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ON THE POWERFULL PART OF n^2+1

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Abstract. We show that n^2+1 is powerfull for $O(x^{2/5+\epsilon})$ integers $n \leq x$ at most, thus answering a question of P. Ribenboim.

The distribution of powerfull integers, i.e. integers such that every prime factor occurs at least twice, is quiet obscure. In [4], P. Ribenboim posed the following problem: Show that for almost all m, m^4-1 is not powerfull. In his review, D. R. Heath-Brown [2] pointed out that this and the more general statement, that for every polynomial f, not powerfull as a polynomial, $f(m)$ is not powerfull for almost all m, can be obtained using a simple sieve. In fact, if n is powerfull and p prime, $n \bmod p^2$ is restricted to p^2-p+1 residue classes. By a standard application of the arithmetic large sieve one gets that the number N of $m \leq x$ such that $f(m)$ is powerfull is $N \ll \frac{x}{\log x}$. In this note we will use a diferent approach to this problem to prove the following theorem. For an integer n we write $P(n)$ for the powerfull part of n, i.e. the product of all p^k with $k \geq 2$, where $p^k|n$, but $p^{k+1} \not| n$, $\omega(n)$ for the number of distinct prime divisors of n, and $d^+(n)$ for the number of squarefree divisors of n.

Theorem 1. *Let A and x be real numbers. Then there are at most $cx^{2/5}A^{4/5}\log^C x$ integers $n \leq x$, such that $P(n^2+1) > n^2A^{-1}$ where $C = 18730$.*

Choosing $A = 2$ resp. $A = x^{2/3-\epsilon}$ we obtain the following statements.

Corollary 2. *For almost all n we have $P(n^2+1) < n^{4/3+\epsilon}$.*

Corollary 3. *There are $\ll x^{2/5}\log^C x$ integers $m \leq x$ such that m^2+1 is powerfull or twice a powerfull integer.*

Note that $\limsup \frac{P(n^2+1)}{n} = \infty$, thus the exponent $4/3$ is not too bad. It seems that the gap stems from the fact that the equation $x^2+1 = D \cdot z^3$ considered in Lemma 5 may very well have no integral solutions at all for many values of D.

To prove our theorem, we need some Lemmata. First we have to count solutions of diophantine equations.

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Lemma 4. *For any D, the equation $x^2 - Dy^2 = -1$ has ≤ 4 solutions with x, y integers and $X \leq x \leq 2X$, X arbitrary real.*

Proof. We may assume that D is not a perfect square, since for $D = 1$ there are only the solutions $x = 0, y = \pm 1$, and for $D > 1$, $x + \sqrt{D}y$ would be a rational integral divisor of -1. The solutions of the equation correspond to units in $\mathbb{Q}(\sqrt{D})$. If (x_1, y_1) is a minimal solution, all solutions are obtained by the recursion $x_{n+1} = x_n x_1 + D y_n y_1, y_{n+1} = x_1 y_n + y_1 x_n$. We may assume that x_1, y_1 are positive, thus $x_{n+1} > x_n x_1$. Further we trivially have $x_1 \geq 2$, thus in every interval of the form $[X, 2X]$, there is at most one solution with both variables positive. Taking signs into account, the total number of solutions with $x_n \leq X$ is therefore ≤ 4.

Lemma 5. *For any D, the equation $x^2 + 1 = Dz^3$ has $c \cdot d^+(D)^{c_0}$ solutions at most, where $c_0 = \frac{2 \log 17 + 4 \log 3}{\log 2} \leq 14.6$.*

Proof. This is a special case of theorem 1 in [1], proven by J. H. Evertse and J. H. Silverman. In their notation we have $n = 3, d = 2, m = 1, L = \mathbb{Q}(i), M = 2$ and $K_3(L) = 0$. We consider the equation $\frac{x^2+1}{D} = y^3$, which is integral at all but $\omega(D)$ places, thus $s = \omega(D) + 1$. Applying their theorem we obtain for the number N of solutions the bound $N \leq 17^{14+2\omega(D)} 3^{4+4\omega(D)} \ll (17^2 3^4)^{\omega(D)}$. Since $d^+(D) = 2^{\omega(D)}$, we get $N \ll d^+(n)^{c_0}$, where $c_0 = \frac{2 \log 17 + 4 \log 3}{\log 2} \leq 14.6$.

Note that the actual value of c_0 is of lesser importance, since only the exponent of the logarithm is concerned. In fact, we have $C = 2^{c_0}$. Note further that we can prove theorem 1 with a bound of $x^{2/3} A^{2/3}$ without appealing to the very deep theorem of Evertse and Silverman.

Lemma 6. *We have for any positive real number c the bound $\sum_{n \leq x} d(n)^c \ll_c x \log^{2^c - 1} x$.*

This was proven by C. Mardjanichvili [3].

Now we can prove theorem 1. Every integer $k \geq 2$ can be written as a nonnegative integral linear combination of 2 and 3, thus every powerfull number n can be written as $n = y^2 z^3$ with y, z integral. Thus every integer n can be written as $n = ay^2 z^3$ with y, z integral and $a = \frac{n}{P(n)}$. Thus to prove theorem 1, it suffices to show that the equation

$$n^2 + 1 = ay^2 z^3 \tag{1}$$

has $\ll x^{2/5} A^{2/5} \log^C x$ integral solutions with $n \leq x$ and $a \leq A$. Now we count the solutions within the range $Y \leq y < 2Y$, $B \leq a < 2B$ and $Z \leq z < 2Z$.

Fix a and z, and set $D = az^3$. Now n is restricted to an interval of the form $[x, 8x]$, thus by lemma 4 there are $\ll 1$ solutions of the equation $n^2 - Dy^2 = -1$ with these restrictions. Thus the total number of solutions is $\ll BZ$.

Now we fix a and y, and set $D = ay^2$. Then by lemma 5 the equation $n^2 + 1 = Dz^3$ has $\ll d^+(D)^{c_0}$ solutions, where c_0 is defined as above. We set $c_1 = 2^{c_0} =$

23709. Thus the total number of solutions in this range is therefore bounded by

$$\ll \sum_{B\le a<2B} \sum_{Y\le y\le 2Y} d^+(ay^2)^{c_0} \le \sum_{B\le a\le 2B} d(a)^{c_0} \sum_{Y\le y<2Y} d(y)^{c_0}.$$

Using Lemma 6 and replacing the occuring log-factors by $\log x$, these sums are $\ll BY \log^{2c_1-2} x$. With these two estimates we obtain for the total number N of solutions the estimate

$$\begin{aligned} N &\ll \log^3 x \max_{\substack{Y,Z>1\\ B<A\\ AY^2Z^3<x}} \min\left(BY \log^{2c_1-2} x, BZ\right) \\ &\ll \log^3 x \max_{Y>1} \min\left(AY \log^{2c_1-2} x, A\left(\frac{x^2}{AY^2}\right)^{1/3}\right) \\ &\ll A^{4/5} x^{2/5} \log^{\frac{4}{5}(c_1-1)+3} x \end{aligned}$$

which gives the bound of theorem 1, since $\frac{4}{5}(c_1-1)+3 = 18729.4$.

References

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