is a $(d-1)$-ball, then $S^\vee$ is contractible.*

Proof. Let T be a triangulation obtained by the pulling refinement of S. As mentioned in [5], S (considered as a simplicial complex) collapses to T and this implies that $T^\vee$ collapses to $S^\vee$. Since T is homeomorphic to (the cell complex) S, the homotopy type of $T^\vee$ is given by Theorem 9. □

It is not known whether Cases (1) and (2) of Theorem 10 cover all subdivisions of oriented matroids. They cover, at least, all subdivisions of realizable ones and all lifting subdivisions of non-realizable ones.

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