

Singularities of the Gauss map and the binormal surface

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Abstract. We consider the bifurcation set of the Gauss map of an immersed n -dimensional manifold M^n in $n+k$ -dimensional Euclidean space. We focus specifically on the case of a surface immersed in 4-space, where the bifurcation set is the *binormal surface* contained in S^3 . Conditions for singularities on this surface are listed, stressing conditions that lead to easy computations of examples. The binormal surface is then related to the evolute for surfaces in 3-space. This relation allows us to generalize many classical results of surface theory. In particular, we show that when the normal curvature is zero, asymptotic curves on the original surface lift to geodesics on the binormal surface.

1 Introduction

In this paper, we consider the singularities of the Gauss map of an immersed n -dimensional manifold M^n in $n+k$ -dimensional Euclidean space. These singularities are directly related to singularities of height functions on M , and also related to the contact between M and hyperplanes. Singularities will occur at normal vectors called *binormal vectors*.

In particular, we will focus on the case of an immersed surface in \mathbb{R}^4 . In this situation, the set of binormal vectors will form a surface, called the *binormal surface*. If we consider this as a subset of S^3 , we can study the singularities of the surface and see how they connect to the local geometry of M .

In Section 5, we show a connection between the evolute of surfaces in \mathbb{R}^3 and the binormal surface for surfaces in \mathbb{R}^4 . Specifically, for a surface M in S^3 , the singularities of the binormal surface are in direct correlation with the singularities of the evolute of the stereographic projection of M . We find a connection between the plane evolute of a surface in S^3 and its binormal surface, and then use this to find the evolute in a limiting process of surfaces.

Establishing a connection between evolutes and binormal surfaces allows us to ask questions about binormals that we typically ask about evolutes. As a particular example, we study the configuration of asymptotic lines on surfaces in \mathbb{R}^4 . In particular, when the normal curvature is zero, then asymptotic lines on the surface lift to geodesics on the binormal surface.

2 Local geometry of surfaces in \mathbb{R}^4

For the preliminary material, we will follow the results of [9], [11], and [12]. Let M be a closed surface without boundary immersed in \mathbb{R}^4 , with $s : M \rightarrow \mathbb{R}^4$ the immersion. Let p be a point of M , and \mathbf{n} a unit normal vector at p . As a general rule of notation, bold face will be used for vectors and maps when we want to emphasize that they lie in \mathbb{R}^4 , while normal font will be used for points on manifolds and vectors not in \mathbb{R}^4 (this notation is partially adopted from [13]). The *second fundamental form* with respect to \mathbf{n} at p is the quadratic form $\mathbb{I}\mathbf{n} : T_pM \times T_pM \rightarrow \mathbb{R}$ defined by $\mathbb{I}\mathbf{n} = \mathbf{n} \cdot d^2s$. We can define the vector valued second fundamental form $\mathbb{I} : T_pM \times T_pM \rightarrow N_pM$ as the projection of d^2s into the normal plane. This quadratic form will map the unit tangent circle at p to an ellipse in the normal plane. This ellipse is called the *curvature ellipse*.

We characterize the points of a manifold by the relation between the curvature ellipse and the origin of the normal plane ([9]):

- p is *elliptic* if the origin is inside the curvature ellipse.
- p is *hyperbolic* if the origin lies outside the curvature ellipse.
- p is *parabolic* if the curvature ellipse is not a line segment and it passes through the origin.
- p is an *imaginary inflection* if the curvature ellipse is a radial line segment not containing the origin.
- p is a *real inflection* if the curvature ellipse is a radial line segment containing the origin, but does not have the origin as an endpoint.
- p is a *flat inflection* if the curvature ellipse is a radial line segment with the origin as one of its endpoints.

A normal vector is a *binormal vector* at p if the second fundamental form $\mathbb{I}\mathbf{n}$ is parabolic. A parabolic form will only have one root, and this direction in the tangent plane is called an *asymptotic direction*. The above classification by the curvature ellipse can be reworded into a classification by binormal vectors:

Proposition 2.1. *Let M be an manifold immersed in \mathbb{R}^4 , and p a point on M .*

- p is *elliptic* if there are no binormal vectors at p .
- p is *hyperbolic* if there are exactly two binormal vectors at p (and hence two asymptotic directions).
- p is *parabolic* if there is exactly one binormal vector at p (and hence one asymptotic direction).
- p is an *imaginary inflection* if $\mathbb{I}\mathbf{n} = 0$ for some \mathbf{n} (hence \mathbf{n} is a binormal vector, with all tangent directions asymptotic), and $\mathbb{I}\mathbf{m}$ is elliptic for some \mathbf{m} (and hence there are no other binormal vectors).

- p is a real inflection if $\Pi_n = 0$ for some \mathbf{n} (hence \mathbf{n} is a binormal vector, with all tangent directions asymptotic), and Π_m is hyperbolic for all $\mathbf{m} \neq \mathbf{n}$ (and hence there are no other binormal vectors).
- p is a flat inflection if $\Pi_n = 0$ for some normal vector \mathbf{n} , and Π_m is parabolic for all $\mathbf{m} \neq \mathbf{n}$.

Proof. The image of the unit tangent circle by the map Π_n can be characterized by projecting the curvature ellipse into the line spanned by \mathbf{n} . In particular, Π_n will have 2, 1, or 0 roots if the curvature ellipse hits the origin twice, once (meaning the projection ends at the origin), or zero times. The proposition is now implied by the configuration of the curvature ellipse and the origin for each type of point. \square

We now want to characterize these points for a generic immersion:

Theorem 2.2. *For a generic immersion $s : M \rightarrow \mathbb{R}^4$, we have the following results:*

- The set of hyperbolic points is a two-dimensional open subset of M .
- The set of elliptic points is also a two-dimensional open subset of M .
- The set of parabolic points is a one-dimensional open subset of M separating the elliptic points from the hyperbolic points.
- There are a finite number of imaginary inflections, all of which lie in the elliptic region.
- There are a finite number of real inflections, all of which are boundary points of the parabolic curve. Near a real inflection, the closure of the parabolic curve looks like a transversal crossing of two curves, with the real inflection at the point of intersection.
- There are no flat inflections.

For a proof, see [9] or [11].

3 Singularities of the Gauss map

Let M^n be a (closed) manifold immersed in \mathbb{R}^{n+k} . The Gauss map is defined as the map on the unit normal bundle UNM by $\Gamma : UNM \rightarrow S^{n+k-1}$,

$$\Gamma(p, \mathbf{n}) = \mathbf{n}.$$

Note that there are several other ways to consider the Gauss map, most notably considering it as a map to the Grassmannian $G(n, n+k)$ (for instance [8] and [17]). Our interpretation relates closely to the standard idea of a Gauss map in \mathbb{R}^3 ([1]), and is the best interpretation for visual geometry. Let $C \subset UNM$ be the critical set of Γ . We say that \mathbf{n} is a binormal vector of M at p if $(p, \mathbf{n}) \in C$. Further, we define the binormal set B to be the image $\Gamma(C) \subset S^{n+k-1}$.

We can characterize binormal vectors by the second order properties of the immersion:

Proposition 3.1. *A vector \mathbf{n} is a binormal vector at $p \in M$ iff $\Pi_{\mathbf{n}}$ is singular. In particular, if M is two-dimensional, \mathbf{n} is a binormal vector iff $\Pi_{\mathbf{n}}$ is a parabolic quadratic form.*

Proof. Let $\mathbf{s}(x_1, x_2, \dots, x_n)$ be a local parameterization of M , and locally parameterize UNM by

$$\begin{aligned} \Gamma(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{n-1}) \\ = (\mathbf{s}(x_1, x_2, \dots, x_n), \mathbf{n}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{n-1})). \end{aligned}$$

Note that for this to be a parameterization, the partials $\partial \mathbf{n} / \partial y_1, \dots, \partial \mathbf{n} / \partial y_{n-1}$ must be linearly independent.

The Gauss map is singular if there exists a nonzero $\alpha \in \mathbb{R}^{2n-1}$ such that $d\Gamma\alpha = 0$. Write $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_1 \in \mathbb{R}^n$, $\alpha_2 \in \mathbb{R}^{n-1}$.

Since $\Gamma \cdot d\mathbf{s} \equiv 0$, we can differentiate to get

$$d\Gamma \cdot d\mathbf{s} \equiv -\Gamma \cdot d^2\mathbf{s}.$$

If Γ is singular at a point (p, \mathbf{n}) , then the left hand side of this equation is zero when evaluated at α , and so the right hand side is zero when evaluated at α_1 . This implies $\mathbf{n} \cdot d^2\mathbf{s}\alpha_1 = 0$, i.e., $\Pi_{\mathbf{n}}$ is parabolic.

Now assume $\Gamma \cdot d^2\mathbf{s}\alpha_1 = 0$ for some $\alpha_1 \in \mathbb{R}^n$. Then $d\Gamma(\alpha_1, \cdot)$ can be considered as a system of $n - 1$ linear equations in $n - 1$ unknowns. This system is nonsingular, since $\{\mathbf{n}_{y_1}, \mathbf{n}_{y_2}, \dots, \mathbf{n}_{y_{n-1}}\}$ is linearly independent, and therefore this system has a solution $\alpha_2 \in \mathbb{R}^{n-1}$, and so $d\Gamma(\alpha_1, \alpha_2) = d\Gamma\alpha = 0$. □

We can also characterize binormal vectors using height functions and contact with hyperplanes:

Corollary 3.2. *A vector \mathbf{n} is a binormal vector at $p \in M$ iff the height function $f_{\mathbf{n}} : M \rightarrow \mathbb{R}$, $f_{\mathbf{n}}(x) = x \cdot \mathbf{n}$, has a degenerate (non-Morse) singularity at p .*

Proof. Again, let \mathbf{s} be a local parameterization of M . We have a singularity when

$$0 = d(f_{\mathbf{n}} \circ \mathbf{s}) = \mathbf{n} \cdot d\mathbf{s}$$

which means \mathbf{n} is a normal vector. We have a degenerate singularity when $d^2(f_{\mathbf{n}} \circ \mathbf{s}) = \mathbf{n} \cdot d^2\mathbf{s}$ is singular, which is the condition for \mathbf{n} to be a binormal vector. □

Corollary 3.3. *Given a normal vector \mathbf{n} at a point p , let A be the affine hyperplane perpendicular to \mathbf{n} and containing p . The vector \mathbf{n} is a binormal vector at p iff A has A_2 or worse contact with M at p .*

Proof. The contact map between A and M is just $f_{\mathbf{n}}$, and so the corollary follows directly from Corollary 3.2. □

As a map between two $(n+k-1)$ -dimensional manifolds, the critical set C , and hence its image $B = \Gamma(C)$, will be in general an $(n+k-2)$ -dimensional set, which will be locally a manifold except for a finite number of cone configurations (see [15]). We can be even more precise about the structure of the binormal set, because we can show that it is the bifurcation set of the family of height functions:

Theorem 3.4. *Let $F : M \times S^{n+k-1} \rightarrow \mathbb{R}$ be the $(n+k-1)$ -dimensional family of height functions: $F(x, \mathbf{n}) = f_{\mathbf{n}}(x)$. Then the bifurcation set of F is the binormal set $\Gamma(C)$.*

Proof. The bifurcation set is nothing more than the set of vectors \mathbf{n} where $F(p, \mathbf{n}) = f_{\mathbf{n}}(p)$ has a degenerate singularity for some $p \in M$, which is precisely the set of binormal vectors. \square

The benefit of this is that if F versally unfolds $f_{\mathbf{n}}$ (which it generically will), the structure of C depends only on the singularity type of $f_{\mathbf{n}}$. We will use this fact heavily in the next section.

Remark. We call these vectors *binormal vectors* to associate these concepts with the binormal vector in the Frenet frame of space curves. Indeed, if you considered the Gauss map $\Gamma : UNM \rightarrow S^2$ of a space curve, then this map has a fold exactly at the binormal vectors.

4 The binormal surface

We will now focus our attention on surfaces immersed in four-dimensional Euclidean space. In this case, we will call the binormal set the binormal surface, even though it technically is not a surface at its cone points. Some connections between the geometry of the surface and the critical points of height functions were established in [11] and [5]. In particular, in [5] the authors study the bifurcation set of the family of projections into 3-spaces, which by duality is the same as the binormal surface. In light of Corollary 3.2, we have some connections between the geometry of a surface and the binormal surface. The following theorem gives a different interpretation of the geometry, depending only on the derivatives of the immersion and mirroring classifications found in [14]. This is a convenient theorem for computing examples.

Theorem 4.1. *Let $B \subset S^3$ be the binormal surface of a surface M . Let $s : M \rightarrow \mathbb{R}^4$ be a local parameterization of M about a point p . Then:*

- *A vector \mathbf{n} is a binormal vector of M at p iff the second fundamental form $\Pi_{\mathbf{n}}$ is parabolic.*
- *B has a cuspidal edge (or worse) at \mathbf{n} iff $\Pi_{\mathbf{n}}$ is parabolic and $\mathbf{n} \cdot d^3 \mathbf{sa}^3 = 0$, where $\mathbf{a} \in T_p M$ is the unique root (up to a scalar multiple) of $\Pi_{\mathbf{n}}$.*
- *B has a swallowtail (or worse) at \mathbf{n} iff, in addition to the above condition, there is also a nonzero vector $\mathbf{b} \in T_p M$ such that $\mathbf{n} \cdot d^4 \mathbf{sa}^4 + 3\mathbf{n} \cdot d^3 \mathbf{sa}^2 \mathbf{b} = 0$.*

- B has an umbilic singularity at \mathbf{n} iff $\mathbf{II}_n = 0$. The singularity is elliptic or hyperbolic depending on whether the cubic form $\mathbf{n} \cdot d^3s$ is elliptic or hyperbolic.

For a generic immersion, these are all the possible singularities of the binormal surface.

Proof. Since B is the bifurcation set of the unfolding F of f_n , the singularity of B at a certain vector \mathbf{n} is determined by the singularity type of f_n . Since F is a three-dimensional unfolding, we will generically only get singularity types of codimension three or less, namely A_2, A_3, A_4 and D_4 . For a two-dimensional map, the probe structure (see [13] and [14]) for these singularities are the following:

A_2 : There exists a nonzero vector $\mathbf{a} \in T_pM$ such that $d^2f_n\mathbf{a} = 0$.

A_3 : There exists nonzero vectors $\mathbf{a}, \mathbf{b} \in T_pM$ such that $d^2f_n\mathbf{a} = 0$ and $d^3f_n\mathbf{a}^2 + d^2f_n\mathbf{b} = 0$.

A_4 : There exists nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in T_pM$ such that $d^2f_n\mathbf{a} = 0, d^3f_n\mathbf{a}^2 + d^2f_n\mathbf{b} = 0$, and $d^4f_n\mathbf{a}^3 + 3d^3f_n\mathbf{a}\mathbf{b} + d^2f_n\mathbf{c} = 0$.

D_4 : Probe structure is equivalent to $d^2f_n = 0$.

Note that the probe definition for the D_4 singularity does not generalize if M has dimension higher than two, but the definitions for the A_k singularities do generalize for higher dimensions.

To complete the theorem, replace $f_n(p)$ with $\mathbf{n} \cdot s(p)$, and then evaluate the equations in the A_3 and A_4 definitions at \mathbf{a} . □

We can now determine the following information about the structure of B :

Corollary 4.2. *The binormal surface satisfies the following properties:*

- There are no points on B associated to an elliptic point of M .
- For each hyperbolic point of M , there are four associated points on B (two antipodal pairs of binormal vectors).
- For each parabolic point, there are two associated points on B (one antipodal pair of binormal vectors).
- A point p is an inflection iff the associated points on B are umbilic singularities.
- The sheets of the binormal surface meet at parabolic points and inflection points.

Since we are dealing with surfaces in \mathbb{R}^4 , the binormal surface is a subset of S^3 , and so we can view it by stereographic projection.

Example 4.3. Perturbed (z, z^3) . Being a minimal surface, a complex function graph will have a trivial binormal surface, consisting only of circles corresponding to flat points. However, slight perturbations can break these flat points into patches of hyperbolic points. For example, we can start with the complex function graph (z, z^3) ,

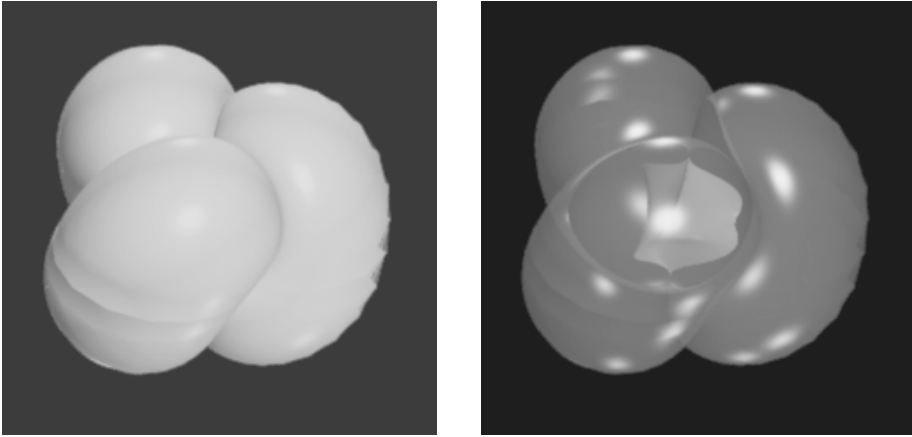


Figure 1. Binormal surface for the perturbed (z, z^3)

and then alter it by adding a quadratic term. Specifically, consider the map $s : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ defined by

$$s(u, v) = (u, v, \varepsilon(u^2 + v^2) + u^3 - 3uv^2, 3u^2v - v^3)$$

A point (u, v) is hyperbolic if $0 < u^2 + v^2 < \varepsilon/3$, parabolic if $u^2 + v^2 = \varepsilon/3$, and elliptic if $u^2 + v^2 > \varepsilon/3$. The point $(0, 0)$ is an imaginary inflection. The vector $\mathbf{n} = (0, 0, 0, 1)$ is the binormal at the origin, and the cubic $\mathbf{n} \cdot d^3s$ at the origin is $3(du)^2 dv - (dv)^3$. Since this cubic has three real roots, we expect three cuspidal edges of the singularity set to converge at $(0, 0, 0, 1)$. So each sheet of the binormal surface should be a topological disk with three cuspidal edges converging together to an umbilic singularity at the center of the disk, the boundary of the two sheets should be the same, and their umbilic singularities should touch. We can see this behavior in Figure 1. Note that in S^3 , all the components of the singularity set look the same. It is only because of stereographic projection that some of the pieces are larger than others, and some pieces are inside others. In S^3 , all four sheets of the binormal surface have exactly the same shape.

5 Evolutes and the binormal surface

Since we can consider the binormal surface as a bifurcation set associated to a given surface, there are reasons to search for connections between the ideas of a binormal surface and evolutes of surfaces in \mathbb{R}^3 . We will now explore this connection.

We begin with a surface M immersed in S^3 , and then look at this surface after stereographic projection $\pi : S^3 \rightarrow \mathbb{R}^3$. It is fairly easy to show the following:

Proposition 5.1. *If $M \subset S^3$, then the normal curvature of M is identically zero, and every point of M is either a hyperbolic point or an imaginary inflection.*

Proof. Let s be a local parameterization of M . Since $s(p)$ is the unit normal direction to S^3 at $s(p)$, it is also a normal vector to M at $s(p)$. Furthermore, since $s \cdot s \equiv 1$, the second derivative implies $s \cdot d^2s = -ds \cdot ds$, and so $\Pi_s \equiv -I$. Therefore the image of the unit tangent circle by Π_s is the value -1 , and so the curvature ellipse is a line segment. Since N is proportional to the oriented area of the curvature ellipse ([9]), $N = 0$. \square

In particular, every point of M contributes to the binormal surface. The structure of the binormal surface will be the same as the structure of the evolute of $\pi(M)$:

Theorem 5.2. *If $M \subset S^3$, then the contact at a point $p \in M$ between a hyperplane and M is the same as the contact between a sphere and $\pi(M)$.*

Proof. Let α be a hyperplane, and consider its contact with M at a point p . Any stereographic projection π is the restriction of an inversion map $inv : \mathbb{R}_p^4 \rightarrow \mathbb{R}_p^4$, where \mathbb{R}_p^4 means \mathbb{R}^4 minus the point p . In particular, inv is a diffeomorphism, so the contact between $inv(\alpha)$ and $inv(M)$ is equivalent to the contact between α and M . Now inversion will map α to a hypersphere. Since $inv(M)$ is contained entirely in \mathbb{R}^3 , the contact between $inv(\alpha)$ and $inv(M)$ is solely determined by the contact between $inv(M) = \pi(M)$ and $inv(\alpha) \cap \mathbb{R}^3$, which is a sphere. \square

We can be even more explicit than this, describing the evolute as the limit of a collection of binormal surfaces. To do this, we first look at the *plane evolute* P of a surface M , i.e., the envelope of the family of affine normal planes of M . In general, the plane evolute will be a three-dimensional submanifold of \mathbb{R}^4 . This manifold has an interesting structure when M is immersed in a sphere:

Theorem 5.3. *Let M be a surface immersed in $S^3(c)$, the 3-sphere of radius c centered at the origin. Then the plane evolute of M is equal to the cone generated by the binormal surface of M .*

Proof. Let s be a local parameterization of M . Since $M \subset S^3(c)$, s is a normal vector. Let n be the unit normal vector field which is perpendicular to s . Taking derivatives of the relations $s \cdot s = c^2$ and $s \cdot n = 0$, we can determine that $ds \cdot s = 0$ and $dn \cdot s = 0$.

Parameterize the plane bundle as $s + xs + yn$. We are looking for values where this map is singular. The condition reduces to finding values x_0 and y_0 such that $(1 + x_0)ds + y_0dn$ is singular. Set $m = (1 + x_0)s + y_0n$, normalized to length one. Note that the plane evolute is the cone generated by m . Since $dma = 0$ for some vector a , Π_m is parabolic and m is a binormal vector. Since $(1 + x)s + yn$ span the entire normal plane, all binormal vectors are in this set. \square

In particular, this theorem requires a surface whose position vector is always a normal vector, which of course requires the surface to be immersed in a sphere.

Now begin with a surface M immersed in \mathbb{R}^3 , and let $\pi_c : S(c) \rightarrow \mathbb{R}^3$ be stereo-

graphic projection from the sphere of radius c , centered at $(0, 0, 0, c)$. As $c \rightarrow \infty$, $\pi_c^{-1}(M)$ converges pointwise to M . However, the binormal surface of $\pi_c^{-1}(M)$ converges to two points (e_4 and $-e_4$) as $c \rightarrow \infty$. But with proper scaling, we can make the binormal surface converge to the evolute.

Theorem 5.4. *Let M be a surface immersed in \mathbb{R}^3 , E its evolute, $\pi_c : S^3(c) \rightarrow \mathbb{R}^3$ stereographic projection, B_c the binormal surface of $\pi_c^{-1}(M)$ and $f_c : S^3 \rightarrow S^3(c)$ the map defined by $f(x, y, z, w) = (cx, cy, cz, cw + c)$. Then away from the parabolic points of M :*

$$\lim_{c \rightarrow \infty} f_c(\pm B_c) = E$$

where the convergence is considered pointwise and the sign is dependent on the sign of the curvature at a point p on M .

The point of this result is that while the binormal surface may converge to a pair of points, the shape of the binormal surface converges to the shape of the evolute. We need to magnify B_c by a factor of c to keep the proper size.

We avoid parabolic points because there are no points on the evolute associated with parabolic points (or we consider them as points at infinity). This means that if p is parabolic, then the binormal vector associated with $\pi_c^{-1}(p)$ converges to a point on the equator of the sphere.

Theorem 5.4 follows primarily from the following lemma:

Lemma 5.5. *Let P_c be the plane evolute of $\pi_c^{-1}(M)$. As $c \rightarrow \infty$, P_c approaches the cylinder over E .*

Proof. Since $\lim \pi_c^{-1}(M) = M$, the limit of P_c will be the plane evolute of M . This is precisely the cylinder over E . \square

Proof of Theorem 5.4. Since $\pi_c^{-1}(M)$ is a surface immersed in a sphere, its plane evolute P_c will be the cone generated by the binormal surface. If we translate the plane evolute c units in the e_4 direction, then the intersection between P_c and $S^3(c)$ will simply be $f_c(B_c)$. In the limiting process, P_c converges to the cylinder over E (translation in the e_4 no longer an issue), and so $f_c(B_c)$ will converge to points on the evolute.

However, each binormal vector has an antipodal pair; one of the two will converge to a point in \mathbb{R}^3 , while the other will diverge to infinity. Whichever vector converges depends on the curvature of the surface at the associated point. \square

6 Asymptotic curves and binormal curves

Now that we have established a connection between evolutes and binormal surfaces, we can connect ideas related to evolutes to ideas about surfaces in \mathbb{R}^4 . For example, objects such as principal directions, lines of curvature, ridges, ribs, and subparabolic

lines may have corresponding objects on surfaces in \mathbb{R}^4 . We will use this to motivate our study of liftings of asymptotic curves to the binormal surface.

Associated with each binormal vector \mathbf{n} is a tangent direction \mathbf{a} which is the unique root of $\Pi_{\mathbf{n}}$. Since this direction is the one used in the probe structure definition of the singularities, it corresponds to a principal curvature direction for surfaces in \mathbb{R}^3 . We define asymptotic curves to be the integral curves of the asymptotic directions. These correspond to the lines of curvature for surfaces in \mathbb{R}^3 .

The generic structure of lines of curvature are well known: away from umbilics, there are two lines of curvatures through any point on the surface. These two curves will always be orthogonal. The only singularities in the configuration of the lines of curvature occur at umbilics, where the curves have a lemon, monstar, or star singularity ([2]).

The generic structure of asymptotic curves are similar, but there are extra complications. On the hyperbolic region, there will still be two curves through every point, but they will not necessarily intersect orthogonally. On surfaces in S^3 , imaginary inflections correspond to umbilics, and so we expect the configuration of asymptotic curves at imaginary inflections to be a lemon, monstar, or star again. The structure of the asymptotic curves have been classified: partially in [7] and completely in [6].

Theorem 6.1 ([6]). *On a generic surface, the asymptotic curves have the following structure:*

- *At a hyperbolic point, the two asymptotic curves intersect transversally.*
- *At a parabolic point, the two asymptotic curves come together to form a cusp.*
- *At an imaginary inflection, the asymptotic curves will form a lemon, monstar, or star configuration (see Figure 2).*
- *At a real inflection, the asymptotic curves will form one of the five structures: \mathcal{U}_1 , \mathcal{U}_2 , \mathcal{U}_3 , \mathcal{U}_4 , or \mathcal{U}_5 .*
- *At a parabolic A_3 point (i.e., a parabolic point which is also an A_3 singularity with respect to one of its normal vectors), the asymptotic curves will have one of the three structures: well-folded node, well-folded focus, or well-folded saddle.*

For surfaces in \mathbb{R}^3 , lines of curvature are lifted to their corresponding sheets of the evolute, the resulting curves called focal curves. These curves are regular when the evolute is regular. They have cusps when they cross a cuspidal edge, a kink when they go through a swallowtail singularity, and they have the three basic configurations (lemon, monstar, star) about an umbilic singularity.

Mirroring the idea, we can lift the asymptotic curves to their corresponding sheets on the binormal surface. We will call these liftings the *binormal curves*. All the results mentioned for focal curves will also be true for binormal curves. However, since different sheets of the binormal surface meet above the parabolic curve as well as above the inflection points, we expect a few more interesting features than we get with focal curves.

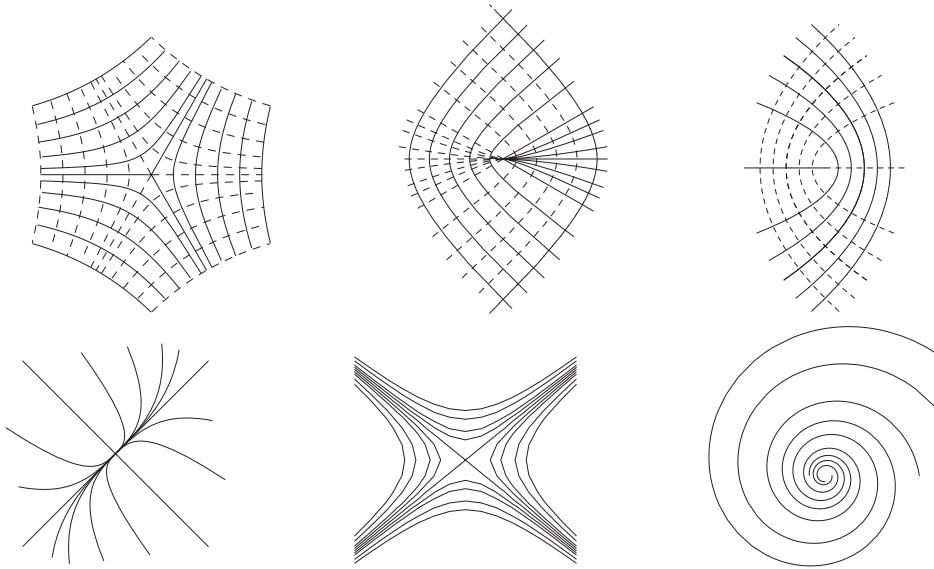


Figure 2. Top: star, monstar, and lemon; Bottom: node, saddle, and focus

Theorem 6.2. *For a generic surface, let α be a lifting of an asymptotic curve to the appropriate sheet of the binormal surface.*

- *The curve α has an ordinary cusp ($\alpha' = 0$ but $\alpha'' \neq 0$) when it passes through a cuspidal edge.*
- *α has an ordinary kink ($\alpha' = 0$ and $\alpha'' = 0$ but $\alpha''' \neq 0$) when it passes through a swallowtail point.*
- *About an umbilic singularity, the binormal curves will have the structure of one half of a lemon, monstar, or star.*
- *Above the parabolic curve and away from A_3 points, the binormal curves of one sheet connect smoothly to the binormal curves of the other sheet.*
- *Above a parabolic A_3 point, the binormal curves will be topologically equivalent to a node, focus, or saddle (see Figure 2) depending on the corresponding topological structure of the asymptotic curves. The singularity itself will sit on a cuspidal edge.*

Proof. We begin with the cusp and the kink. If α is the lifting of an asymptotic curve, then we can set $\alpha = \mathbf{n} \circ \beta$, where β is a curve in \mathbb{R}^2 and $s \circ \beta$ is a parametrization of the asymptotic curve. Then by definition, we have $\mathbf{n} \cdot d^2s\beta' \equiv 0$. Differentiating this equation once, we find that the condition for a cuspidal edge from Theorem 4.1 is equivalent to $d\mathbf{n}\beta' = \alpha' = 0$. Differentiating the equation twice, we find that the condition for a swallowtail from Theorem 4.1 is equivalent to $d\mathbf{n}\beta' = \alpha' = 0$ and $d^2\mathbf{n}\beta'' + d\mathbf{n}\beta' = \alpha'' = 0$. Thus the first two parts of the theorem are proved.

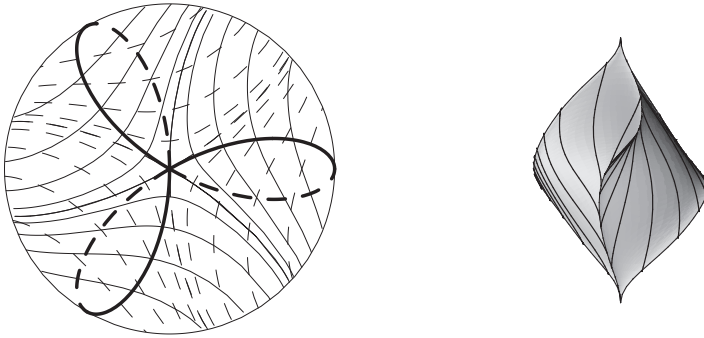


Figure 3. Asymptotic, A_3 , and binormal curves for perturbed z^3

The remaining three parts of the theorem are standard results of liftings of bi-valued curves sets to double coverings. The tangent vectors of the two asymptotic curves meeting at a parabolic point are parallel, so their liftings to a smooth surface will give a regular curve (since it is a bifurcation set, we can be assured that the binormal surface will be smooth above a parabolic point away from the A_3 points, even if two sheets of the surface meet there). The umbilic configurations will lift directly, and as the names imply, the well-folded node, saddle, and focus will lift to a node, saddle and focus. Finally, note that we have a node, saddle, or focus only when we are above a parabolic A_3 point, and so the singularity must sit on a cuspidal edge. \square

The perturbed z^3 provides a nice example of the asymptotic and binormal curves. Half of its binormal surface is shown in Figure 3 along with its binormal curves. The binormal curves have three focus configurations corresponding to the three A_3 parabolic points, though the cuspidal edge passing through the focus makes the configuration difficult to see. Note also that the binormal curves have cusps as they pass through the cuspidal edge.

7 Surfaces with zero normal curvatures

We end by looking at the asymptotic curves and binormal curves of a special class of surfaces: surfaces whose normal curvature is identically zero. (See [16] for more information on these surfaces. In particular, Lemma 7.1 and Propositions 7.2 and 7.3 can be found in [16], though approached from a different direction.) While this is a strong requirement, there are still some interesting examples that satisfy this condition: any tangent developable of a regular curve in \mathbb{R}^4 , any surface immersed in a 3-sphere or hyperplane, and any surface which is the cross product of two curves immersed in orthogonal planes. The condition of zero normal curvature allows us to reproduce a few extra features of lines of curvature and focal curves, including a result about geodesics of the binormal surface.

There are several ways to characterize zero normal curvature. The normal curvature is proportional to the area of the curvature ellipse, so zero normal curvature

means the curvature ellipse is a straight line. In terms of equations, this is equivalent to a normal vector \mathbf{n} with $\Pi_{\mathbf{n}} = k\mathbf{I}$ for some number k . Another expression of zero normal curvature is $d(d\mathbf{n} \cdot \mathbf{m}) = 0$, where \mathbf{n} and \mathbf{m} are a local orthonormal frame field of the normal bundle. Here, the outside d is the exterior derivative.

We begin by showing that some desirable features of the asymptotic directions are satisfied when we have $N = 0$. But first a general lemma:

Lemma 7.1. *Let p be a hyperbolic point of M , and let \mathbf{v} and \mathbf{w} be the two asymptotic directions at p . Then $\Pi_{\mathbf{n}}(\mathbf{v}, \mathbf{w}) = 0$ for any $\mathbf{n} \in N_pM$.*

Proof. Let \mathbf{n}_1 and \mathbf{n}_2 be the two binormal vectors corresponding to \mathbf{v} and \mathbf{w} . Any other normal vector can be written as a linear combination of these two. Since $\Pi_{\mathbf{n}_1}(\mathbf{v}, \cdot) = 0$ and $\Pi_{\mathbf{n}_2}(\mathbf{w}, \cdot) = 0$, we know that $\Pi_{\mathbf{n}_1}(\mathbf{v}, \mathbf{w}) = \Pi_{\mathbf{n}_2}(\mathbf{v}, \mathbf{w}) = 0$, and therefore any linear combination is also zero. \square

Proposition 7.2. *A hyperbolic point $p \in M$ has perpendicular asymptotic directions iff the normal curvature of M at p is zero.*

Proof. If $N = 0$, then there is some normal vector \mathbf{n} such that $\Pi_{\mathbf{n}} = k\mathbf{I}$ for some nonzero constant k . Since $\Pi_{\mathbf{n}}(\mathbf{v}, \mathbf{w}) = 0$, we have $\mathbf{I}(\mathbf{v}, \mathbf{w}) = 0$.

Now assume $\mathbf{I}(\mathbf{v}, \mathbf{w}) = 0$. The set of quadratic forms A on T_pM with $A(\mathbf{v}, \mathbf{w}) = 0$ is two-dimensional, and it is spanned by $\Pi_{\mathbf{n}_1}$ and $\Pi_{\mathbf{n}_2}$ from Lemma 7.1. Hence $\mathbf{I} = k\Pi_{\mathbf{n}}$ for some unit normal vector \mathbf{n} , and this implies $N = 0$. \square

So asymptotic curves are orthogonal iff $N = 0$. In particular, since surfaces immersed in a sphere have $N \equiv 0$, their asymptotic curves are always orthogonal. We already expected this, since they are also the images of lines of curvature through inverse stereographic projection.

Points with zero normal curvature have the additional benefit of having the same eigenvectors for all $\Pi_{\mathbf{n}}$:

Proposition 7.3. *A hyperbolic point $p \in M$ has zero normal curvature iff there are two linearly independent vectors \mathbf{v} and \mathbf{w} which are eigenvectors of $\Pi_{\mathbf{n}}$ for all normal vectors at p .*

Proof. If $N = 0$ at p , then there exists a normal vector \mathbf{n} with $\Pi_{\mathbf{n}} = k\mathbf{I}$. In particular, every vector is an eigenvector of $\Pi_{\mathbf{n}}$. If \mathbf{v} is an eigenvector associated with another normal vector \mathbf{m} , then it is an eigenvector for any linear combination of \mathbf{n} and \mathbf{m} , and hence an eigenvector of every second fundamental form.

Now assume there are two linearly independent vectors \mathbf{v} and \mathbf{w} which are eigenvectors of all second fundamental forms. Take any two normal vectors \mathbf{m} and $\tilde{\mathbf{m}}$, and say they have eigenvalues λ_1, λ_2 and κ_1, κ_2 respectively. Since they have the same eigenvectors, the eigenvalues associated with $a\mathbf{m} + b\tilde{\mathbf{m}}$ will be $a\lambda_1 + b\kappa_1$ and $a\lambda_2 + b\kappa_2$. We can always choose a and b to make these values equal, and so there will be a vector \mathbf{n} with $\Pi_{\mathbf{n}} = k\mathbf{I}$ (keep in mind that all eigenvalues and vectors are with respect to the first fundamental form). \square

In particular, for a surface with zero normal curvature, the asymptotic directions (and hence asymptotic curves) can be defined using any second fundamental form, not just the two parabolic forms.

Finally, a surprising theorem for surfaces in \mathbb{R}^3 is that away from the singularities of the evolute, the lines of curvature lift to geodesics on the corresponding sheet of the evolute. Unfortunately, this result does not generalize completely for surfaces in \mathbb{R}^4 , but it is true for surfaces with normal curvature zero.

Lemma 7.4. *Let α be a parametrization of a binormal curve, and let β be the curve is \mathbb{R}^2 such that $\alpha = \mathbf{n} \circ \beta$. If the binormal surface is regular at $\alpha(0)$, then the normal plane of the surface \mathbf{n} at the point $\alpha(t)$ is spanned by the vectors $\mathbf{n} \circ \beta(0)$ and $ds\beta'(0)$.*

Proof. Differentiating $\mathbf{n} \cdot \mathbf{n} \equiv 1$, it follows immediately that $\alpha(0)$ is a normal vector to \mathbf{n} at $\alpha(0)$. Next, begin with the symmetric relation $\mathbf{n} \cdot d^2\mathbf{s} + d\mathbf{n} \cdot d\mathbf{s} \equiv 0$ and evaluate it at $\beta'(t)$ to get $\mathbf{n} \cdot d^2s\beta'(t) + d\mathbf{n} \cdot ds\beta'(t) \equiv 0$. Since $\beta'(t)$ is an asymptotic direction, the first term is zero, and so we find $ds\beta'(t)$ is a normal vector. It is clear that $ds\beta'(t)$ and \mathbf{n} are linearly independent. □

Theorem 7.5. *Using the notation above, $\alpha(t)$ is a geodesic of \mathbf{n} at $t = 0$ iff $\alpha'(0) \neq 0$ and the normal curvature N is zero at $s \circ \beta(0)$.*

In particular, if $N \equiv 0$, then away from the singularities of \mathbf{n} , the corresponding asymptotic curves lift to geodesics on \mathbf{n} .

Proof. A curve is a geodesic at a point if its second derivative lies in the span of the first derivative and the normal plane, so we need to show that $(\alpha)''(0)$ is in the span of the vectors $(\alpha)'(0)$, $\alpha(0)$, and $ds\beta'(0)$.

The easiest way of doing this is to find a nonzero vector which is orthogonal to all four vectors. Let dsv be the other corresponding asymptotic vector at $s \circ \beta(0)$. We check for orthogonality:

- It is obvious that dsv is orthogonal to \mathbf{n} .
- Since $d\mathbf{n}\beta'(0) \cdot dsv = -\mathbf{n} \cdot d^2s\mathbf{v}\beta'(0) = 0$, dsv is orthogonal to $d\mathbf{n}\beta'(0)$.
- First, since $d\mathbf{n}\beta'(t) \cdot ds\beta'(t) = -\mathbf{n} \cdot d^2s\beta'(t)^2 = 0$, we know that $d\mathbf{n}\beta'(t) = \alpha'(t)$ is a normal vector at $s \circ \beta(t)$ for all t . Next, differentiate the relation $(\alpha)'(t) \cdot ds \circ \beta(t) \equiv 0$ with respect to t and evaluate at v to get $\alpha''(t) \cdot dsv + \alpha'(t) \cdot d^2s\mathbf{v}\beta'(t) = 0$. From Proposition 7.1, the second term is zero, so the first term is zero, and $(\alpha)''(t)$ is orthogonal to dsv . Since $\alpha''(0) \cdot dsv = (d^2\mathbf{n}(\beta'(0))^2 + d\mathbf{n}\beta''(0)) \cdot dsv$.
- Finally, $dsv \cdot ds\beta'(0) = I(v, \beta'(0))$, which is zero iff $N = 0$ (Proposition 7.2). □

8 Conclusion and further directions of research

As mentioned at the beginning of Section 5, establishing connections between evolutes and binormal surfaces leads to a number of interesting questions. Objects studied on

surfaces in \mathbb{R}^3 , such as ribs, ridges, principal curvatures, parabolic curves and sub-parabolic lines have some corresponding objects on surfaces in \mathbb{R}^4 . For some surfaces (such as surfaces in S^3), we expect most of the results from \mathbb{R}^3 to extend without difficulty. But when we allow ourselves the full freedom of \mathbb{R}^4 , the results should be even richer.

We can also extend the results of dynamic surfaces. For instance, the birth and death of umbilics and parabolic curves have been studied ([10] and [3], [4], respectively) for surfaces in \mathbb{R}^3 . Again, the extra dimension will give dynamic surfaces in \mathbb{R}^4 a more complicated and a more interesting structure. The transitions in the parabolic curve, A_3 curve, inflections and asymptotic curves have been studied in [6], but there is still much to study in this direction.

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