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SYMMETRIES IN HEXAGONAL QUASIGROUPS

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Abstract. Hexagonal quasigroup is idempotent, medial and semisymmetric quasigroup. In this article we define and study symmetries about a point, segment and ordered triple of points in hexagonal quasigroups. The main results are the theorems on composition of two and three symmetries.

1. Introduction

Hexagonal quasigroups are defined in [3].

Definition. A quasigroup $(Q,\cdot)$ is said to be *hexagonal* if it is idempotent, medial and semisymmetric, i.e. if its elements a, b, c satisfy:

(id) $$a\cdot a = a$$

(med) $$(a\cdot b)\cdot(c\cdot d) = (a\cdot c)\cdot(b\cdot d)$$

(ss) $$a\cdot(b\cdot a) = (a\cdot b)\cdot a = b\,.$$

From (id) and (med) easily follows distributivity

(ds) $$a\cdot(b\cdot c) = (a\cdot b)\cdot(a\cdot c),\quad (a\cdot b)\cdot c = (a\cdot c)\cdot(b\cdot c)$$

When it doesn't cause confusion, we can omit the sign "$\cdot$", e.g. instead of $(a\cdot b)\cdot(c\cdot d)$ we may write $ab\cdot cd$.

In this article, Q will always be a hexagonal quasigroup.

The basic example of hexagonal quasigroup is formed by the points of Euclidean plane, with the operation $\cdot$ such that the points a, b and $a\cdot b$ form a positively oriented regular triangle. This structure was used for all the illustrations in this article.

Motivated by this example, Volenec in [3] and [4] introduced some geometric terms to any hexagonal quasigroup. Some of these terms can be defined in any idempotent medial quasigroup (see [2]) or even medial quasigroup (see [1]).

The elements of hexagonal quasigroup are called *points*, and pairs of points are called *segments*.

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Definition. We say that the points a, b, c and d form a *parallelogram*, and we write $\mathrm{Par}\,(a,b,c,d)$ if $bc \cdot ab = d$ holds. (Fig. 1)

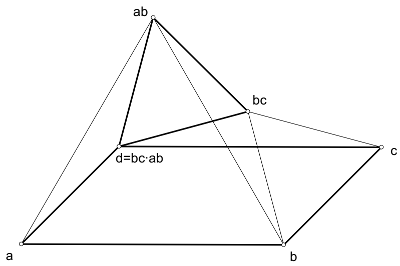

FIGURE 1. Parallelogram (definition)

Accordingly to [3], the structure $(Q, \mathrm{Par}\,)$ is a parallelogram space. In other words, Par is a quaternary relation on Q (instead of $(a,b,c,d) \in \mathrm{Par}$ we write $\mathrm{Par}\,(a,b,c,d)$) such that:

1. Any three of the four points a, b, c, d uniquely determine the fourth, such that $\mathrm{Par}\,(a,b,c,d)$.
2. If (e,f,g,h) is any cyclic permutation of (a,b,c,d) or (d,c,b,a), then $\mathrm{Par}\,(a,b,c,d)$ implies $\mathrm{Par}\,(e,f,g,h)$.
3. From $\mathrm{Par}\,(a,b,c,d)$ and $\mathrm{Par}\,(c,d,e,f)$ it follows $\mathrm{Par}\,(a,b,f,e)$. (Fig. 2)

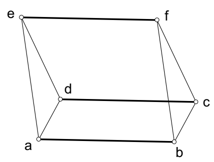

FIGURE 2. Property 3 of the relation Par

Accordingly to [3]:

Theorem 1. *From* $\mathrm{Par}\,(a_1,b_1,c_1,d_1)$ *and* $\mathrm{Par}\,(a_2,b_2,c_2,d_2)$ *it follows* $\mathrm{Par}\,(a_1a_2, b_1b_2, c_1c_2, d_1d_2)$.

In the rest of this section we present some definitions and results from [5].

Definition. The point m is a *midpoint* of the segment $\{a,b\}$, if $\mathrm{Par}\,(a,m,b,m)$ holds. This is denoted by $\mathrm{M}\,(a,m,b)$.

Remark. For given a,b such m can exist or not; and it can be unique or not.

Theorem 2. *Let* $\mathrm{M}\,(a,m,c)$. *Then* $\mathrm{M}\,(b,m,d)$ *and* $\mathrm{Par}\,(a,b,c,d)$ *are equivalent.*

Definition. The point m is called a *center of a parallelogram* $\mathrm{Par}\,(a,b,c,d)$ if $\mathrm{M}\,(a,m,c)$ and $\mathrm{M}\,(b,m,d)$.

Definition. The function $T_{a,b} : Q \to Q$,

$$T_{a,b}(x) = ab \cdot xa$$

is called *transfer* by the vector $[a, b]$. (Fig. 3)

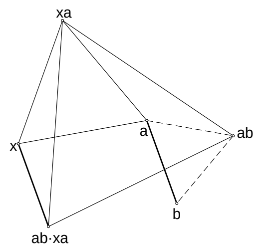

FIGURE 3. Transfer by the vector $[a, b]$

Lemma 1. *For any* $a, b, x \in Q$*,* $\operatorname{Par}\big(x, a, b, T_{a,b}(x)\big)$*. The equality* $T_{a,b} = T_{c,d}$ *is equivalent to* $\operatorname{Par}(a, b, d, c)$*.*

Theorem 3. *The set of all transfers is a commutative group. Specially, the composition of two transfers is a transfer. The inverse of* $T_{a,b}$ *is* $T_{b,a}$*.*

2. Symmetries in hexagonal quasigroup

Lemma 2. *For any points* $a, b, c, x \in Q$*, the following equalities hold*

$$(xa \cdot a)a = a(a \cdot ax) = xa \cdot ax$$
$$(xa \cdot b)a = a(b \cdot ax) = xa \cdot bx = xb \cdot ax = b(a \cdot bx) = (xb \cdot a)b$$
$$(xa \cdot b)c = a(b \cdot cx) = (x \cdot ac) \cdot bx = xb \cdot (ac \cdot x)$$

Proof. Since Q is semisymmetric quasigroup, $pq = r$ is equivalent to $qr = p$.

First, we prove the last set of equalities.

From $(b \cdot cx) \cdot (xa \cdot b)c \overset{\text{(med)}}{=} b(xa \cdot b) \cdot (cx \cdot c) \overset{\text{(ss)}}{=} xa \cdot x \overset{\text{(ss)}}{=} a$, it follows $a(b \cdot cx) = (xa \cdot b)c$.

From $(ac \cdot x) \cdot (x(ac) \cdot bx) \overset{\text{(med)}}{=} (ac \cdot x(ac))(x \cdot bx) \overset{\text{(ss)}}{=} xb$, it follows $(x \cdot ac) \cdot bx = xb \cdot (ac \cdot x)$.

From $\big((x \cdot ac) \cdot bx\big)(xa \cdot b) \overset{\text{(med)}}{=} \big((x \cdot ac) \cdot xa\big)(bx \cdot b) \overset{\text{(med)}}{=} \big((xx) \cdot (ac)a\big)(bx \cdot b) \overset{\text{(id,ss)}}{=} (xc)x \overset{\text{(ss)}}{=} c$, it follows $(xa \cdot b)c = (x \cdot ac) \cdot bx$.

Now putting $a = b = c$ we obtain the first line of equalities, and putting $a = c$ the second line. □

Definition. *Symmetry with respect to the point* a is the function $\sigma_a : Q \to Q$ defined by (see Fig. 4)

$$\sigma_a(x) = a(a \cdot ax) = (xa \cdot a)a = xa \cdot ax\,.$$

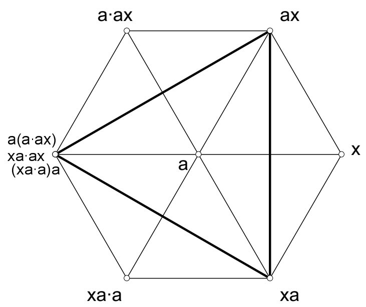

FIGURE 4. Symmetry with respect to the point a

From $\sigma_a(x) = xa \cdot ax$ it follows Par $(a, x, a, \sigma_a(x))$, so we have:

Corollary 1. *The equality* $\sigma_m(a) = b$ *is equivalent to* $\mathrm{M}\,(a, m, b)$.

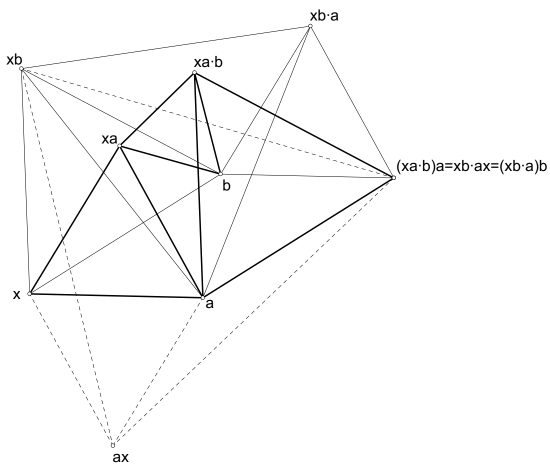

FIGURE 5. Symmetry with respect to the line segment $\{a, b\}$

The function $\sigma_a(x) = xa \cdot ax$ can be generalised this way:

Definition. The function $\sigma_{a,b} : Q \to Q$ defined by

$$\sigma_{a,b}(x) = xa \cdot bx,$$

is called *symmetry with respect to the segment* $\{a, b\}$. (Fig. 5)

It follows immediately:

Corollary 2. *For any* $a, b, x \in Q$

$$\sigma_{a,a} = \sigma_a\,, \qquad \sigma_{a,b} = \sigma_{b,a}\,, \qquad \mathrm{Par}\,\big(a, x, b, \sigma_{a,b}(x)\big)\,.$$

Theorem 4. *The equality* $\sigma_{a,b} = \sigma_m$ *is equivalent to* $\mathrm{M}\,(a, m, b)$.

Proof. Let $\mathrm{M}(a, m, b)$ and let $x \in Q$. From $\mathrm{Par}\big(a, x, b, \sigma_{a,b}(x)\big)$ and $\mathrm{M}(a, m, b)$ and Theorem 2 we obtain $\mathrm{M}\big(x, m, \sigma_{a,b}(x)\big)$, and now from Corollary 1 $\sigma_m(x) = \sigma_{a,b}(x)$.

Inversely, from $\sigma_{a,b} = \sigma_m$ it follows $\sigma_m(a) = \sigma_{a,b}(a) = aa \cdot ba = b$, and now Corollary 1 implies $\mathrm{M}(a, m, b)$. □

The function $\sigma_a(x) = a(a \cdot ax)$ can be generalised in another way:

Definition. The function $\sigma_{a,b,c}(x) = (xa \cdot b)c$ is called *symmetry with respect to the ordered triple of points* (a, b, c). (Fig. 6)

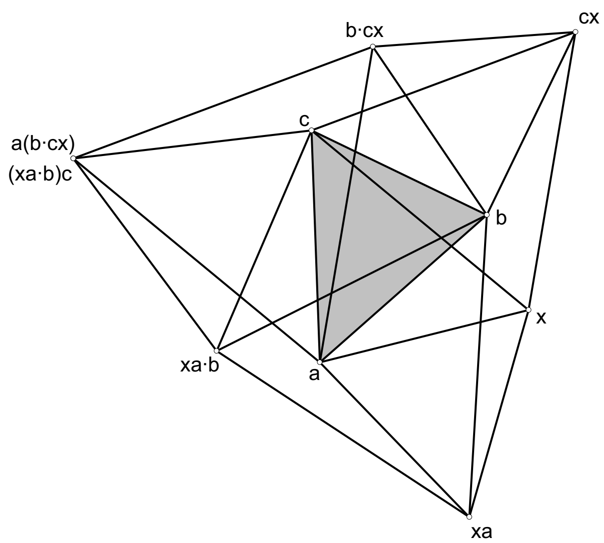

FIGURE 6. Symmetry with respect to the ordered triple of points (a, b, c)

Lemma 2 implies

$$\sigma_{a,b,c}(x) = (xa \cdot b)c = a(b \cdot cx) = (x \cdot ac) \cdot bx = xb \cdot (ac \cdot x)\,.$$

It immediately follows:

Corollary 3. *For any* $a, b, c, x \in Q$

$$\sigma_a = \sigma_{a,a,a}, \quad \sigma_{a,b} = \sigma_{a,b,a} = \sigma_{b,a,b}\,, \quad \sigma_{a,b,c} = \sigma_{ac,b}\,, \quad \mathrm{Par}\big(ac, x, b, \sigma_{a,b,c}(x)\big)\,.$$

Note that different order of points (e.g. (b, a, c)) produces different symmetry.

Theorem 5. *The symmetry* $\sigma_{a,b,c}$ *is an involutory automorphism of the hexagonal quasigroup* $(Q, \cdot)$.

Proof. We first show that $\sigma_{a,b,c} \circ \sigma_{a,b,c}$ is identity:

$$\sigma_{a,b,c}\big(\sigma_{a,b,c}(x)\big) = \sigma_{a,b,c}\big((xa \cdot b)c\big) = a \cdot b\big(c \cdot (xa \cdot b)c\big) \overset{(ss)}{=} a \cdot b(xa \cdot b) \overset{(ss)}{=} a \cdot xa \overset{(ss)}{=} x\,.$$

It follows that $\sigma_{a,b,c}$ is a bijection. Further:

$$\sigma_{a,b,c}(xy) = (xy \cdot a)b \cdot c \overset{\text{(ds)}}{=} (xa \cdot ya)b \cdot c$$
$$\overset{\text{(ds)}}{=} (xa \cdot b)(ya \cdot b) \cdot c \overset{\text{(ds)}}{=} (xa \cdot b)c \cdot (ya \cdot b)c = \sigma_{a,b,c}(x) \cdot \sigma_{a,b,c}(y)\,,$$

so $\sigma_{a,b,c}$ is an automorphism. □

From Theorem 5 and Corollary 3, it follows:

Corollary 4. *Symmetries σ_a and $\sigma_{a,b}$ are involutory automorphisms of the hexagonal quasigroup $(Q, \cdot)$.*

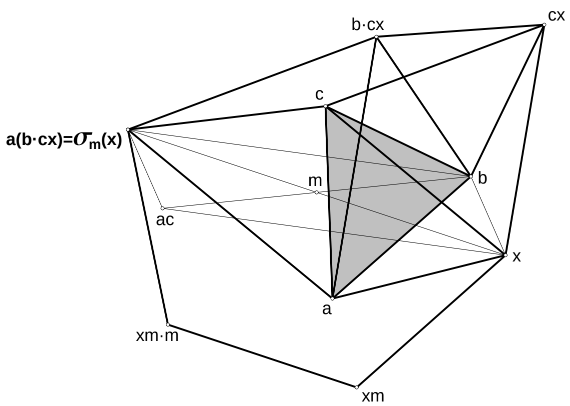

FIGURE 7. Theorem 6

Theorem 6. *The equality $\sigma_{a,b,c} = \sigma_m$ is equivalent to $\mathrm{M}\,(ac, m, b)$. (Fig. 7)*

Proof. The statement follows immediately from $\sigma_{a,b,c} = \sigma_{ac,b}$ (Corollary 3) and Theorem 4. □

The following two theorems are about the compositions of two and three symmetries.

Theorem 7. *The composition of two symmetries is a transfer (Fig. 8). More precisely, for any $a_1, a_2, a_3,\ b_1, b_2, b_3$*

$$\sigma_{b_1,b_2,b_3} \circ \sigma_{a_1,a_2,a_3} = T_{a_1a_3,b_1b_3} \circ T_{a_2,b_2}\,.$$

Proof. Since composition of two transfers is a transfer (Theorem 3), it's enough to prove the above equality.

Let $x \in Q$ be any point, and let $y = \sigma_{a_1,a_2,a_3}(x)$, $z = \sigma_{b_1,b_2,b_3}(y)$, and $w = T_{a_2,b_2}(x)$. We need to prove that $T_{a_1a_3,b_1b_3}(w) = z$.

Lemma 1 implies $\mathrm{Par}\,(x, a_2, b_2, w)$, and from Corollary 3 it follows $\mathrm{Par}\,(a_1a_3, x, a_2, y)$ and $\mathrm{Par}\,(b_1b_3, y, b_2, z)$.

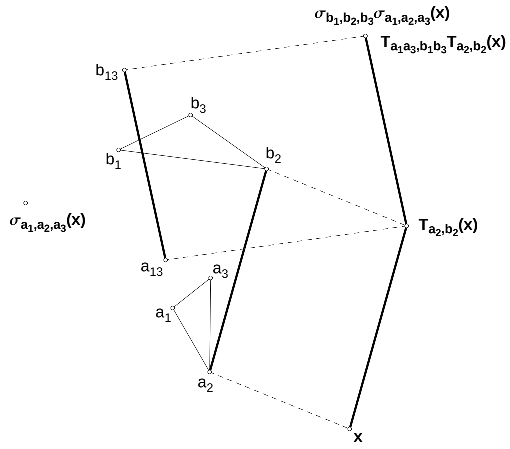

FIGURE 8. Theorem 7

Property 2 of Par implies Par (b_2, w, x, a_2) and Par (x, a_2, y, a_1a_3), and now from Property 3 it follows Par (b_2, w, a_1a_3, y).

Similarly, Property 2 implies Par (w, a_1a_3, y, b_2) and Par (y, b_2, z, b_1b_3), and because of Property 3 it follows Par (w, a_1a_3, b_1b_3, z).

From this relation and Lemma 1 it finally follows $z = T_{a_1a_3,b_1b_3}(w)$. □

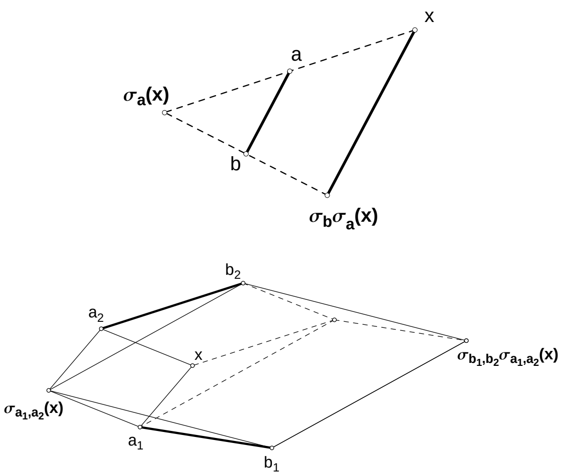

FIGURE 9. Corollary 5

Using Corollary 3 we obtain (see Fig. 9):

Corollary 5. *For* $a, b \in Q$, $\quad \sigma_b \circ \sigma_a = T_{a,b} \circ T_{a,b}$.
For $a_1, a_2, b_1, b_2 \in Q$, $\quad \sigma_{b_1,b_2} \circ \sigma_{a_1,a_2} = T_{a_1,b_1} \circ T_{a_2,b_2}$.

Corollary 6. *The equation* $\sigma_{a_1,a_2,a_3} = \sigma_{b_1,b_2,b_3}$ *is equivalent to* $\mathrm{Par}\,(a_1a_3, b_1b_3, a_2, b_2)$.

Proof. By Theorem 5, $\sigma_{a_1,a_2,a_3} = \sigma_{b_1,b_2,b_3}$ is equivalent to $\sigma_{b_1,b_2,b_3} \circ \sigma_{a_1,a_2,a_3} =$ identity. From Theorem 7 we know $\sigma_{b_1,b_2,b_3} \circ \sigma_{a_1,a_2,a_3} = T_{a_1a_3,b_1b_3} \circ T_{a_2,b_2}$, so the initial equality is equivalent to $T_{a_1a_3,b_1b_3} \circ T_{a_2,b_2} =$ identity. Because of Theorem 3 this is equivalent to $T_{a_1a_3,b_1b_3} = T_{b_2,a_2}$, and further because of Lemma 1 to $\mathrm{Par}\,(a_1a_3, b_1b_3, a_2, b_2)$. □

Theorem 8. *The composition of three symmetries is a symmetry. More precisely, for any* $a_1, a_2, a_3,$ $b_1, b_2, b_3,$ $c_1, c_2, c_3,$ *and for* d_1, d_2, d_3 *such that* $\mathrm{Par}\,(a_i, b_i, c_i, d_i)$, *for* $i = 1, 2, 3$,

$$\sigma_{c_1,c_2,c_3} \circ \sigma_{b_1,b_2,b_3} \circ \sigma_{a_1,a_2,a_3} = \sigma_{d_1,d_2,d_3}\,.$$

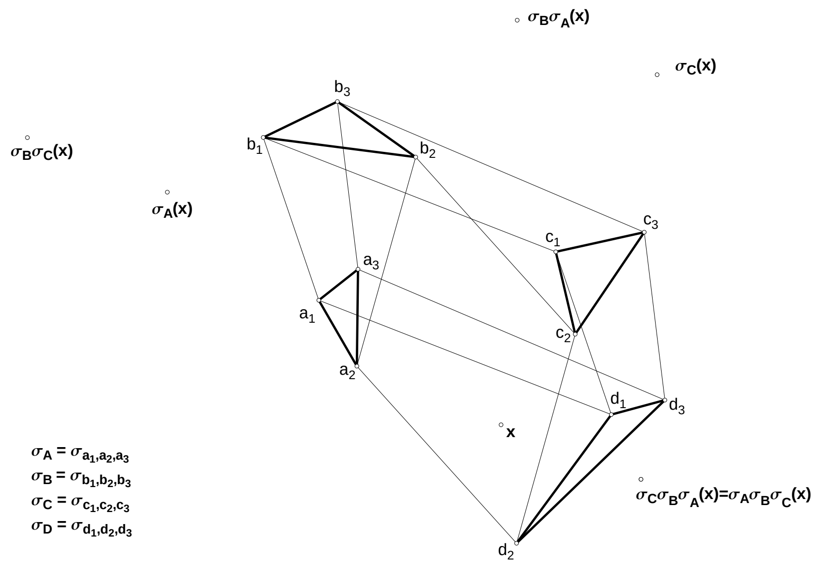

FIGURE 10. Corollary 7

Proof. Let $x \in Q$ be any point, and let $y, z, t \in Q$ be such that

$$y = \sigma_{a_1,a_2,a_3}(x) \quad \text{i.e.} \quad \mathrm{Par}\,(a_1a_3, x, a_2, y)$$
$$z = \sigma_{b_1,b_2,b_3}(y) \quad \text{i.e.} \quad \mathrm{Par}\,(b_1b_3, y, b_2, z)$$
$$t = \sigma_{c_1,c_2,c_3}(z) \quad \text{i.e.} \quad \mathrm{Par}\,(c_1c_3, z, c_2, t)$$

and let $w \in Q$ be such that $\mathrm{Par}\,(d_2, a_2, y, w)$. We need to prove that $\sigma_{d_1,d_2,d_3}(x) = t$, i.e. $\mathrm{Par}\,(d_1d_3, x, d_2, t)$.

From $\mathrm{Par}\,(a_1, b_1, c_1, d_1)$ and $\mathrm{Par}\,(a_3, b_3, c_3, d_3)$, because of Theorem 1 we get $\mathrm{Par}\,(a_1a_3, b_1b_3, c_1c_3, d_1d_3)$.

Now we use Property 3 of the relation Par to conclude:

$$\begin{aligned}
\operatorname{Par}(b_2,c_2,d_2,a_2),\ \operatorname{Par}(d_2,a_2,y,w) \quad &\Rightarrow \quad \operatorname{Par}(b_2,c_2,w,y)\,,\\
\operatorname{Par}(z,b_1b_3,y,b_2),\ \operatorname{Par}(y,b_2,c_2,w) \quad &\Rightarrow \quad \operatorname{Par}(z,b_1b_3,w,c_2)\,,\\
\operatorname{Par}(b_1b_3,w,c_2,z),\ \operatorname{Par}(c_2,z,c_1c_3,t) \quad &\Rightarrow \quad \operatorname{Par}(b_1b_3,w,t,c_1c_3)\,,\\
\operatorname{Par}(d_1d_3,a_1a_3,b_1b_3,c_1c_3),\ \operatorname{Par}(b_1b_3,c_1c_3,t,w) \quad &\Rightarrow \quad \operatorname{Par}(d_1d_3,a_1a_3,w,t)\,,\\
\operatorname{Par}(a_1a_3,x,a_2,y),\ \operatorname{Par}(a_2,y,w,d_2) \quad &\Rightarrow \quad \operatorname{Par}(a_1a_3,x,d_2,w)\,,\\
\operatorname{Par}(x,d_2,w,a_1a_3),\ \operatorname{Par}(w,a_1a_3,d_1d_3,t) \quad &\Rightarrow \quad \operatorname{Par}(x,d_2,t,d_1d_3)\,.
\end{aligned}$$

The relations on the left hand side are valid because of the assumptions, previous conclusions and Property 2 of Par.

The last obtained relation is equivalent to $\operatorname{Par}(d_1d_3,x,d_2,t)$. □

Corollary 7. *For any $a_i,b_i,c_i \in Q$, $i=1,2,3$* (see Fig. 10)

$$\sigma_{a_1,a_2,a_3}\circ\sigma_{b_1,b_2,b_3}\circ\sigma_{c_1,c_2,c_3}=\sigma_{c_1,c_2,c_3}\circ\sigma_{b_1,b_2,b_3}\circ\sigma_{a_1,a_2,a_3}\,.$$

Corollary 8. *For any $a,b,c\in Q$, $\sigma_a\circ\sigma_b\circ\sigma_c=\sigma_c\circ\sigma_b\circ\sigma_a$.*

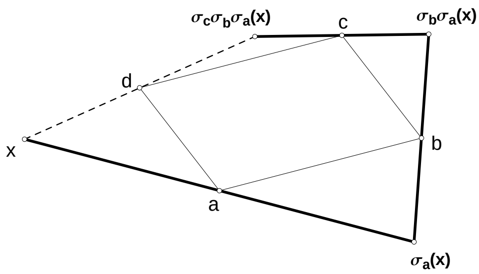

FIGURE 11. Corollary 9

Corollary 9. *For $a,b,c,d\in Q$, if* $\operatorname{Par}(a,b,c,d)$ *then $\sigma_c\circ\sigma_b\circ\sigma_a=\sigma_d$.* (Fig. 11)

It is known (in Euclidean geometry) that midpoints of sides of any quadrilateral form a parallelogram. We can state the same fact in terms of hexagonal quasigroup in the following way:

Theorem 9. *From* $\mathrm{M}(x,a,y)$, $\mathrm{M}(y,b,z)$, $\mathrm{M}(z,c,t)$ *and* $\operatorname{Par}(a,b,c,d)$ *it follows* $\mathrm{M}(x,d,t)$.

Proof. $\mathrm{M}(x,a,y)$, $\mathrm{M}(y,b,z)$ and $\mathrm{M}(z,c,t)$ are equivalent to $\sigma_a(x)=y$, $\sigma_b(y)=z$ and $\sigma_c(z)=t$ respectively. Therefore, the three assumptions can be written as: $\sigma_c\big(\sigma_b(\sigma_a(x))\big)=t$. From the preceding corollary it follows $\sigma_d(x)=t$, i.e. $\mathrm{M}(x,d,t)$. □

Theorem 10. *Let a_i,b_i,c_i,d_i, $i=1,2,3$ be points such that* $\operatorname{Par}(a_i,b_i,c_i,d_i)$, *for $i=1,2,3$, and a,b,c,d points satisfying* $\operatorname{Par}(a,b,c,d)$. *Then*

$$\operatorname{Par}\big(\sigma_{a_1,a_2,a_3}(a),\sigma_{b_1,b_2,b_3}(b),\sigma_{c_1,c_2,c_3}(c),\sigma_{d_1,d_2,d_3}(d)\big)\,.$$

Proof. From $\mathrm{Par}(a_1, b_1, c_1, d_1)$ and $\mathrm{Par}(a_3, b_3, c_3, d_3)$ and Theorem 1 it follows $\mathrm{Par}(a_1a_3, b_1b_3, c_1c_3, d_1d_3)$, and from $\mathrm{Par}(a, b, c, d)$ and $\mathrm{Par}(a_2, b_2, c_2, d_2)$ it follows $\mathrm{Par}(a_2a, b_2b, c_2c, d_2d)$. Similarly we obtain $\mathrm{Par}(a \cdot a_1a_3, b \cdot b_1b_3, c \cdot c_1c_3, d \cdot d_1d_3)$, and finally $\mathrm{Par}\big((a \cdot a_1a_3) \cdot a_2a, (b \cdot b_1b_3) \cdot b_2b, (c \cdot c_1c_3) \cdot c_2c, (d \cdot d_1d_3) \cdot d_2d\big)$, which proves the Theorem. □

We immediately have:

Corollary 10. *From* $\mathrm{Par}(a, b, c, d)$ *and* $\mathrm{Par}(p, q, r, s)$ *it follows*

$$\mathrm{Par}(\sigma_p(a), \sigma_q(b), \sigma_r(c), \sigma_s(d)).$$

Corollary 11. *For* $p, q, r \in Q$, *from* $\mathrm{Par}(a, b, c, d)$ *it follows*

$$\mathrm{Par}(\sigma_{p,q,r}(a), \sigma_{p,q,r}(b), \sigma_{p,q,r}(c), \sigma_{p,q,r}(d)).$$

Corollary 12. *For* $p \in Q$, *from* $\mathrm{Par}(a, b, c, d)$ *it follows*

$$\mathrm{Par}\left(\sigma_p(a), \sigma_p(b), \sigma_p(c), \sigma_p(d)\right).$$

Corollary 13. *For* $p, q, r \in Q$, *from* $\mathrm{M}(a, b, c)$ *it follows*

$$\mathrm{M}\left(\sigma_{p,q,r}(a), \sigma_{p,q,r}(b), \sigma_{p,q,r}(c)\right).$$

Corollary 14. *For* $p \in Q$, *from* $\mathrm{M}(a, b, c)$ *it follows* $\mathrm{M}\big(\sigma_p(a), \sigma_p(b), \sigma_p(c)\big)$.

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