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GENERALIZED VERMA MODULE HOMOMORPHISMS IN SINGULAR CHARACTER

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ABSTRACT. In this paper we study invariant differential operators on manifolds with a given parabolic structure. The model for the parabolic geometry is the quotient of the orthogonal group by a maximal parabolic subgroup corresponding to crossing of the k-th simple root of the Dynkin diagram. In particular, invariant differential operators discussed in the paper correspond (in a flat model) to the Dirac operator in several variables.

1. Introduction

1.1. **Definitions and notation.** Let G be a real or complex semisimple Lie group, P a parabolic subgroup of G, $\mathfrak{g}$ and $\mathfrak{p}$ their Lie algebras. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set of positive roots Φ^+ for $(\mathfrak{g}, \mathfrak{h})$. Let $\Phi = \Phi^+ \cup -\Phi^+$ be the set of all roots. Because $\mathfrak{h} \subset \mathfrak{p}$, $\mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{p}$ as well and there is a set of roots $\Phi_{\mathfrak{p}} \subset \Phi$ so that the corresponding root spaces are contained in $\mathfrak{p}$. Let W be the Weyl groups associated to the triple $(\mathfrak{g}, \mathfrak{h}, \Phi^+)$. The choice of the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ determines a gradation $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_k$ with $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$. Let $\mathfrak{g}_- := \oplus_{i<0} \mathfrak{g}_i$ and $\mathfrak{p}^+ := \oplus_{i>0} (\mathfrak{g}_i)$.

We say that an element of $\mu \in \mathfrak{h}^*$ is P-dominant, resp. $\mathfrak{p}$-dominant, if it is a highest weight of an irreducible finite dimensional representation of P, resp. $\mathfrak{p}$. Such a representation is unique up to an isomorphism and will be denoted by $\mathbb{V}_\mu$. Similarly, we define G- and $\mathfrak{g}$-dominance. A nonzero highest weight vector in $\mathbb{V}_\lambda$ will be denoted by v_λ. Note, that μ is $\mathfrak{p}$-dominant iff for each $\alpha \in \Delta_{\mathfrak{p}}$ $\mu(H_\alpha) \in \mathbb{Z}_0^+$, where H_α is the coroot corresponding to α. Each P-dominant weight μ is also $\mathfrak{p}$ dominant and the P-module $\mathbb{V}_\mu$ is a $\mathfrak{p}$-module as well.

Let $P_{\mathfrak{p}}^{++} \subset \mathfrak{h}^*$ be the set of all $\mathfrak{p}$-dominant elements and P_P^{++} the set of P-dominant elements. The homogenous space G/P is a principal fiber bundle and for each $\mu \in P_P^{++}$, there is an associated vector bundle $\mathcal{V}_\mu := G \times_P \mathbb{V}_\mu$. The group G has a natural left action on $\mathcal{V}_\mu$ and on its sections $\Gamma(\mathcal{V}_\mu)$ (we consider smooth sections in case of real lie groups G, P and holomorphic sections in the complex case).

The paper is in final form and no version of it will be submitted elsewhere.

An invariant differential operator of order k is a map

$$D : \Gamma(\mathcal{V}_\lambda) \to \Gamma(\mathcal{V}_\mu)$$

that commutes with the natural G-action on sections and $D(s)(x)$ depends only on derivations of s in x up to order k.

1.2. **Invariant operators and generalized Verma modules.** Let $\mathcal{U}(\mathfrak{g})$ resp. $\mathcal{U}(\mathfrak{p})$ be the universal enveloping algebra of $\mathfrak{g}$ resp. $\mathfrak{p}$. For each P-dominant weight μ, $\mathbb{V}_\mu$ is also a representation of $\mathcal{U}(\mathfrak{p})$ and we define the generalized Verma module

$$M_{\mathfrak{p}}(\mu) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}_\mu$$

where the left $\mathfrak{g}$-action is simply the left multiplication in $\mathcal{U}(\mathfrak{g})$. As a $\mathfrak{g}_-$-module (and also $\mathfrak{g}_0$-module), $M_{\mathfrak{p}}(\mu) \simeq \mathcal{U}(\mathfrak{g}_-) \otimes \mathbb{V}_\mu$.

To each linear invariant differential operator of order k we can assign a map $\phi : \mathcal{J}^k_{eP}(\mathcal{V}_\lambda) \to \mathbb{V}_\mu, \quad j^k_{eP} s \mapsto D(s)(eP)$, where $\mathcal{J}^k_{eP}$ is the space of k-jets in the point $eP \in G/P$ and the fiber in $\mathcal{V}_\mu$ over eP is identified with $\mathbb{V}_\mu$ via $[e,v]_P \mapsto v$. The space $\mathcal{J}^k_{eP}$ can be given a structure of P-module in a natural way, so that ϕ is a P-homomorphism. Moreover, there is a $1-1$ correspondence between such operators of order $\leq k$ and $\mathrm{Hom}_P(\mathcal{J}^k_{eP}(\mathcal{V}_\lambda), \mathbb{V}_\mu)$. Each section $s \in \Gamma(\mathcal{V}_\lambda)$ can be represented by a P-equivariant function $f \in C^\infty(G, \mathbb{V}_\lambda)^P$. The space $\mathcal{J}^k_{eP}$ is dual to $\mathcal{U}_k(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}^*_\lambda$, ($\mathcal{U}_k(\mathfrak{g}_-)$ is the k-th filtration of $\mathcal{U}(\mathfrak{g})$) and the duality is given by

$$\langle Y_1 \dots Y_l \otimes A, j^k_e f\rangle = A\big((L_{Y_1} \dots L_{Y_l} f)(e)\big) \tag{1}$$

for $l \leq k$, $A \in \mathbb{V}^*_\lambda$, $Y_j \in \mathfrak{g}$, L_{Y_j} the derivation with respect to the left invariant vector fields given by Y_j.

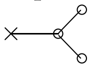
It follows that there is a natural duality between invariant linear differential operators $D : \Gamma(\mathcal{V}_\lambda) \to \Gamma(\mathcal{V}_\mu)$ of any finite order and $(\mathfrak{g}, P)$-homomorphisms of generalized Verma modules $M_{\mathfrak{p}}(\mathbb{V}^*_\mu) \to M_{\mathfrak{p}}(\mathbb{V}^*_\lambda)$ (for details, see [1], [2]).

1.3. **Homomorphisms of generalized Verma modules.** Let us define the affine action of the Weyl groups by $w \cdot \mu := w(\mu+\delta)-\delta$, where $\delta := 1/2 \sum_{\beta \in \Phi^+} \beta$. A necessary condition for existence of a nonzero $\mathfrak{g}$-homomorphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is that $\mu = w \cdot \lambda$ for some $w \in W$.

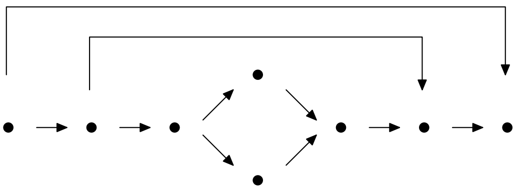
The Weyl orbit of $\lambda + \delta$ (and $\mu + \delta$) contains a unique dominant weight. The Weyl orbit is called regular if this weight is in the interior of the dominant Weyl chamber and singular otherwise. Writing this dominant weight as $\tilde{\lambda} + \delta$, regularity is equivalent to the fact that $\tilde{\lambda}$ is $\mathfrak{g}$-dominant. In the singular case, $\tilde{\lambda}$ need not even have to be $\mathfrak{p}$-dominant.

There is a subset $W^{\mathfrak{p}}$ of W of elements that take $\mathfrak{g}$-dominant elements to $\mathfrak{p}$-dominant elements. The Hasse diagram for $(\mathfrak{g}, \mathfrak{p})$ is the set $W^{\mathfrak{p}}$ of vertices so that there is an arrow $w \to w'$ if and only if $w = s_\beta w'$ (root reflection) for some $\beta \in \Phi^+$ and the length $l(w') = l(w) + 1$.

2. Dirac operator in the parabolic setting

2.1. **Example in low dimension.** Let us consider a homogeneous space G/P of type , i.e. $\mathfrak{g} = so(8, \mathbb{C})$, and $\mathfrak{p}$ consists of the Cartan subalgebra and those root spaces, the determining roots of which could be written as a linear combination of simple roots having nonnegative coefficient in the first simple root α_1. The pair $\mathfrak{g}, \mathfrak{p}$ determines a gradation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

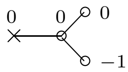
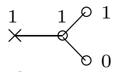
If $\tilde{\lambda} \in P_{\mathfrak{g}}^{++}$, then the structure of generalized Verma module homomorphisms on the affine orbit of $\tilde{\lambda}$ looks as follows:



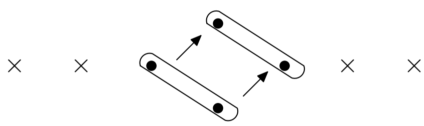
The generalized Verma module $M_{\mathfrak{p}}(\tilde{\lambda})$ is on the top, the others are of the form $M_{\mathfrak{p}}(\mu)$, where μ is only $\mathfrak{p}$-dominant. The form of the diagram does not depend on a choice of a $\mathfrak{g}$-dominant weight $\tilde{\lambda}$.

The same graph with reversed arrows describes the structure of invariant differential operators between sections of homogeneous vector bundles, associated to dual representations $\mathbb{V}_{\mu}^{*}$. The dual graph will be called the *regular BGG graph for* $(\mathfrak{g}, \mathfrak{p}, \lambda)$.

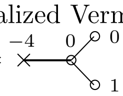
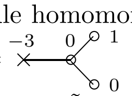
The long arrows are not in the Hasse graph of $(\mathfrak{g}, \mathfrak{p})$ and the corresponding operators are called nonstandard.

Further, let as consider a weight $\tilde{\lambda} :=$ . We see that $\tilde{\lambda} + \delta =$  is on the wall of fundamental Weyl chamber, so it has a singular affine orbit.

The structure of generalized Verma module homomorphisms on the affine Weyl orbit of $\tilde{\lambda}$ looks like



The crosses $\times$ correspond to weights that are not $\mathfrak{p}$-dominant and so there are no associated generalized Verma modules for them. The nodes $\bullet$ are $\mathfrak{p}$-dominant and the encircled weights coincide in this case.

It means there is only one possible generalized Verma module homomorphism in this case, $D : M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$, where $\mu =$  and $\lambda =$ 

We see the general fact that the affine orbit of a singular weight $\tilde{\lambda}$ (i.e. $\tilde{\lambda} + \delta$ is on the wall of the fundamental Weyl chamber) is smaller than the regular one: some weights are "glued together" and some are not $\mathfrak{p}$-dominant.

We will show the existence of D from the example. Let as represent the elements of $so(8,\mathbb{C})$ as matrices antisymmetric with respect to the antidiagonal and the cartan subalgebra is the algebra of diagonal matrices, see e.g. [3].

In the standard basis ϵ_i of $\mathfrak{h}^*$, $\mu = \frac{1}{2}[-7|1,1,-1]$ and $\lambda = \frac{1}{2}[-5|1,1,1]$.

Note that $\delta = [3|2,1,0]$, $\mu+\delta = \frac{1}{2}[-1|5,3,-1]$, $\lambda+\delta = \frac{1}{2}[1|5,3,1]$, so we see that $\lambda+\delta$ and $\mu+\delta$ are connected by a root reflection.

The homomorphism $D : M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is completely determined by the image of the highest weight vector in $M_{\mathfrak{p}}(\mu)$. This is a vector in $M_{\mathfrak{p}}(\lambda)$ of weight μ, annihilated by all positive root spaces in $\mathfrak{g}$.

Let $y_{i,j}$ resp. $Y_{i,j}$ be a matrix $E_{i,j} - E_{9-j,9-i}$ so that $y_{i,j} \in \mathfrak{g}_-$ and $Y_{i,j} \in \mathfrak{g}_0$ ($E_{i,j}$ is a matrix having 1 in i-th row and j-th column and 0 on other places). These are exactly generators of negative root spaces in $\mathfrak{g}$. Similarly, generators of positive root spaces will be denoted by $x_{i,j}$ and $X_{i,j}$ and the generators of the Cartan subalgebra by $h_i = E_{i,i} - E_{9-i,9-i}$:

$$
\left(\begin{array}{c|cccccc|c}
h_1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & 0 \\
\hline
y_{21} & h_2 & X_{23} & X_{24} & X_{25} & X_{26} & 0 & -x_{17} \\
y_{31} & Y_{32} & h_3 & X_{34} & X_{35} & 0 & -X_{26} & -x_{16} \\
y_{41} & Y_{42} & Y_{43} & h_4 & 0 & -X_{35} & -X_{25} & -x_{15} \\
y_{51} & Y_{52} & Y_{53} & 0 & -h_4 & -X_{34} & -X_{24} & -x_{14} \\
y_{61} & Y_{62} & 0 & -Y_{53} & -Y_{43} & -h_3 & -X_{23} & -x_{13} \\
y_{71} & 0 & -Y_{62} & -Y_{52} & -Y_{42} & -Y_{32} & -h_2 & -x_{12} \\
\hline
0 & -y_{71} & -y_{61} & -y_{51} & -y_{41} & -y_{31} & -y_{21} & -h_1
\end{array}\right) \tag{2}
$$

The module $\mathbb{V}_\lambda$ is a highest weight module, hence from the PBW theorem it follows that the vectors

$$y_{i_1,j_1} \dots y_{i_n,j_n} \otimes Y_{k_1,l_1} \dots Y_{k_m,l_m} v_\lambda \tag{3}$$

generate $M_{\mathfrak{p}}(\lambda)$.

Lemma 1. *There is exactly one vector (up to a multiple) in $M_{\mathfrak{p}}(\lambda)$ of weight μ that is extremal, i.e. annihilated by all positive root spaces in $\mathfrak{g}$. The vector has a form*

$$y_{5,1} \otimes v_\lambda - y_{31} \otimes Y_{53} v_\lambda - y_{21} \otimes Y_{52} v_\lambda$$

(under the identification $M_{\mathfrak{p}}(\lambda) \simeq \mathcal{U}(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$).

Proof. The vector $y_{i_1,j_1} \dots y_{i_n,j_n} \otimes Y_{k_1,l_1} \dots Y_{k_m,l_m} v_\lambda$ is a weight vector with weight $\lambda - \sum_k weight\,(y_{i_k,j_k}) - \sum_{k'} weight\,(Y_{i_{k'},j_{k'}})$, where weight($y$) is its weight in the adjoint representation (i.e. a root). The difference $\lambda - \mu$ is equal to $[-1|0,0,-1]$ in our case, so the μ-weight space in $M_{\mathfrak{p}}(\lambda)$ is generated by vectors of type (3), where the sum $\sum_k weight\,(y_{i_k,j_k}) + \sum_{k'} weight\,(Y_{i_{k'},j_{k'}})$ is $[-1|0,0,-1]$. There are only 4 possibilities how to obtain $[-1|0,0,-1]$ as a sum of negative roots in $\mathfrak{g}$:

- $[-1|0,0,-1]$ itself – corresponds to y_{51}, so the weight vector is $y_{51} \otimes v_\lambda$
- $[0|-1,0,-1] + [-1|1,0,0]$ – weight vector $y_{21} \otimes Y_{52} v_\lambda$
- $[0|0,-1,-1] + [-1|0,1,0]$ – weight vector $y_{31} \otimes Y_{53} v_\lambda$
- $[0|0,-1,-1] + [0|-1,1,0] + [-1|1,0,0,]$ – weight vector $y_{21} \otimes Y_{53} Y_{32} v_\lambda$

The last vector is zero because $\lambda = \frac{1}{2}[-5|1,1,1]$, Y_{32} is the negative root space of the root $\beta = \epsilon_2 - \epsilon_3$ and the copy of $sl(2,\mathbb{C})$ in $\mathfrak{g}$ generated by $h_2 - h_3, X_{23}, Y_{32}$ acts trivial on v_λ, because $\beta(h_2 - h_3) = 1 - 1 = 0$ (if we denote this copy of $sl(2,\mathbb{C})$ by $\mathfrak{g}'$, then $\mathbb{V}_\lambda$ contains an irreducible $\mathfrak{g}'$-submodule generated by v_λ that has highest weight $\lambda(\beta) = 0$, so this submodule is trivial). Therefore, $Y_{32}v_\lambda = 0$.

We have identified a 3-dimensional μ-weight space in $M_\mathfrak{p}(\lambda)$ and are looking for a vector in this space that is extremal, i.e. annihilated by all positive root spaces in $\mathfrak{g}$. The action of the positive root spaces can be computed using just the commutation relation in $\mathcal{U}(\mathfrak{g})$ and the fact that we know the action of $\mathfrak{p}$ on v_λ. For example,

$$\begin{aligned} x_{12}(y_{51} \otimes v_\lambda) &= y_{51}x_{12} \otimes v_\lambda + [x_{12}, y_{51}] \otimes v_\lambda \\ &= y_{51} \otimes x_{12}v_\lambda + [x_{12}, y_{51}] \otimes v_\lambda = 0 + (-Y_{52}) \otimes v_\lambda \\ &= 1 \otimes (-Y_{52}v_\lambda) \end{aligned}$$

$$\begin{aligned} x_{12}(y_{21} \otimes Y_{52}v_\lambda) &= y_{21}x_{12} \otimes Y_{52}v_\lambda + [x_{12}, y_{21}] \otimes Y_{52}v_\lambda \\ &= y_{21} \otimes x_{12}Y_{52}v_\lambda + (h_1 - h_2) \otimes Y_{52}v_\lambda = y_{21} \otimes Y_{52}x_{12}v_\lambda \\ &\quad + y_{21} \otimes [x_{12}, Y_{52}]v_\lambda + 1 \otimes (h_1 - h_2)Y_{52}v_\lambda = 0 + 0 \\ &\quad + 1 \otimes Y_{52}(h_1 - h_2)v_\lambda + 1 \otimes [h_1 - h_2, Y_{52}] \\ &= 1 \otimes (-\frac{5}{2} - \frac{1}{2})v_\lambda + 1 \otimes Y_{52}v_\lambda = -2 \otimes Y_{52}v_\lambda \end{aligned}$$

$$\begin{aligned} x_{12}(y_{31} \otimes Y_{53}v_\lambda) &= y_{31} \otimes x_{12}Y_{53}v_\lambda + [x_{12}, y_{31}] \otimes Y_{53}v_\lambda \\ &= y_{31} \otimes Y_{53}x_{12}v_\lambda + y_{31} \otimes [x_{12}, Y_{53}]v_\lambda + (-Y_{32}) \otimes Y_{53}v_\lambda \\ &= 0 + 0 - 1 \otimes Y_{32}Y_{53}v_\lambda = -Y_{53}Y_{32}v_\lambda - [Y_{32}, Y_{53}]v_\lambda \\ &= 0 - 1 \otimes (-Y_{52})v_\lambda = 1 \otimes Y_{52}v_\lambda \end{aligned}$$

where $\otimes$ means product over $\mathcal{U}(\mathfrak{p})$.

Similarly, we compute the action of the other positive root spaces x_{ij} and X_{ij} on each of the 3 nonzero vectors of weight μ. The condition that their combination is annihilated by all of them yields the unique (up to multiple) vector from the lemma. In fact, it suffices that it is annihilated by $x_{12}, X_{23}, X_{34}, X_{35}, X_{26}$ and x_{17} because the others can be obtained by commuting those. □

This proves that there exists a unique nonzero $\mathfrak{g}$-homomorphism of generalized Verma modules $M_\mathfrak{p}(\mu) \to M_\mathfrak{p}(\lambda)$.

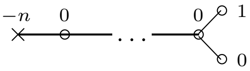
2.2. **Generalization of the example.** The previous example can be generalized to higher dimensions:

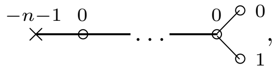
Lemma 2. *Let* $\mathfrak{g} = so(2n+2,\mathbb{C})$*,* $\mathfrak{p}$ *a parabolic subalgebra corresponding to*

Then choosing

$$\lambda = \frac{1}{2}[-2n+1|1,1,\dots,1] \quad \text{and} \quad \mu = \frac{1}{2}[-2n-1|1,1,\dots,1,-1]\,,$$

or, in the language of Dynkin diagrams,

$\lambda =$ 

$\mu =$ 

,

there exists a unique (up to a multiple) nonzero homomorphisms of generalized Verma modules

$$M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)\,.$$

Representing elements of $\mathfrak{g}$ *as matrices like in* (2)*, the image of the highest weight vector of* $M_{\mathfrak{p}}(\mu)$ *in* $M_{\mathfrak{p}}(\lambda)$ *is*

(4) $y_{n+2,1} \otimes v_\lambda - y_{n,1} \otimes Y_{n+2,n} v_\lambda - y_{n-1,1} \otimes Y_{n+2,n-1} v_\lambda - \dots - y_{2,1} \otimes Y_{n+2,2} v_\lambda\,.$

Similarly, for the weights

$$\lambda' = \frac{1}{2}[-2n+1|1,1,\dots,1,-1] \quad \text{and} \quad \mu' = \frac{1}{2}[-2n-1|1,1,\dots,1,1]\,,$$

there also exists a unique (up to multiple) nonzero homomorphisms of generalized Verma modules

$$M_{\mathfrak{p}}(\mu') \to M_{\mathfrak{p}}(\lambda')$$

and the image of the highest weight vector of $M_{\mathfrak{p}}(\mu')$ *in* $M_{\mathfrak{p}}(\lambda')$ *is*

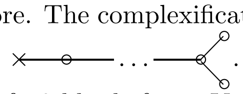
(5) $y_{n+1,1} \otimes v_{\lambda'} - y_{n,1} \otimes Y_{n+1,n} v_{\lambda'} - y_{n-1,1} \otimes Y_{n+1,n-1} v_{\lambda'} - \dots - y_{2,1} \otimes Y_{n+1,2} v_{\lambda'}\,.$

Proof. The line of arguments is described in the proof of lemma 1. The computations of the extremal vector of the proper weight is very technical but straightforward. □

2.3. **The real version.** Let as now suppose that $\mathfrak{g} = so(2n+1,1;\mathbb{R})$ is the real Lie algebra consisting of matrices invariant with respect to the quadratic form $x_0 x_\infty + \sum_{j=1}^{2n} x_j^2$ and $\mathfrak{p}$ is the (real) parabolic subalgebra stabilizing a line in the null-cone. In matrices,

$$\left(\begin{array}{c|c|c} \mathbb{R} & \mathfrak{g}_1 & 0 \\ \hline \mathfrak{g}_{-1} & so(2n) & \mathfrak{g}_1 \\ \hline 0 & \mathfrak{g}_{-1} & \mathbb{R} \end{array}\right)$$

The negative part $\mathfrak{g}_{-1} \simeq \mathbb{R}^{2n}$ is the fundamental defining representation of $so(2n) \subset \mathfrak{g}_0$ via the adjoint action and $\mathfrak{g}_0 = so(2n) \oplus \mathbb{R}$.

We assume that $\mathfrak{g}$ is naturally embedded into its complexification $\mathfrak{g}^c = so(2n+2,\mathbb{C})$ and that the Cartan subalgebra, positive roots and fundamental weights of the complexification are given like before. The complexification of $\mathfrak{p}$ is exactly the parabolic subalgebra corresponding to  .

Let $\mathbb{V}_\lambda$ and $\mathbb{V}_\mu$ be representation of $\mathfrak{p}^c$ like before. Via restriction, they are (complex) representations of the real form $\mathfrak{p}$ as well.

As vector spaces, the generalized Verma modules for real Lie algebras and complex inducing representation are isomorphic to the generalized Verma modules for the complex Lie algebras:

$$M_{\mathfrak{p}^c}(\mu) = \mathcal{U}(\mathfrak{g}^c) \otimes_{\mathcal{U}(\mathfrak{p}^c)} \mathbb{V}_\mu \simeq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}_\mu \,.$$

The first product is over the complex universal enveloping algebra and the second is over the real algebra.

This vector space homomorphism is compatible with the action of $\mathfrak{g} \subset \mathfrak{g}^c$ on both spaces, i.e. it is an $\mathfrak{g}$-isomorphism.

Because we know from the previous section that there exists a unique (up to multiple) $\mathfrak{g}^c$-homomorphism $M_{\mathfrak{p}^c}(\mu) \to M_{\mathfrak{p}^c}(\lambda)$, it follows that there exist a unique (up to multiple) nonzero homomorphism of the real generalized Verma modules $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ in this case as well.

2.4. **Description of the differential operator.** Let $\mathfrak{g}, \mathfrak{p}$ be as in the last section, $G = \mathrm{S}pin(2n+1, 1)$ the real Lie group with Lie algebra $\mathfrak{g}$, P the parabolic subgroup of G whose lie algebra is $\mathfrak{p}$. Let V_λ and V_μ be representations of $\mathfrak{p}^c$ like in lemma 2. The duality between homomorphisms of generalized Verma modules and invariant differential operators yields a nonzero invariant differential operator $D : \Gamma(\mathcal{V}_{\lambda^*}) \to \Gamma(\mathcal{V}_{\mu^*})$ where $\mathcal{V}_{\nu^*} = G \times_P \mathbb{V}^*_\nu$, $\nu = \lambda, \mu$ ($\Gamma(\mathcal{V})$ is the set of smooth sections).

Lemma 3. *The operator D is of first order.*

Proof. The homomorphism from lemma 2 sends $v_\mu \mapsto uv_\lambda$, where $u \in \mathcal{U}(\mathfrak{g}^c)$ is given by (4) resp. (5). We see from (4) resp. (5) that $uv_\lambda \in \mathcal{U}_1(\mathfrak{g}^c_-) \otimes \mathbb{V}_\lambda$. For $u_0 \in \mathcal{U}(\mathfrak{g}^c_0)$, $u_0 v_\mu \mapsto u_0 u v_\lambda$ which is also in $\mathcal{U}_1(\mathfrak{g}^c_-) \otimes \mathbb{V}_\lambda$ and we see that the homomorphism $M_{\mathfrak{p}^c}(\mu) \to M_{\mathfrak{p}^c}(\lambda)$ maps $1 \otimes_{\mathbb{C}} \mathbb{V}_\mu$ to $\mathcal{U}_1(\mathfrak{g}^c_-) \otimes_{\mathbb{C}} \mathbb{V}_\lambda$ (but not to $\mathcal{U}_0(\mathfrak{g}^c_-) \otimes_{\mathbb{C}} \mathbb{V}_\lambda$). It follows that the homomorphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ takes $1 \otimes_{\mathbb{R}} \mathbb{V}_\mu$ to $\mathcal{U}_1(\mathfrak{g}_-) \otimes_{\mathbb{R}} \mathbb{V}_\lambda$. Dualizing this (see the introduction, (1) or [1]), we get a map $\mathcal{J}^1_{eP}(\mathcal{V}_{\lambda^*}) \to \mathbb{V}^*_\mu$ which yields an invariant differential operator of order 1. ☐

Now we want to describe the P-homomorphism $\varphi : \mathcal{J}^1_{eP}(\mathcal{V}_{\lambda^*}) \to \mathbb{V}^*_\mu$. This is a $\mathfrak{p}$-homomorphism, hence also a $\mathfrak{g}_0$-homomorphism. As a $\mathfrak{g}_0$-module, $\mathcal{J}^1_{eP}(\mathcal{V}_{\lambda^*}) \simeq \mathbb{V}^*_\lambda \oplus (\mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{V}^*_\lambda)$ ($\mathfrak{g}_1$ is dual to $\mathfrak{g}_{-1}$, the model for the tangent space in eP of G/P).

As $\mathrm{s}o(2n, \mathbb{C})$-modules, the spaces $\mathbb{V}_\mu$ and $\mathbb{V}_\lambda$ are called basic spinor modules and can be realized as subspaces of the Clifford algebra $\mathrm{C}liff(2n, \beta)$, where $\beta(x, y) = \sum_j x_j y_{2n-j}$ is the form defining the matrices in $\mathrm{s}o(2n, \mathbb{C})$. We will denote S^+ the representation $\mathbb{V}_\lambda$ with highest weight $[\frac{1}{2}, \dots, \frac{1}{2}]$ and S^- the representation V_μ with highest weight $[\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}]$ (as $\mathrm{s}o(2n, \mathbb{C})$-modules). It can be shown that $(S^+)^* \simeq S^-$. Further, because S^+ and S^- are subspaces of the Clifford algebra, the Clifford multiplication $\mathbb{R}^{2n} \otimes S^\pm \to S^\mp$ is defined.

So, as a representation of $\mathfrak{g}^{ss}_0 \simeq \mathrm{s}o(2n)$, $\mathbb{V}^*_\lambda \simeq S^-$ and $\mathbb{V}^*_\mu \simeq S^+$. It can be shown easily that $\mathfrak{g}_1 \simeq \mathbb{R}^{2n}$, the defining representation of $\mathrm{s}o(2n)$ and therefore, as a $\mathfrak{g}^{ss}_0$-module homomorphism,

$$\varphi : \mathcal{J}^1_{eP}(\mathcal{V}_{\lambda^*}) \simeq S^- \oplus ((\mathbb{R}^{2n}) \otimes_{\mathbb{R}} S^-) \to S^+ \,. \tag{6}$$

It is a well-known fact that

$$\mathbb{R}^{2n} \otimes_{\mathbb{R}} S^- \simeq S^+ \oplus T\,, \tag{7}$$

where S^+ and T are the spinor and twistor representations (see [5]), so the operator is given just by the projection π of the second summand in (6) to S^+.

It is well-known that $\mathfrak{g}_-$ can be imbedded into G/P as an open dense subspace by

$$i : \mathfrak{g}_- \to G/P, \quad y \mapsto \exp(y)P.$$

We will identify $\mathfrak{g}_-$ with its image under i. To any section $s \in \Gamma(\mathcal{V})$ given by $gP \mapsto [g, v]_P$ we can assign a $\mathbb{V}$-valued function f on $\mathfrak{g}_-$ given by

$$f : \mathfrak{g}_- \to \mathbb{V}, y \mapsto v, \quad \text{where } s(i(y)) = [\exp(y), v]_P\,. \tag{8}$$

The space $\mathfrak{g}_-$ is endowed with a basis

$$\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline e_j & 0 & 0 \\ \hline 0 & -e_j^T & 0 \end{array}\right)$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)^T$ is the j-th vector of the standard basis of $\mathbb{R}^{2n}$, and with the standard metric $\sum_j x_j^2$. Let ∇ be the flat Levi-Civita connection on $\mathfrak{g}_-$ induced by this metric. The map $i : \mathfrak{g}_- \to G/P = S^{2n}$ is a conformal map.

Theorem 1. *Let $s, s' \in \Gamma(\mathcal{V}_{\lambda^*})$ are sections and $f, f' : \mathfrak{g}_- \to S^+$ (S^-) the spinor valued functions corresponding to s and s' under the above identification. Assume that $s' = Ds$. Then $f' = \sum_{i=1}^{2n} e_i \nabla_{e_i} f$.*

Proof. Take $s \in \Gamma(G \times_P S^-)$ and denote by ∇ any Weyl covariant derivative on $G \times_P S^-$. Then $(s, \nabla s)$ is a section of $\mathcal{J}^1(\mathcal{V}_\lambda^*)$. The last bundle is the associated bundle to the P-module $J^1(S^-) \simeq S^- \oplus \mathbb{C}^{2n} \otimes S^-$, where we identify $\mathfrak{g}_- \simeq \mathfrak{g}/\mathfrak{p}$ (as a $\mathfrak{p}$-module) via the Killing form. It follows from the classification of the first order invariant operators in [4], that there is a P-homomorphism $\pi : J^1(S^-) \to S^+$ such that $D = \tilde{\pi} \circ \nabla$, where $\tilde{\pi} : \mathcal{J}^1(\mathcal{V}_\lambda^*) \to \mathcal{V}_\mu^*$ is induced by π. But there is clearly a unique $\mathfrak{g}_0^{ss}$-homomorphism from $J^1(S^-)$ to S^+, given by the invariant projection from $\mathfrak{g}_+ \otimes_{\mathbb{R}} S^-$ to S^+. Hence π should be equal to this projection (up to a multiple).

Let us restrict now ourselves to the big cell $\mathfrak{g}_-$. We can take the flat connection for ∇ on $\mathfrak{g}_-$. The explicit form of the projection π was computed in [5]. Its form is, up to a multiple, equal to

$$\pi(\sum_j e_j \otimes s_j) = \sum_j e_j s_j \tag{9}$$

where $e_j s_j$ is the Clifford multiplication. So, we get $Df = \sum_{i=1}^{n} e_i \nabla_{e_i} f$. □

In the case that the quadratic form is not specified, we get a more general formula for D. Suppose that $b(x, y)$ is a scalar product corresponding to a given quadratic form $\beta(x)$ on $\mathfrak{g}_-$. Take any basis $\{e_i\}$ of $\mathfrak{g}_-$ and denote by $\{e_i'\}$ the dual basis with respect to b.

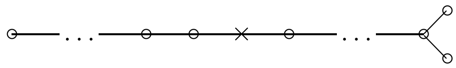
By an easy modification of calculations in [5], we get the following claim.

Let $s, s' \in \Gamma(\mathcal{V}_{\lambda^*})$ *are sections and* $f, f' : \mathfrak{g}_- \to S^+$ (S^-) *the spinor valued functions corresponding to* s *and* s' *under the above identification. Assume that* $s' = Ds$. *Then* $f' = \sum_{i=1}^n e_i \nabla_{e_i'} f$

Therefore, we can call the operator D the Dirac operator.

3. More Dirac operators

3.1. **Verma modules in higher grading.** Consider now a pair of real Lie algebras $(\mathfrak{g}, \mathfrak{p})$ with complexification described by the Dynkin diagram



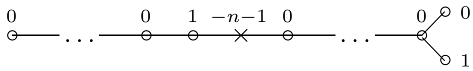
The real form is chosen to be $\mathfrak{g} = so(k, 2n + k; \mathbb{R})$ and the k-th node is crossed. We can choose $\mathfrak{p}$ to be the parabolic subalgebra corresponding to the following gradation:

$$\left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right)$$

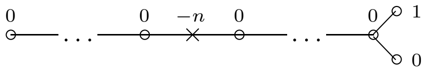
where $\mathfrak{g}_0 = sl(k, \mathbb{R}) \oplus so(2n) \oplus \mathbb{R}E$ and, as a $\mathfrak{g}_0$-module, $\mathfrak{g}_{-1} \simeq ((\mathbb{R}^k)^* \otimes \mathbb{R}^{2n})$, the product of dual resp. defining representations of $sl(k, \mathbb{R})$ resp. $so(2n)$. The part $\mathfrak{g}_{-2}$ is commutative.

The real generalized Verma modules are again the same as the complex, due to the fact that we consider complex representations of $\mathfrak{p}$.

Theorem 2 (generalization of Lemma 2). *Independent of the dimension, there is a Verma module homomorphism* $D : M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ *for*

$\mu =$ 

and

$\lambda =$ 

and the corresponding (dual) differential operator is of first order. Analogous statement holds for the weights μ' *and* λ' *having interchanged* 0 *and* 1 *over the last positions in the Dynkin diagram.*

Proof. Using the technique of lemma 1, it can be shown that the only extremal vector of weight μ in $M_{\mathfrak{p}^c}(\lambda)$ is the vector

$$y_{n+2,k} \otimes v_\lambda - y_{n,k} \otimes Y_{n+2,n} v_\lambda - y_{n-1,k} \otimes Y_{n+2,n-1} v_\lambda - \ldots - y_{2,k} \otimes Y_{n+2,2} v_\lambda$$

We see again that it lies in $\mathcal{U}_1(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$, so only first derivations are involved. $\square$

3.2. **Description of the operator.** As before, we associate to the graded Lie algebra from the last paragraph a real form $\mathfrak{g}$ of real matrices fixing the inner product $\sum_{i=1}^{k} x_i x_{2(n+k)+1-i} + \sum_{j=1}^{2n} x_{k+j}^2$ (it has signature $(2n+k,k)$). The parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is such that $\mathfrak{g}_0 = sl(k,\mathbb{R}) \oplus so(2n) \oplus \mathbb{R}E$. The complexification $(\mathfrak{g}^c, \mathfrak{p}^c)$ is the even orthogonal complex Lie algebra of rank $n+k$ with the k-th nod crossed in the Dynkin diagram.

As a $\mathfrak{g}_0$-module, $\mathfrak{g}_{-1} \simeq ((\mathbb{R}^k)^* \otimes \mathbb{R}^{2n})$, the product of dual resp. defining representations of $sl(k,\mathbb{R})$ resp. $so(2n)$. The part $\mathfrak{g}_{-2}$ is commutative. Let μ, λ be weights like before and consider $\mathbb{V}_\mu$ and $\mathbb{V}_\lambda$ to be complex representations of the real Lie algebra $\mathfrak{p}$ with highest weight μ resp. λ. We see that, as a $\mathfrak{g}_0^{ss}$-module, $\mathbb{V}_\mu \simeq \mathbb{C}^{k^*} \otimes S^-$ and $\mathbb{V}_\lambda \simeq \mathbb{C} \otimes S^+$ where $\mathbb{C}^k$ resp. $\mathbb{C}$ are the defining resp. trivial representation of $sl(k,\mathbb{R})$.

We know from 2.3 and the previous paragraph that there is a nonzero homomorphism of generalized Verma modules $M_\mathfrak{p}(\mu) \to M_\mathfrak{p}(\lambda)$ in this case as well.

The corresponding dual differential operator acts between sections of dual representation:

$$D : \Gamma(G \times_P \mathbb{C} \otimes S^-) \to \Gamma(G \times_P \mathbb{C}^k \otimes S^+)$$

(we identified $(S^-)^* \simeq S^+$).

Assume that s is a section of $G \times_P (\mathbb{C} \otimes S^-)$ and f is a $\mathbb{C} \otimes S^- \simeq S^-$-valued functions on $\mathfrak{g}_-$ defined as in (8). The coordinates on $\mathfrak{g}_{-1}$ can be chosen to be $y_{11}, \dots, y_{1n}, \dots, y_{k1}, \dots, y_{kn}$ and on $\mathfrak{g}_{-2}$ $y_1, \dots, y_l$. We assign a function $Df : \mathfrak{g}_- \to \mathbb{C}^k \otimes S^+$ to each section Ds and Df can be naturally identified with k S^+-valued functions $D_1(f), \dots, D_k(f)$.

Assume that f is constant in the $\mathfrak{g}_{-2}$ variables $y_1, \dots, y_l$, so, it can be considered as a function of $y_{i,j}$ only.

As before, the corresponding differential operator D can be written in the form $D = \tilde{\pi} \circ \nabla$, where ∇ is the covariant derivative of the Weyl connection on the tangent bundle induced by the trivialization of the tangent bundle by left invariant vector fields.

On a function f that does not depend on the $\mathfrak{g}_{-2}$-variables, covariant derivative $\nabla_{e_{i,j}}$ coincide with the ordinary flat derivations $\frac{\partial}{\partial y_{ij}}$ of f. Restricting to such functions, the operator can be considered as

$$C^\infty((\mathbb{R}^k)^* \otimes \mathbb{R}^{2n}, S^-) \to C^\infty((\mathbb{R}^k)^* \otimes \mathbb{R}^{2n}, \mathbb{C}^k \otimes S^+)$$

It is given by the projection

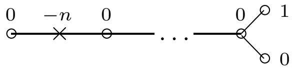
$$\pi : (\mathbb{R}^k \otimes (\mathbb{R}^{2n})^*) \otimes_\mathbb{R} S^- \simeq \mathbb{C}^k \otimes_\mathbb{C} (\mathbb{C}^{2n} \otimes_\mathbb{C} S^-) \to \mathbb{C}^k \otimes_\mathbb{C} S^+$$

The projection should be a $\mathfrak{g}_0$-homomorphism, what yields $\pi = (\pi_1, \dots, \pi_k)$ where $\pi_i : \mathbb{R}^{2n} \otimes S^- \to S^+$ is given by (9). Therefore, the operator $D = (D_1, \dots, D_k)$ where $D_i = \sum_j e_j \nabla_{e_{ij}}$.

4. A complex of homomorphisms of Verma modules

4.1. **The singular orbit.** In the case $k \geq 1$ (more Dirac operators in the sense of the last section), there are also other $\mathfrak{p}$-dominant weights on the affine orbit of

λ and μ. Let us return now to the case of complex Lie algebras and consider the case $k = 2$. Then

$\lambda =$ 

and $\lambda + \delta = \frac{1}{2}[3, 1| \dots, 5, 3, 1]$.

There are exactly 3 other $\mathfrak{p}-$dominant weights μ, ν, ξ on the same affine orbit:

$$\mu + \delta = \frac{1}{2}[3, -1 \mid \dots, 5, 3, -1]$$

$$\nu + \delta = \frac{1}{2}[1, -3 \mid \dots, 5, 3, -1]$$

$$\xi + \delta = \frac{1}{2}[-1, -3 \mid \dots, 5, 3, 1]$$

We know from Theorem 2 that there is a homomorphism of generalized Verma modules $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$.

Theorem 3. *There exists a unique* (*up to a multiple*) *nonzero homomorphism of generalized Verma modules* $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$ *and a unique* (*up to a multiple*) *nonzero homomorphism* $M_{\mathfrak{p}}(\xi) \to M_{\mathfrak{p}}(\nu)$ *so that the composition* $M_{\mathfrak{p}}(\xi) \to M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$ *is zero and the composition* $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ *is zero. So, there is a complex of generalized Verma modules described by the highest weights* λ, μ, ν, ξ

$$\begin{array}{c} \lambda + \delta = \frac{1}{2}[3, 1| \dots, 5, 3, 1] \\ \uparrow \\ \mu + \delta = \frac{1}{2}[3, -1| \dots, 5, 3, -1] \\ \uparrow \\ \nu + \delta = \frac{1}{2}[1, -3| \dots, 5, 3, -1] \\ \uparrow \\ \xi + \delta = \frac{1}{2}[-1, -3| \dots, 5, 3, -1] \end{array}$$

Proof. The existence and uniqueness of the homomorphism can be shown using the technique of lemma 1. We will outline the proof only in low dimension, for $k = 2, n = 3$. In this case, elements of $\mathfrak{g}$ are matrices 10×10 antisymmetric with respect to the antidiagonal and we can denote the root spaces by y_{ij} and Y_{ij} similar to (2).

The extremal vector in $w \in M_{\mathfrak{p}}(\mu)$ of the weight ν is described by some $u \in \mathcal{U}(\mathfrak{g})$ so that $w = u\beta$, where β is the highest weight vector in $M_{\mathfrak{p}}(\mu)$. It can be checked that the only possibility is (up to multiple)

$$\begin{aligned} u = {} & (y_{61} - y_{41}Y_{64} - y_{31}Y_{63})(y_{52} - y_{42}Y_{54} - y_{32}Y_{53}) \\ & + (y_{62} - y_{42}Y_{64} - y_{32}Y_{63})(y_{52} - y_{42}Y_{54} - y_{32}Y_{53})Y_{21} - y_{91}\,. \end{aligned}$$

The multiplication of the brackets is multiplication in $\mathcal{U}(\mathfrak{g})$. Similarly, it can be checked that the extremal vector in $M_{\mathfrak{p}}(\nu)$ of the weight ξ is $w\gamma$, where β is the highest weight vector in $M_{\mathfrak{p}}(\nu)$ and

$$w = (y_{51} - y_{41}Y_{54} - y_{31}Y_{53}) + (y_{52} - y_{42}Y_{54} - y_{32}Y_{53})Y_{21}\,.$$

The composition of two homomorphisms of generalized Verma modules $M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is zero exactly if it sends the highest weight vector $\gamma \in M_{\mathfrak{p}}(\nu)$ to zero. We know that γ is mapped to $u\beta$ where β is the highest weight vector in $M_{\mathfrak{p}}(\mu)$. The second homomorphism sends β to $u'\alpha$, where α is the highest weight vector in $M_{\mathfrak{p}}(\lambda)$ and u' is determined by the proof of Theorem 2. Therefore, $u\beta$ is mapped to $uu'\alpha$ by the homomorphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$. It remains to show that $uu'\alpha = 0$ in $M_{\mathfrak{p}}(\lambda)$. This can be done using commutation relation in $\mathcal{U}(\mathfrak{g})$ and basic representation theory.

Similarly we check that $M_{\mathfrak{p}}(\xi) \to M_{\mathfrak{p}}(\nu) \to M_{\mathfrak{p}}(\mu)$ is zero.

In higher dimension, the extremal vectors are similar, just instead of $(y_{61} - y_{41}Y_{63} - y_{31}Y_{63})$ one has to write $(y_{2n,1} - y_{2n-2,1}Y_{2n,2n-2} - \ldots - y_{31}Y_{2n-2,3})$ etc. □

Remark 1. Choosing proper real forms of $\mathfrak{g}, \mathfrak{p}$, these homomorphisms can be translated to invariant differential operators. The first one is the Dirac operator in two variables, as we already showed. The second and third operator together with the first form a complex of differential operators.

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