ARCHIVUM MATHEMATICUM (BRNO)
Tomus 43 (2007), 197 – 229

THE GEOMETRY OF NEWTON'S LAW AND RIGID SYSTEMS

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Abstract. We start by formulating geometrically the Newton's law for a classical free particle in terms of Riemannian geometry, as pattern for subsequent developments. For constrained systems we have intrinsic and extrinsic viewpoints, with respect to the environmental space. Multi–particle systems are modelled on n-th products of the pattern model. We apply the above scheme to discrete rigid systems. We study the splitting of the tangent and cotangent environmental space into the three components of center of mass, of relative velocities and of the orthogonal subspace. This splitting yields the classical components of linear and angular momentum (which here arise from a purely geometric construction) and, moreover, a third non standard component. The third projection yields a new explicit formula for the reaction force in the nodes of the rigid constraint.

Introduction

The original approach to classical mechanics is based on the Newton's law. This is still used and popular mainly in the literature devoted to applied sciences and engineering, even if it is not very sophisticated from the mathematical and geometrical viewpoint (see, for instance, [9, 12, 13, 26]).

On the other hand, an approach to mechanics based on modern differential geometry has been developed and became more and more popular in the last decades. This viewpoint is achieved in terms of Riemannian, Lagrangian, Hamiltonian, symplectic, variational and jet geometry. A very huge literature exists in this respect (see, for instance, [1, 2, 3, 4, 6, 8, 10, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24]). These methods have been very successful for understanding several theoretical aspects and for the solution of several concrete problems, and have stimulated a large number of further classical and quantum theories.

2000 *Mathematics Subject Classification*: 70G45, 70B10, 70Exx.

Key words and phrases: classical mechanics, rigid system, Newton's law, Riemannian geometry.

Acknowledgements. This work has been partially supported by MIUR (Progetto PRIN 2003 "Sistemi integrabili, teorie classiche e quantistiche"), GNFM and GNSAGA of INdAM, and the Universities of Florence and Lecce.

Received October 4, 2006.

In this paper, our aims are more specific and foundational. Namely, we reformulate classical mechanics of a system with a finite number of particles and rigid systems, in terms of the Newton's law, in a way which, on one hand, is closer to the classical treatment of the subjects and, on the other hand, is expressed through the modern language of differential geometry.

Our approach is addressed both to differential geometers, who could easily get mechanical concepts written in their language, and to mathematical physicists, who are interested in a mathematically rigourous foundation of mechanics.

In fact, several ideas have been achieved independently by differential geometers and mathematical physicists in different contexts and with different purposes and languages. Sometimes, facts which appear in one of the two disciplines as easy and elementary may correspond to more difficult and fundamental facts in the other discipline. We believe that linking those facts provides a new insight on classical matters and yields new results as well. For the above reasons, from time to time, we recall some classical facts of one of the two areas which are possibly not very familiar to experts of the other area.

Thus, this paper, in spite of the sophisticated mathematical language, in comparison to the standard literatures of mechanics, analyses concrete mechanical contents.

On the other hand, this paper provides the classical background for a covariant approach to the quantisation of a rigid body, which is the subject of a subsequent paper [20].

The guideline of our approach is the description of mechanics of a system of n free and constrained particles, including a rigid system, in terms of the Riemannian formulation of mechanics of one particle.

We start by recalling the mechanics of one free particle moving in an affine Euclidean configuration space. We express the Newton's law in terms of covariant derivative. In several respects, it is convenient to introduce forces as forms (instead as vector fields) from the very beginning.

Then, we can naturally apply this Riemannian approach to the mechanics of a constrained particle. We have an intrinsic and an extrinsic viewpoint related to the embedding of the constrained configuration space into the environmental space. In particular, we use the Gauss' Theorem concerning the splitting of the Riemannian connection in order to get an explicit expression of the reaction force via the 2nd fundamental form of the constrained configuration space.

Next, we describe the mechanics of a system of n free particles, as one free particle moving in a higher dimensional product configuration space. For this purpose, it is necessary to introduce a weighted metric (besides the standard product metric). Of course, in the case on n free particles, we have the additional projection on the single particle spaces. Furthermore, we have the splitting of the configuration space into the affine component of the center of mass and the vector component of relative distances. The 1st splitting can be used to achieve information on the single particles and is orthogonal with respect to both metrics. The 2nd splitting has a fundamental role and is orthogonal only with respect to the weighted metric.

The systematic use of the weighted metric and of the above orthogonal splitting as a fundamental scheme seems to be original. In particular, we show that the classical concepts of total kinetic energy, total kinetic momentum, total force, etc. can be regarded as a direct consequence of the above geometric scheme.

Then, the formulation of a a constrained system of n particles can be easily obtained from the above scheme, by repeating the scheme of one free and constrained particle.

Eventually, a particular care is devoted to the analysis of a system of n particles with a rigid constraint.

First we study the geometry of the rigid configuration space, distinguishing the non degenerate and degenerate cases. Then, we formulate the kinematics and mechanics of a rigid system according to the above scheme. In particular, we show (Section 4.2) that the classical formula of the velocity of a rigid system (well–known in mechanics) can be regarded as the parallelisation of a Lie group (well–known in differential geometry). This fact yields an interpretation of the inertia tensor as a representative of the weighted metric induced by the parallelisation. In this context, we exhibit a new explicit intrinsic expression of the angular velocity via the inertia tensor (Corollary 4.8).

By combining the splitting of center of the mass and the splitting of the constrained configuration space, we obtain a splitting of the tangent and cotangent environmental configuration spaces into three components: the component of center of mass, the rotational component and a further orthogonal component to the configuration space (Theorem 4.10 and Theorem 4.11). This splitting is reflected on all objects of the rigid system mechanics, providing a clear geometric interpretation of some classical constructions of mechanics and new results as well. For instance, the total momentum of forms arises from our geometric scheme via the projection on the rotational component (Corollary 4.12). Moreover, a special application of the above splitting is the explicit expression of the reaction force on every node (Corollary 4.21). This formula seems to be new and possibly useful in engineering applications.

We assume all manifolds and maps to be C^∞. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are manifolds, then the sheaf of local smooth maps $\boldsymbol{M} \to \boldsymbol{N}$ is denoted by $\operatorname{map}(\boldsymbol{M}, \boldsymbol{N})$.

1. Preliminaries

In this paper we use a few non-standard mathematical constructions. In order to make the paper self-contained, we start with some introductory notions.

Scale spaces and units of measurement. In order to describe in a rigorous mathematical way the units of measurements and the coupling scales, we introduce the notion of "scale space" [11].

We define a *scale space* $\mathbb{U}$ as "positive 1-dimensional semi-vector space" over $I\!R^+$. Roughly speaking, this has the same algebraic structure as $I\!R^+$, but no distinguished generator over $I\!R^+$. We can naturally define the tensor product between scale spaces and ordinary vector spaces. Moreover, we can naturally define the rational powers $\mathbb{U}^{p/q}$ of a scale space $\mathbb{U}$. Rules analogous to those of real numbers hold for scale spaces; accordingly, we adopt analogous notation. In particular, we shall write $\mathbb{U}^0 := I\!R$, $\mathbb{U}^{-1} := \mathbb{U}^*$, $\mathbb{U}^p := \otimes^p \mathbb{U}$.

In our theory, these spaces will appear tensorialised with spacetime tensors. The scale spaces appearing in tensor products are not effected by differential operators, hence their elements can be treated as constants.

A *coupling scale* is defined to be a scale factor needed for allowing the equality of two scaled objects and a *unit of measurements* is defined to be a basis of a scale space.

We introduce the scale spaces $\mathbb{T}$ of *time intervals*, $\mathbb{L}$ of *lengths* and $\mathbb{M}$ of *masses*.

We will consider time units of measurement $u_0 \in \mathbb{T}$, or their duals $u^0 \in \mathbb{T}^*$.

Generalised affine spaces. In this paper, we need a more general definition of the standard notion. Namely, we introduce generalised affine spaces associated with (possibly non Abelian) groups. This generalisation is suitable for the description of the configuration space of rigid systems.

A (*left*) *generalised affine space* is defined to be a triple $(A, \mathrm{D}A, l)$, where A is a set, $\mathrm{D}A$ is a group and $l \colon \mathrm{D}A \times A \to A$ is a free and transitive left action. For the sake of simplicity, we often denote the generalised affine space $(A, \mathrm{D}A, l)$ just by A.

For each $o \in A$, the *left translation* $l_o \colon \mathrm{D}A \to A \colon g \mapsto go$ is invertible.

A *generalised affine map* is defined to be a map $f \colon A \to A'$ between generalised affine spaces, such that, for a certain $o \in A$, we have $f(a) = \mathrm{D}f(ao^{-1})f(o)$, for all $a \in A$, where $\mathrm{D}f \colon \mathrm{D}A \to \mathrm{D}A'$ is a group morphism. We can easily prove that, if such a $\mathrm{D}f$ exists, then it is unique and independent of the choice of o. We say $\mathrm{D}f$ to be the *generalised derivative* of f. For example, if $o \in A$, then the *left translation* $l_o \colon \mathrm{D}A \to A$ is a generalised affine map and its derivative is just the identity.

Now, let us consider a generalised affine space A associated with a Lie group G. Then, there is a unique smooth structure of A, such that the left translation $l \colon G \times A \to A$ be smooth. Let $\mathfrak{g}$ be the Lie algebra of G. It is easily proved that the affine space A is parallelisable through a natural isomorphism $TA \simeq A \times \mathfrak{g}$.

We recall that any manifold M which is endowed with a parallelisation $T\boldsymbol{M} \simeq \boldsymbol{M} \times \boldsymbol{F}$, has a natural linear connection $\nabla \colon TT\boldsymbol{M} \to VT\boldsymbol{M}$, where $VT\boldsymbol{M}$ is the space of vectors which are tangent to the fibres of the natural projection $T\boldsymbol{M} \to \boldsymbol{M}$. More precisely, we have $TT\boldsymbol{M} \simeq \boldsymbol{M} \times \boldsymbol{F} \times \boldsymbol{F} \times \boldsymbol{F}$, and ∇ is just the natural projection on the subspace $VT\boldsymbol{M} \simeq \boldsymbol{M} \times \boldsymbol{F} \times \{0\} \times \boldsymbol{F}$.

2. Mechanics of one particle

First, we review the one free and constrained particle mechanics as an introduction to our formalism and a pattern for next generalisations.

2.1. Free particle.

Configuration space. We define the *time* to be a 1-dimensional affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes I\!R$. We shall always refer to an affine chart (x^0) induced by an origin $t_0 \in \boldsymbol{T}$ and a time unit of measurement $u^0 \in \mathbb{T}$.

We define the *pattern configuration space* to be a 3-dimensional affine space $\boldsymbol{P}$ associated with an oriented vector space $\boldsymbol{S}$. We shall refer to a (local) chart (x^i) on $\boldsymbol{P}$. Latin indices $i\,, j\,, h\,, k$ will run from 1 to 3.

We shall also be involved with the tangent space $T\boldsymbol{P} = \boldsymbol{P} \times \boldsymbol{S}$ and the cotangent space $T^*\boldsymbol{P} = \boldsymbol{P} \times \boldsymbol{S}^*$. We shall refer to the local charts $(x^i, \dot{x}^i)$ of $T\boldsymbol{P}$ and $(x^i, \dot{x}_i)$ of $T^*\boldsymbol{P}$ and to the corresponding local bases of vector fields ∂_i and forms d^i. We also denote by $(x^i, \dot{x}^i, \overset{\shortmid}{x}{}^i, \ddot{x}^i)$ the induced chart of $TT\boldsymbol{P}$, with $(\partial_i, \dot{\partial}_i)$ and $(d^i, \dot{d}^i)$ the corresponding bases of vector fields and 1-forms; we have the chart $(x^i, \dot{x}^i, \ddot{x}^i)$ of $VT\boldsymbol{P}$.

The parallelisation of $\boldsymbol{P}$ induced by the affine structure yields a flat linear connection ∇ (see the Preliminaries).

We equip $\boldsymbol{S}$ with a scaled Euclidean metric $g \in \mathbb{L}^2 \otimes (\boldsymbol{S}^* \otimes \boldsymbol{S}^*)$, called *pattern metric*, which can be regarded as a scaled Riemannian metric of $\boldsymbol{P}$

$$g \colon \boldsymbol{P} \to \mathbb{L}^2 \otimes (\boldsymbol{T}^*\boldsymbol{P} \otimes \boldsymbol{T}^*\boldsymbol{P})\,.$$

We denote by $\bar{g}$ the corresponding contravariant metric. We have the coordinate expressions $g = g_{ij}\, d^i \otimes d^j$ and $\bar{g} = g^{ij}\, \partial_i \otimes \partial_j$, with $g_{ij} \in \operatorname{map}(\boldsymbol{P}, \mathbb{L}^2 \otimes I\!\!R)$ and $g^{ij} \in \operatorname{map}(\boldsymbol{P}, \mathbb{L}^{-2} \otimes I\!\!R)$. The associated *flat isomorphism* and its inverse, the *sharp isomorphism*, are denoted by $g^\flat \colon T\boldsymbol{P} \to \mathbb{L}^2 \otimes T^*\boldsymbol{P}$ and $g^\sharp \colon T^*\boldsymbol{P} \to \mathbb{L}^{-2} \otimes T\boldsymbol{P}$. The metric g and an orientation of $\boldsymbol{S}$ yield the scaled volume form $\eta \in \mathbb{L}^3 \otimes \Lambda^3 \boldsymbol{S}^*$ and its inverse $\bar{\eta} \in \mathbb{L}^{*3} \otimes \Lambda^3 \boldsymbol{S}$.

The Riemannian connection associated with g coincides with ∇. We denote the vertical projection associated with ∇ by $\nu \colon TT\boldsymbol{P} \to T\boldsymbol{P}$ and the Christoffel symbols by Γ^i_{hk}.

Kinematics. We define the *phase space* as the 1st jet space of maps $\boldsymbol{T} \to \boldsymbol{P}$

$$J_1\boldsymbol{P} := \boldsymbol{T} \times (\mathbb{T}^{-1} \otimes T\boldsymbol{P}) = \boldsymbol{T} \times \boldsymbol{P} \times (\mathbb{T}^{-1} \otimes \boldsymbol{S})\,.$$

The induced chart of $J_1\boldsymbol{P}$ is $(x^0,\, x^i,\, x^i_0)$.

A *motion* is defined to be a map $s \colon \boldsymbol{T} \to \boldsymbol{P}$.

The *1st differential*, the *2nd differential*, the *velocity* and the *acceleration* of a motion s are defined to be, respectively, the maps

$$ds \colon \boldsymbol{T} \to \mathbb{T}^{-1} \otimes T\boldsymbol{P}\,, \qquad d^2 s \colon \boldsymbol{T} \to \mathbb{T}^{-1} \otimes T(\mathbb{T}^{-1} \otimes T\boldsymbol{P})\,, \qquad j_1 s \colon \boldsymbol{T} \to J_1\boldsymbol{P}\,,$$
$$\nabla ds \colon \boldsymbol{T} \to \mathbb{T}^{-2} \otimes T\boldsymbol{P}\,.$$

By definition, we have $j_1 s(t) = (t, ds(t))$ and $\nabla ds = \nu \circ d^2 s$.

Moreover, by taking into account the splittings

$$\mathbb{T}^{-1} \otimes T\boldsymbol{P} \simeq \boldsymbol{P} \times (\mathbb{T}^{-1} \otimes \boldsymbol{S})\,,$$
$$\mathbb{T}^{-1} \otimes T(\mathbb{T}^{-1} \otimes T\boldsymbol{P}) \simeq \big(\boldsymbol{P} \times (\mathbb{T}^{-1} \otimes \boldsymbol{S})\big) \times \big((\mathbb{T}^{-1} \otimes \boldsymbol{S}) \times (\mathbb{T}^{-2} \otimes \boldsymbol{S})\big)\,,$$

we can write $ds = (s, Ds)$, $d^2 s = (s, Ds, Ds, D^2 s)$, $\nabla ds = (s, D^2 s)$, where $Ds \colon \boldsymbol{T} \to \mathbb{T}^{-1} \otimes \boldsymbol{S}$ is the standard derivative of s.

We have the coordinate expressions

$$(x^i, \dot{x}^i_0) \circ ds = (s^i,\, \partial_0 s^i)\,, \quad (x^i, \dot{x}^i_0, \overset{\shortmid}{x}{}^i_0, \ddot{x}^i_{00}) \circ d^2 s = (s^i,\, \partial_0 s^i,\, \partial_0 s^i,\, \partial^2_{00} s^i)\,,$$
$$(x^0, x^i, x^i_0) \circ j_1 s = (x^0,\, s^i,\, \partial_0 s^i)\,, \quad (x^i, \ddot{x}^i_{00}) \circ \nabla ds = (s^i,\, \partial^2_0 s^i + (\Gamma^i_{hk} \circ s)\, \partial_0 s^h\, \partial_0 s^k).$$

With reference to a mass $m \in \mathbb{M}$, we define the *kinetic energy* and the *kinetic momentum*, respectively, to be the maps

$$\mathcal{K} : \mathbb{T}^{-1} \otimes T\boldsymbol{P} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R \quad : v \mapsto \tfrac{1}{2}\, m\, g(v, v)$$
$$\mathcal{P} := D\mathcal{K} : \mathbb{T}^{-1} \otimes T\boldsymbol{P} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P} : v \mapsto m\, g^\flat(v)\,.$$

We have the coordinate expressions $\mathcal{K} = \frac{1}{2}\, m\, g_{ij}\, \dot{x}^i_0\, \dot{x}^j_0$ and $\mathcal{P} = m\, g_{ij}\, \dot{x}^i_0\, d^j$.

Dynamics. In our context, the force acting on a particle is given a priori on the the phase space. Moreover, it is convenient to introduce the force as a co–vector.

Thus, a *force* is defined to be a map

$$F : J_1\boldsymbol{P} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}\,.$$

The force F is said to be *conservative* if it factorises through $J_1\boldsymbol{P} \to \boldsymbol{P}$ and can be derived from a *potential* $\mathcal{U} \colon \boldsymbol{P} \to \mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M} \otimes I\!R$ by the equality $F = d\mathcal{U}$. If the force is conservative, then we define the associated *Lagrangian* to be the map $\mathcal{L} := \mathcal{K} + \mathcal{U}$.

We say that a motion s fulfills the *Newton's law of motion* if

$$m\, g^\flat\, (\nabla ds) = F \circ j_1 s\,. \tag{2.1}$$

It is remarkable that we can link the formulation of dynamics in terms of the connection ∇ with the Lagrangian approach, directly without any reference to variational or Lagrangian calculus. In fact, the following *Lagrange's formula* holds

$$m\, g^\flat\, (\nabla ds) = \big(D(\dot{\partial}_i \mathcal{K} \circ ds) - \partial_i \mathcal{K} \circ ds\big)\, (d^i \circ s)\,.$$

By the way, the above formula provides quickly the Christoffel's symbols of ∇.

Hence, the coordinate expression of the Newton's law is

$$m\, g_{ij}\, \big(\partial^2_0 s^j + (\Gamma^j_{hk} \circ s)\, \partial_0 s^h\, \partial_0 s^k\big) \;\equiv\; D(\dot{\partial}_i \mathcal{K} \circ ds) - \partial_i \mathcal{K} \circ ds = F_i \circ j_1 s\,.$$

In the particular case when the force is conservative, the Newton's law of motion is expressed by the *Lagrange equations*

$$D(\dot{\partial}_i \mathcal{L} \circ ds) - \partial_i \mathcal{L} \circ ds = 0\,.$$

2.2. **Constrained particle.**

We assume an embedded submanifold of the pattern Euclidean affine space as configuration space of a constrained particle.

The mechanics of a constrained particle has two features: an *intrinsic* and an *extrinsic* one. According to the intrinsic viewpoint, the particle behaves as a 'free' particle moving in an l-dimensional Riemannian manifold; hence, according to the intrinsic viewpoint, we can repeat the scheme of the previous section. On the other hand, the environment space adds an exterior geometric structure: the 2nd fundamental form, which measures the deviation of the submanifold from being an affine subspace of the environmental space. Then, according to the extrinsic viewpoint, we interpret the reaction force in terms of the 2nd fundamental form of the constrained space.

Configuration space. We define the *configuration space* for a constrained particle to be an embedded submanifold of dimension $1 \le l \le 3$

$$i_{\mathrm{con}} \colon \boldsymbol{P}_{\mathrm{con}} \hookrightarrow \boldsymbol{P}\,.$$

Thus, by definition of embedded submanifold, for each $p \in \boldsymbol{P}_{\mathrm{con}}$, there exists a chart (x^i) of $\boldsymbol{P}$ in a neighbourhood of p, such that $\boldsymbol{P}_{\mathrm{con}}$ is locally characterised by the constraint $\{x^{l+1} = 0, \dots, x^3 = 0\}$. Then, $(y^1, \dots, y^l) := (x^1|_{\boldsymbol{P}_{\mathrm{con}}}, \dots, x^l|_{\boldsymbol{P}_{\mathrm{con}}})$ turns out to be a local chart of $\boldsymbol{P}_{\mathrm{con}}$. The functions $y^1, \dots, y^l$ are said to be local *Lagrangian coordinates* and the functions $x^{l+1}, \dots, x^3$ to be local *constraints.* From now on, we shall refer to such adapted charts.

For practical reasons, we shall adopt the following convention:

– indices i, j, h, k will run from 1 to 3;

– indices a, b, c, d will run from 1 to l;

– indices r, s, t will run from l to 3.

We have $T\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} = \boldsymbol{P}_{\mathrm{con}} \times \boldsymbol{S}$ and $T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} = \boldsymbol{P}_{\mathrm{con}} \times \boldsymbol{S}^*$.

We have the natural injection $Ti_{\mathrm{con}} \colon T\boldsymbol{P}_{\mathrm{con}} \hookrightarrow T\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} \subset T\boldsymbol{P}$ and its dual projection $\pi := T^* i_{\mathrm{con}} \colon T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} \to T^*\boldsymbol{P}_{\mathrm{con}}$, with coordinate expressions $Ti_{\mathrm{con}}(X^a\, \partial_a) = X^a\, \partial_a$ and $\pi(\omega_i\, d^i) = \omega_a\, d^a$.

We consider the orthogonal subspaces

$$\begin{aligned} T^{\perp}\boldsymbol{P}_{\mathrm{con}} &:= \{X \in T\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} \mid g(X, T\boldsymbol{P}_{\mathrm{con}}) = 0\} \subset T\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}}\,,\\ T_{\perp}\boldsymbol{P}_{\mathrm{con}} &:= \{\alpha \in T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} \mid \alpha(T\boldsymbol{P}_{\mathrm{con}}) = 0\} \quad\ \subset T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}}\,. \end{aligned}$$

The vector fields ∂_a are tangent to $\boldsymbol{P}_{\mathrm{con}}$, while the vector fields ∂_r are transversal. If $\partial_r \in T^{\perp}\boldsymbol{P}_{\mathrm{con}}$, then the adapted chart (x^i) is said to be *orthogonal* to the submanifold.

The subspace $T_{\perp}\boldsymbol{P}_{\mathrm{con}}$ consists of the forms of the type $\omega = \sum_{l+1 \le r \le n} \omega_r\, d^r$, i.e. of forms whose "tangent" components vanish.

The restriction

$$g_{\mathrm{con}} := i^*_{\mathrm{con}} g \colon \boldsymbol{P}_{\mathrm{con}} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{P}_{\mathrm{con}} \otimes T^*\boldsymbol{P}_{\mathrm{con}})$$

of the pattern metric g to $\boldsymbol{P}_{\mathrm{con}}$ is a scaled Riemannian metric, which will be called the *intrinsic metric.* Its coordinate expression is $g_{\mathrm{con}} = (g_{\mathrm{con}})_{ab}\, d^a \otimes d^b$, where we have set $(g_{\mathrm{con}})_{ab} := g_{ab}|_{\boldsymbol{P}_{\mathrm{con}}}$. The contravariant form of g_{con} will be denoted by $\bar{g}_{\mathrm{con}}$. We stress that, in general, the $l \times l$ "tangent" submatrix of (g^{ij}) is different from the inverse of the matrix (g_{ab}); they are equal if and only if the adapted chart (x^i) is orthogonal.

The intrinsic metric g_{con} yields the Riemannian connection ∇_{con}.

With reference to a mass $m \in \mathbb{M}$, we define the *intrinsic kinetic energy* and the *intrinsic kinetic momentum*

$$\begin{aligned} \mathcal{K}_{\mathrm{con}} &:= i^*\mathcal{K} \colon \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{con}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!\!R \quad : v \mapsto \tfrac{1}{2}\, m\, g_{\mathrm{con}}(v, v)\,,\\ \mathcal{P}_{\mathrm{con}} &:= i^*\mathcal{P} \colon \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{con}} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P} \colon v \mapsto m\, g^{\flat}{}_{\mathrm{con}}(v)\,, \end{aligned}$$

with coordinate expressions $\mathcal{K}_{\mathrm{con}} = \frac{1}{2}\, m\, g_{\mathrm{con}\, ab}\, \dot{y}^a\, \dot{y}^b$ and $\mathcal{P}_{\mathrm{con}} = m\, g_{\mathrm{con}\, ab}\, \dot{y}^a\, d^b$.

Let us analyse the orthogonal splittings of the tangent and cotangent spaces induced by the metric.

The metric yields the injective map $\jmath\colon T^*\boldsymbol{P}_{\text{con}} \hookrightarrow T^*\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}}$ through the identity $j = g^\flat \circ \pi \circ g^\sharp{}_{\text{con}}$.

We have the mutually dual orthogonal splittings

$$T\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}} = T\boldsymbol{P}_{\text{con}} \underset{\boldsymbol{P}_{\text{con}}}{\oplus} T^\perp\boldsymbol{P}_{\text{con}}\,, \quad T^*\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}} = T^*\boldsymbol{P}_{\text{con}} \underset{\boldsymbol{P}_{\text{con}}}{\oplus} T_\perp\boldsymbol{P}_{\text{con}}\,,$$

with projections

$$\begin{aligned} \pi^\parallel &\colon T\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}} \to T\boldsymbol{P}_{\text{con}}\,, &\qquad \pi^\perp &\colon T\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}} \to T^\perp\boldsymbol{P}_{\text{con}} \\ \pi &\colon T^*\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}} \to T^*\boldsymbol{P}_{\text{con}}\,, &\qquad \pi_\perp &\colon T^*\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}} \to T_\perp\boldsymbol{P}_{\text{con}}\,. \end{aligned}$$

As the projection π has a very simple expression, it is convenient to compute the other projections $\pi^\parallel$, $\pi^\perp$, $\pi_\perp$ via the following identities: $\pi^\parallel = g^\sharp{}_{\text{con}} \circ \pi \circ g^\flat$ and $\pi_\perp = g^{\perp\flat} \circ \pi^\perp \circ g^\sharp$. Then, for each $X \in T\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}}$ and $\omega \in T^*\boldsymbol{P}\,|_{\boldsymbol{P}_{\text{con}}}$, we obtain the equalities

$$\begin{gathered} \jmath(\omega) = g_{cb}\, g^{ab}\, \omega_a\, d^c\,, \quad \pi^\parallel(X) = X^i\, g_{ib}\, g^{ba}_{\text{con}}\, \partial_a = (X^a + X^r\, g_{rb}\, g^{ba}_{\text{con}})\, \partial_a\,, \\ \pi^\perp(X) = X^r(\partial_r - g_{rb}\, g^{ba}_{\text{con}}\, \partial_a)\,, \quad \pi_\perp(\omega) = (\omega_r - \omega_a\, g^{ab}_{\text{con}}\, g_{br})\, d^r\,. \end{gathered}$$

Of course, the above formulas simplify considerably if the adapted chart is orthogonal, i.e. if $g_{rb}|_{\boldsymbol{P}_{\text{con}}} = 0$.

Kinematics. We define the *intrinsic phase space* as the 1st jet space of maps $\boldsymbol{T} \to \boldsymbol{P}_{\text{con}}$

$$J_1\boldsymbol{P}_{\text{con}} := \boldsymbol{T} \times (\mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\text{con}})\,.$$

The induced chart of $J_1\boldsymbol{P}_{\text{con}}$ is $(x^0,\, y^a,\, y^a_0)$.

A *constrained motion* is defined to be a map $s_{\text{con}}\colon \boldsymbol{T} \to \boldsymbol{P}_{\text{con}} \subset \boldsymbol{P}$. Clearly, a constrained motion can be naturally regarded as a motion of the pattern space, via the inclusion i_{con}. Indeed, a motion $s\colon \boldsymbol{T} \to \boldsymbol{P}$ is constrained if and only if $s^r = 0$.

The *1st differential*, the *2nd differential* and the *velocity* of a constrained motion s_{con}, computed in the environment space, turn out to be valued in the corresponding constrained subspaces

$$\begin{gathered} ds_{\text{con}}\colon \boldsymbol{T} \to \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\text{con}}\,, \qquad d^2 s_{\text{con}}\colon \boldsymbol{T} \to \mathbb{T}^{-1} \otimes T(\mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\text{con}})\,, \\ j_1 s_{\text{con}}\colon \boldsymbol{T} \to J_1\boldsymbol{P}_{\text{con}}\,. \end{gathered}$$

We have the coordinate expressions

$$\begin{gathered} (y^a, \dot{y}^a_0) \circ ds_{\text{con}} = (s^a,\, \partial_0 s^a)\,, \quad (y^a, \dot{y}^a_0, \dot{y}'^a_0, \ddot{y}^a_{00}) \circ d^2 s_{\text{con}} = (s^a,\, \partial_0 s^a,\, \partial_0 s^a,\, \partial^2_0 s^a) \\ (x^0, y^a, y^a_0) \circ j_1 s_{\text{con}} = (x^0,\, s^a,\, \partial_0 s^a)\,. \end{gathered}$$

Hence, as far as the above objects are considered, the intrinsic and the extrinsic approaches coincide, up to the natural inclusion of the constrained spaces into the corresponding environmental spaces.

Conversely, the intrinsic and extrinsic approaches of the acceleration of the constrained motion s_{con} do not coincide. The intrinsic viewpoint is suitable for the intrinsic expression of the law of motion and the extrinsic viewpoint provides the constraint reaction force.

We define the *intrinsic acceleration* of a constrained motion s_{con} as the map

$$\nabla_{\mathrm{con}} ds_{\mathrm{con}} \colon \boldsymbol{T} \to \mathbb{T}^{-2} \otimes T\boldsymbol{P}_{\mathrm{con}} \,,$$

with coordinate expression $\nabla_{\mathrm{con}} ds_{\mathrm{con}} = \big(\partial_0^2 s^a + (\Gamma_{\mathrm{con}}{}^a_{bc}) \circ s_{\mathrm{con}}\, \partial_0 s^b\, \partial_0 s^c\big)(\partial_a \circ s_{\mathrm{con}})$.

Analogously to the free case, the intrinsic co-acceleration is given by the Lagrange's formula $m\, g_{\mathrm{con}}{}^{\flat}(\nabla_{\mathrm{con}} ds) = \big(D(\dot\partial_a \mathcal{K}_{\mathrm{con}} \circ ds_{\mathrm{con}}) - \dot\partial_a \mathcal{K}_{\mathrm{con}} \circ ds_{\mathrm{con}}\big)(d^a \circ s_{\mathrm{con}})$.

On the other hand, by regarding the constrained motion s_{con} as a motion of the environmental space, we define the *extrinsic acceleration* as the map

$$\nabla ds_{\mathrm{con}} \colon \boldsymbol{T} \to \mathbb{T}^{-2} \otimes T\boldsymbol{P} \,,$$

with coordinate expression $\nabla ds_{\mathrm{con}} = \big(\partial_0^2 s^i + (\Gamma^i_{bc}) \circ s_{\mathrm{con}}\, \partial_0 s^b\, \partial_0 s^c\big)(\partial_i \circ s_{\mathrm{con}})$.

Then, according to the Gauss' Theorem [7], we have the splitting

$$\nabla ds_{\mathrm{con}} = \pi^{\|}(\nabla ds_{\mathrm{con}}) + \pi^{\perp}(\nabla ds_{\mathrm{con}}) \,,$$

with $\pi^{\|}(\nabla ds_{\mathrm{con}}) = \nabla_{\mathrm{con}} ds_{\mathrm{con}}$ and $\pi^{\perp} \circ \nabla ds_{\mathrm{con}} = N \circ ds_{\mathrm{con}}$, where

$$N \colon T\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} \to T^{\perp}\boldsymbol{P}_{\mathrm{con}}$$

is a quadratic map, called *2nd fundamental form*, whose coordinate expression is $N = \Gamma^r_{bc}\, \dot y^b\, \dot y^c\, (\partial_r - g_{rb}\, g^{ba}_{\mathrm{con}} \partial_a)$. The map N measures how the submanifold $\boldsymbol{P}_{\mathrm{con}}$ deviates, at 1st order, from being an affine subspace of $\boldsymbol{P}$. The quickest way to compute the 2nd fundamental form is the following: compute the covariant expressions of the extrinsic and intrinsic accelerations via the Lagrange's formulas; then pass to the contravariant expressions and take the difference.

Dynamics. Let us consider a force $\tilde F \colon J_1\boldsymbol{P} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}$ in the environment space $\boldsymbol{P}$. As we are dealing with constrained mechanics, we are involved only with its restriction

$$F := \tilde F|_{J_1\boldsymbol{P}_{\mathrm{con}}} \colon J_1\boldsymbol{P}_{\mathrm{con}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} \,.$$

According to the splitting of $T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}}$, we can write $F = F_{\mathrm{con}} + F_{\mathrm{con}\,\perp}$, where

$$F_{\mathrm{con}} = i^* F \colon J_1\boldsymbol{P}_{\mathrm{con}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\mathrm{con}} \,.$$

We call F_{con} the *intrinsic force*.

Let us assume that constraint confines the motion on the configuration space $\boldsymbol{P}_{\mathrm{con}}$, via Newton's law of motion, by means of a suitable additional 'reaction force' defined on the constrained space

$$R \colon J_1\boldsymbol{P}_{\mathrm{con}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}} \,.$$

According to the splitting of $T^*\boldsymbol{P}|_{\boldsymbol{P}_{\mathrm{con}}}$, we can write $R = R_{\mathrm{con}} + R_{\mathrm{con}\,\perp}$, where

$$R_{\mathrm{con}} = i^* R \colon J_1\boldsymbol{P}_{\mathrm{con}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\mathrm{con}} \,.$$

We call R_{con} the *intrinsic reaction force*.

A constrained motion $s_{\mathrm{con}} \colon \boldsymbol{T} \to \boldsymbol{P}_{\mathrm{con}}$ is said to fulfill the *constrained Newton's law of motion* if the following equation holds

$$m\, g^{\flat} \circ (\nabla ds_{\mathrm{con}}) = (F + R) \circ j_1 s_{\mathrm{con}} \,.$$

2.1. Theorem. *A constrained motion* $s \colon \boldsymbol{T} \to \boldsymbol{P}_{\rm con}$ *fulfills the* constrained Newton's law of motion *if and only if*

$$m\, g^{\flat}_{\rm con} \circ (\nabla_{\rm con} ds_{\rm con}) = (F_{\rm con} + R_{\rm con}) \circ j_1 s_{\rm con}$$
$$m\, g^{\flat} \circ (N \circ ds_{\rm con}) = (F_{{\rm con}\,\perp} + R_{{\rm con}\,\perp}) \circ j_1 s_{\rm con}\,.$$

Actually, for each choice of initial data in $J_1\boldsymbol{P}_{\rm con}$*, the 1st equation has* (*locally*) *a unique solution and the 2nd equation is fulfilled if and only if*

$$R_{{\rm con}\,\perp} = m\, g^{\flat} \circ N - F_{{\rm con}\,\perp}\,. \tag{2.2}$$

According with the above result, we make the "minimal assumption" (*virtual works principle*, i.e., *smooth constraint*) $R_{\rm con} = 0$. Then, the explicit coordinate expression of R is

$$\begin{aligned} R = \sum_{1\le a,b,c\le l}^{l+1\le r\le 3} & \big(m\,(\Gamma_{arb} - g_{rc}\, g^{cd}_{\rm con}\, \Gamma_{{\rm con}\; adb})\, \dot y^a\, \dot y^b \\ & - F_r + g^{ba}_{\rm con}\, F_a\, g_{br}\big) \circ i_{\rm con}\, (d^r \circ i_{\rm con})\,. \end{aligned} \tag{2.3}$$

In classical literature (see, for instance, [13]) the computation of reaction force is presented implicitly as the solution of a linear system associated with Lagrange multipliers. Instead, the above formula (which involves an adapted chart) provides an explicit expression of the reaction force in terms of the Christoffel symbols, or of the metric. In the particular case when the adapted chart is orthogonal, this formula becomes very easy.

Thus, the dynamics of a constrained particle can be interpreted as the dynamics of a 'free' particle moving on the Riemannian configuration space $\boldsymbol{P}_{\rm con}$. Then, all main notions and results holding in the free case by using the Riemannian structure of $\boldsymbol{P}$ can be easily rephrased in the constrained case. This is a remarkable conceptual and practical advantage of the present approach.

3. Mechanics of a system of n particles

In this section we generalise the previous concepts and results to systems of many particles. Our guideline will be the interpretation of the multi–particle system as a one–particle moving in a higher dimensional space. In this way, all we have learned for one-particle can be applied directly to multi–particle systems. On the other hand, we have additional concepts, e.g. the center of mass splitting, which follow from the projections on the factor spaces of the different particles.

3.1. **Free particles.**

We shall systematically use the prefix "multi" to indicate objects of the n–system analogous to objects of one-particle.

We assume $n \ge 1$, and consider n masses $m_1\,, \dots, m_n \in \mathbb{M}$. We define the *total mass* as $m_0 := \sum_i m_i \in \mathbb{M}$ and the i-th *weight* as $\mu_i = m_i/m_0$. Clearly, we have $\sum_i \mu_i = 1$. For each $i = 1, \dots, n$. with reference to the i-th particle, it is convenient to consider a copy of the following pattern objects: $\boldsymbol{P}_i \equiv \boldsymbol{P}$ and $\boldsymbol{S}_i \equiv \boldsymbol{S}$.

3.1.1. *Geometry of the multi–configuration space.*
The *multi–configuration space* is defined to be the product space

$$\boldsymbol{P}_{\text{mul}} := \boldsymbol{P}_1 \times \cdots \times \boldsymbol{P}_n \,.$$

Clearly, $\boldsymbol{P}_{\text{mul}}$ is an affine space associated with the vector space $\boldsymbol{S}_{\text{mul}} := \boldsymbol{S}_1 \times \cdots \times \boldsymbol{S}_n$.

A product chart (x^i_{mul}) of $\boldsymbol{P}_{\text{mul}}$ induced by charts $(x^i_1), \dots, (x^i_n)$ of the single components is said to be *without interference of the particles.* Conversely, a chart (x^i_{mul}) of $\boldsymbol{P}_{\text{mul}}$ which cannot be written as a product as above is said to be *with interference of the particles.* A typical notation for the elements of $\boldsymbol{P}_{\text{mul}}$ will be $p_{\text{mul}} = (p_1, \dots, p_n) \in \boldsymbol{P}_{\text{mul}}$, and analogously for $\boldsymbol{S}_{\text{mul}}$ and $\boldsymbol{S}^*_{\text{mul}}$.

We define the *multi–geometrical metric* and the *multi–weighted metric* as

$$g_{\text{mul}} \colon \boldsymbol{P}_{\text{mul}} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{P}_{\text{mul}} \otimes T^*\boldsymbol{P}_{\text{mul}}) \colon (u_{\text{mul}}, v_{\text{mul}}) \mapsto \sum_i g(u_i, v_i)\,; \tag{3.4}$$

$$G_{\text{mul}} \colon \boldsymbol{P}_{\text{mul}} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{P}_{\text{mul}} \otimes T^*\boldsymbol{P}_{\text{mul}}) \colon (u_{\text{mul}}, v_{\text{mul}}) \mapsto \sum_i \mu_i \, g(u_i, v_i)\,. \tag{3.5}$$

The contravariant tensors of g_{mul} and G_{mul} are denoted, respectively, by $\bar{g}_{\text{mul}}$ and $\bar{G}_{\text{mul}}$. Of course, if $n \geq 2$, then the two metrics are distinct.

For a system of n particles, we will rephrase the dynamics of a system of one particle, by replacing the pattern metric g with the weighted metric G_{mul} and the mass m with the total mass m_0. This procedure yields the correct Newton's law of motion, in full analogy with the one particle case.

According to this scheme, we define the *multi–kinetic energy* and the *multi–kinetic momentum* by

$$\begin{aligned} \mathcal{K}_{\text{mul}} &\colon \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\text{mul}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R &&\colon v \mapsto \tfrac{1}{2}\, m_0 \, G_{\text{mul}}(v, v)\,, \\ \mathcal{P}_{\text{mul}} &\colon \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\text{mul}} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\text{mul}} &&\colon v \mapsto m_0 \, G^\flat{}_{\text{mul}}(v)\,, \end{aligned}$$

and recover the standard formulas $\mathcal{K}_{\text{mul}}(v) = \sum_i \frac{1}{2}\, m_i \, g(v_i, v_i)$ and $\mathcal{P}_{\text{mul}}(v) = \big(m_i \, g(v_i)\big)$.

The linear connection ∇_{mul} induced on the multi–configuration space $\boldsymbol{P}_{\text{mul}}$ by the affine structure (see the Preliminaries) coincides with the Riemannian connection induced by both metrics g_{mul} and G_{mul}. This is true although the two metrics need not to be proportional.

The multi–configuration space has two distinguished splittings.

Multi–splitting. We have the obvious affine *multi–splitting* $\boldsymbol{P}_{\text{mul}} = \boldsymbol{P}_1 \times \ldots \times \boldsymbol{P}_n$. The corresponding affine projections $\pi_i \colon \boldsymbol{P}_{\text{mul}} \to \boldsymbol{P}_i$ and the further induced projections can be used to extract information on the single particles from the kinematical and dynamical multi–objects of the multi–system.

The subspaces $\boldsymbol{S}_1, \dots, \boldsymbol{S}_n \subset \boldsymbol{S}_{\text{mul}}$ are mutually orthogonal with respect to both metrics g_{mul} and G_{mul}.

Diagonal splitting. We have the following further *diagonal splitting* of $\boldsymbol{P}_{\text{mul}}$, which has no analogous in the one-particle scheme.

We define the *diagonal affine subspace*, the *diagonal vector subspace* and the *relative vector subspace*, respectively, as

$$\begin{aligned} \boldsymbol{P}_{\mathrm{dia}} &:= \{p_{\mathrm{mul}} \in \boldsymbol{P}_{\mathrm{mul}} \mid p_1 = \cdots = p_n\} \subset \boldsymbol{P}_{\mathrm{mul}}\,, \\ \boldsymbol{S}_{\mathrm{dia}} &:= \{v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}} \mid v_1 = \cdots = v_n\} \subset \boldsymbol{S}_{\mathrm{mul}}\,, \\ \boldsymbol{S}_{\mathrm{rel}} &:= \{v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}} \mid \sum_i \mu_i\, v_i = 0\} \quad \subset \boldsymbol{S}_{\mathrm{mul}}\,. \end{aligned}$$

In the following, the subscripts "${}_{\mathrm{dia}}$" and "${}_{\mathrm{rel}}$" will denote the objects associated with the above spaces.

3.1. Theorem. *We have the affine splitting of the multi–configuration space and the linear splittings of the associated vector and covector multi–spaces*

$$\boldsymbol{P}_{\mathrm{mul}} = \boldsymbol{P}_{\mathrm{dia}} \oplus \boldsymbol{S}_{\mathrm{rel}} : p_{\mathrm{mul}} = (p_0, \dots, p_0) + (p_1 - p_0, \, \dots, \, p_n - p_0)\,, \tag{3.6}$$

$$\boldsymbol{S}_{\mathrm{mul}} = \boldsymbol{S}_{\mathrm{dia}} \oplus \boldsymbol{S}_{\mathrm{rel}} : v_{\mathrm{mul}} = (v_0, \dots, v_0) + (v_1 - v_0, \, \dots, \, v_n - v_0)\,, \tag{3.7}$$

$$\begin{aligned} \boldsymbol{S}^*_{\mathrm{mul}} = \boldsymbol{S}^*_{\mathrm{dia}} \oplus \boldsymbol{S}^*_{\mathrm{rel}} : \alpha_{\mathrm{mul}} &= (\mu_1\, \alpha_0, \, \dots, \, \mu_n\, \alpha_0) \\ &\quad + (\alpha_1 - \mu_1\, \alpha_0, \, \dots, \, \alpha_n - \mu_n\, \alpha_0)\,, \end{aligned} \tag{3.8}$$

where, for each $p_{\mathrm{mul}} \in \boldsymbol{P}_{\mathrm{mul}}$*,* $p_0 \in \boldsymbol{P}$ *is the unique point such that*

$$\sum_i \mu_i\, (p_i - p_0) = 0\,,$$

and where, for each $v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}}$ *and* $\alpha_{\mathrm{mul}} \in \boldsymbol{S}^*_{\mathrm{mul}}$*,* $v_0 := \sum_i \mu_i\, v_i$ *and* $\alpha_0 := \sum_i \alpha_i$*.*

The above splittings are orthogonal with respect to the weighted metric G_{mul}*.*

Proof. Let us prove the splitting $\boldsymbol{S}_{\mathrm{mul}} = \boldsymbol{S}_{\mathrm{dia}} \oplus \boldsymbol{S}_{\mathrm{rel}}$. For each $v_0 \in \boldsymbol{S}$, we have $\sum_i \mu_i v_0 = 0$ if and only if $v_0 = 0$. Hence, $\boldsymbol{S}_{\mathrm{dia}} \cap \boldsymbol{S}_{\mathrm{rel}} = 0$. Moreover, for each $v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}}$, we have $(v_1, \dots, v_n) = (v_0, \dots, v_0) + (v_1 - v_0, \, \dots, \, v_n - v_0)$, with any $v_0 \in \boldsymbol{S}$. Clearly, $(v_0, \dots, v_0) \in \boldsymbol{S}_{\mathrm{dia}}$ and $(v_1 - v_0 \, \dots, \, v_n - v_0) \in \boldsymbol{S}_{\mathrm{rel}}$, if and only if $v_0 = \sum_i \mu_i v_i$. Hence, $\boldsymbol{S}_{\mathrm{dia}} + \boldsymbol{S}_{\mathrm{rel}} = \boldsymbol{S}_{\mathrm{mul}}$ and the expression of the splitting is expressed by the 2nd formula of the statement.

The splitting $\boldsymbol{P}_{\mathrm{mul}} = \boldsymbol{P}_{\mathrm{dia}} \oplus \boldsymbol{S}_{\mathrm{rel}}$ may be proved in a similar way.

The splitting $\boldsymbol{S}_{\mathrm{mul}} = \boldsymbol{S}_{\mathrm{dia}} \oplus \boldsymbol{S}_{\mathrm{rel}}$ implies the splitting $\boldsymbol{S}^*_{\mathrm{mul}} = \boldsymbol{S}^*_{\mathrm{dia}} \oplus \boldsymbol{S}^*_{\mathrm{rel}}$. Moreover, for each $\alpha_{\mathrm{mul}} \in \boldsymbol{S}^*_{\mathrm{mul}}$. we have $(\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n) + (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$. with any $\beta_i \in \boldsymbol{S}^*_i$. On the other hand, for each $v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}}$, we obtain $(\beta_1, \dots, \beta_n)(v_1, \dots, v_n) = (\alpha_1, \dots, \alpha_n)(v_0, \dots, v_0)$ if and only if $\sum_i \beta_i\,(v_i) = \sum_i \mu_i\, (\sum_j \alpha_j)(v_i)$, i.e., if and only if $\beta_i = \mu_i\, (\sum_j \alpha_j)$. In virtue of the equality $\sum_i \big(\alpha_i - \mu_i\, (\sum_j \alpha_j)\big)(v_i) = \sum_i \alpha_i (v_i - v_0)$, we obtain

$$\begin{aligned} &\Big(\alpha_1 - \mu_1\, (\sum_j \alpha_j), \dots, \alpha_n - \mu_n\, (\sum_j \alpha_j)\Big)(v_1, \dots, v_n) \\ &\qquad = (\alpha_1, \dots, \alpha_n)(v_1 - v_0, \dots, v_n - v_0)\,. \end{aligned}$$

□

We denote the projection associated with the above splittings by

$$\begin{aligned} \pi_{\mathrm{dia}} &: \boldsymbol{P}_{\mathrm{mul}} \to \boldsymbol{P}_{\mathrm{dia}}\,, & \pi_{\mathrm{rel}} &: \boldsymbol{P}_{\mathrm{mul}} \to \boldsymbol{S}_{\mathrm{rel}}\,, \\ \bar\pi_{\mathrm{dia}} &: \boldsymbol{S}_{\mathrm{mul}} \to \boldsymbol{S}_{\mathrm{dia}}\,, & \bar\pi_{\mathrm{rel}} &: \boldsymbol{S}_{\mathrm{mul}} \to \boldsymbol{S}_{\mathrm{rel}}\,, \\ \bar\pi^*_{\mathrm{dia}} &: \boldsymbol{S}^*_{\mathrm{mul}} \to \boldsymbol{S}^*_{\mathrm{dia}}\,, & \bar\pi^*_{\mathrm{rel}} &: \boldsymbol{S}^*_{\mathrm{mul}} \to \boldsymbol{S}^*_{\mathrm{rel}}\,. \end{aligned}$$

Clearly, the above splittings depend on the choice of the multi–mass and are not orthogonal with respect to the geometrical metric (unless all masses are equal).

We stress that, while we have the natural inclusion $\boldsymbol{P}_{\mathrm{dia}} \hookrightarrow \boldsymbol{P}_{\mathrm{mul}}$, we do not have a natural inclusion $\boldsymbol{S}_{\mathrm{rel}} \hookrightarrow \boldsymbol{P}_{\mathrm{mul}}$.

We have a natural splitting of the weighted multi–metric of the type $G_{\mathrm{mul}} = G_{\mathrm{dia}} \oplus G_{\mathrm{rel}}$. Moreover, the affine structures of $\boldsymbol{P}_{\mathrm{dia}}$ and $\boldsymbol{S}_{\mathrm{rel}}$ yield the flat connections ∇_{dia} and ∇_{rel}, which turn out to be the Riemannian connections induced by G_{dia} and G_{rel}, respectively.

Center of mass splitting. We can describe the diagonal splitting in another way, via the center of mass. According to the above Theorem, we define the *center of mass* of $p_{\mathrm{mul}} \in \boldsymbol{P}_{\mathrm{mul}}$ to be the unique point $p_0 \in \boldsymbol{P}$, such that $\sum_i \mu_i\,(p_i - p_0) = 0$. By considering any $o \in \boldsymbol{P}$, we can write $p_0 = o + \sum_i \mu_i\,(p_i - o)$. With reference to the center of mass, it is convenient to consider a copy of the following pattern objects: $\boldsymbol{P}_{\mathrm{cen}} \equiv \boldsymbol{P}$, $\boldsymbol{S}_{\mathrm{cen}} \equiv \boldsymbol{S}$ and $\boldsymbol{S}^*_{\mathrm{cen}} \equiv \boldsymbol{S}^*$. Thus, we have the *center of mass* affine projection

$$\pi_{\mathrm{cen}} : \boldsymbol{P}_{\mathrm{mul}} \to \boldsymbol{P}_{\mathrm{cen}} : p_{\mathrm{mul}} \mapsto p_0 := o + \sum_i \mu_i\,(p_i - o)\,, \qquad \text{for any } o \in \boldsymbol{P}\,.$$

The linear projections associated with the affine projection $\pi_{\mathrm{cen}} : \boldsymbol{P}_{\mathrm{mul}} \to \boldsymbol{P}_{\mathrm{cen}}$ turn out to be, respectively, the *weighted sum* and the *sum*

$$\bar\pi_{\mathrm{cen}} : \boldsymbol{S}_{\mathrm{mul}} \to \boldsymbol{S}_{\mathrm{cen}} : v_{\mathrm{mul}} \mapsto v_0 := \sum_i \mu_i\, v_i\,,$$

$$\mathsf{S}_{\mathrm{cen}} : \boldsymbol{S}^*_{\mathrm{mul}} \to \boldsymbol{S}^*_{\mathrm{cen}} : \alpha_{\mathrm{mul}} \mapsto \alpha_0 := \sum_i \alpha_i\,.$$

Clearly, we have the natural affine isomorphism and linear isomorphisms

$$\begin{aligned} \boldsymbol{P}_{\mathrm{cen}} \to \boldsymbol{P}_{\mathrm{dia}} &: p_0 \mapsto (p_0, \dots, p_0)\,, \\ \boldsymbol{S}_{\mathrm{cen}} \to \boldsymbol{S}_{\mathrm{dia}} &: v_0 \mapsto (v_0, \dots, v_0)\,, \\ \boldsymbol{S}^*_{\mathrm{cen}} \to \boldsymbol{S}^*_{\mathrm{dia}} &: \beta_0 \mapsto (\mu_1\,\beta_0\,, \dots, \mu_n\,\beta_0)\,. \end{aligned}$$

3.2. Corollary. We have the *center of mass splittings*

$$\begin{aligned} \boldsymbol{P}_{\mathrm{mul}} \simeq \boldsymbol{P}_{\mathrm{cen}} \times \boldsymbol{S}_{\mathrm{rel}} : p_{\mathrm{mul}} &\simeq \big(p_0\,,\ (p_1 - p_0, \dots, p_n - p_0)\big)\,, \\ \boldsymbol{S}_{\mathrm{mul}} \simeq \boldsymbol{S}_{\mathrm{cen}} \times \boldsymbol{S}_{\mathrm{rel}} : v_{\mathrm{mul}} &\simeq \big(v_0\,,\ (v_1 - v_0, \dots, v_n - v_0)\big)\,, \\ \boldsymbol{S}^*_{\mathrm{mul}} \simeq \boldsymbol{S}^*_{\mathrm{cen}} \times \boldsymbol{S}^*_{\mathrm{rel}} : \alpha_{\mathrm{mul}} &\simeq \big(\alpha_0\,,\ (\alpha_1 - \mu_1\,\alpha_0, \dots, \alpha_n - \mu_n\,\alpha_0)\big)\,. \end{aligned}$$

According to our scheme, we define the *center of mass kinetic energy* and the *center of mass kinetic momentum* as

$$\mathcal{K}_{\mathrm{cen}} \colon \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{cen}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R \qquad : v_{\mathrm{cen}} \mapsto \tfrac{1}{2}\, m_0\, g_{\mathrm{cen}}(v_{\mathrm{cen}}, v_{\mathrm{cen}})\,,$$
$$\mathcal{P}_{\mathrm{cen}} \colon \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{cen}} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\mathrm{cen}} \colon v_{\mathrm{cen}} \mapsto m_0\, g_{\mathrm{cen}}(v_{\mathrm{cen}})\,.$$

Moreover, we obtain the equality

$$\mathcal{K}_{\mathrm{mul}}(v_{\mathrm{mul}}) = \mathcal{K}_{\mathrm{cen}}(v_0) + \mathcal{K}_{\mathrm{mul}}(v_{\mathrm{rel}})\,,$$

3.1.2. *Kinematics.*

We define the *multi–phase space* as $J_1\boldsymbol{P}_{\mathrm{mul}} := \boldsymbol{T} \times (\mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{mul}})$.

A *multi–motion* is defined to be a map $s_{\mathrm{mul}} \colon \boldsymbol{T} \to \boldsymbol{P}_{\mathrm{mul}}$. Of course, the multi–motion can be regarded as the family of motions of the system: $s_{\mathrm{mul}} = (s_1, \ldots, s_n)$. This holds also for the derived quantities, like the *multi–velocity* $ds_{\mathrm{mul}} \colon \boldsymbol{T} \to \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{mul}}$ and the *multi–acceleration* $\nabla_{\mathrm{mul}} ds_{\mathrm{mul}} \colon \boldsymbol{T} \to \mathbb{T}^{-2} \otimes T\boldsymbol{P}_{\mathrm{mul}}$. We can relate the multi–motion to the splitting of center of mass (Corollary 3.2) by the equalities

$$s_{\mathrm{mul}} = s_{\mathrm{dia}} + s_{\mathrm{rel}} \simeq (s_{\mathrm{cen}}\,, s_{\mathrm{rel}})\,, \qquad ds_{\mathrm{mul}} = ds_{\mathrm{dia}} + ds_{\mathrm{rel}} \simeq (ds_{\mathrm{cen}}\,, ds_{\mathrm{rel}})\,.$$

$$\nabla_{\mathrm{mul}} ds_{\mathrm{mul}} = \nabla_{\mathrm{dia}}\, ds_{\mathrm{dia}} + \nabla_{\mathrm{rel}}\, ds_{\mathrm{rel}} \simeq (\nabla_{\mathrm{cen}}\, ds_{\mathrm{cen}}\,, \nabla_{\mathrm{rel}}\, ds_{\mathrm{rel}})\,.$$

3.1.3. *Dynamics.*
In analogy with the case of one–particle, we define a *multi–force* to be a map

$$F_{\mathrm{mul}} \colon J_1\boldsymbol{P}_{\mathrm{mul}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\mathrm{mul}}\,.$$

Again, the multi–force can be regarded as the family of forces acting on each particle $F_{\mathrm{mul}} = (F_1, \ldots, F_n)$. In general, each of the components is defined on the whole phase space. In the particular case when each component F_i of the multi–force depends only the i-th phase space, the multi–force is said to be *without interaction*.

We say that a multi–force F_{mul} fulfills the *Newton's 3rd principle* if, for each $1 \le i \le n$,

$$F_i = \sum_{1 \le i \ne j \le n} F_{ij}\,, \qquad F_{ij}(p_i, p_j) = \lambda_{ij}(\|p_j - p_i\|_g)\; g^\flat(p_j - p_i)\,, \qquad \lambda_{ij} = \lambda_{ji}\,,$$

where $F_{ij} \colon \boldsymbol{P}_i \times \boldsymbol{P}_j \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_i$ and $\lambda_{ij} \colon \mathbb{L}^2 \otimes I\!R \to \mathbb{T}^2 \otimes \mathbb{M}$, for each $1 \le i \ne j \le n$.

The *total force* of the system is defined to be the component of the multi–force with respect to the center of mass

$$F_{\mathrm{cen}} := \mathcal{S}_{\mathrm{cen}} \circ F_{\mathrm{mul}} = \sum_i F_i \colon J_1\boldsymbol{P}_{\mathrm{mul}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\mathrm{cen}}\,.$$

The multi–force is said to be *conservative* if it can be derived from a *multi–potential* $\mathcal{U}_{\mathrm{mul}} \colon \boldsymbol{P}_{\mathrm{mul}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R$ as $F_{\mathrm{mul}} = d\mathcal{U}_{\mathrm{mul}}$. In this case, we define the *multi–Lagrangian* to be the map

$$\mathcal{L}_{\mathrm{mul}} := \mathcal{K}_{\mathrm{mul}} + \mathcal{U}_{\mathrm{mul}} \colon T\boldsymbol{P}_{\mathrm{mul}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R\,.$$

We say that a multi–motion s_{mul} fulfills the *Newton's law of motion* if

$$m_0\, G^\flat_{\text{mul}} \circ (\nabla_{\text{mul}}\, ds_{\text{mul}}) = F_{\text{mul}} \circ j_1 s_{\text{mul}}\,. \tag{3.9}$$

We can split the Newton's law with respect to the multi–splitting and to the splitting of the center of mass. In the former case, we simply obtain the system of coupled equations $m_i\, g^\flat \circ ds_i = F_i \circ j_1 s$. In the latter case, we have the following Theorem.

3.3. Theorem. *The Newton's equation is equivalent to the system*

$$\begin{aligned} m_0\, g^\flat_{\text{cen}} \circ (\nabla_{\text{cen}}\, ds_{\text{cen}}) &= F_{\text{cen}} \circ j_1 s_{\text{mul}}\,, \\ m_0\, G^\flat_{\text{rel}} \circ (\nabla_{\text{rel}}\, ds_{\text{rel}}) &= F_{\text{rel}} \circ j_1 s_{\text{mul}}\,. \end{aligned} \tag{3.10}$$

If F_{cen} factors through $J_1\boldsymbol{P}_{\text{cen}}$, then the 1st equation can be integrated independently of the 2nd one and can be interpreted as the equation of motion of the center of mass. As in the one-particle case, the coordinate expression of the Newton's law is, with reference to any chart (x^i_{mul}) of $\boldsymbol{P}_{\text{mul}}$,

$$D(\dot\partial_i \mathcal{K}_{\text{mul}} \circ ds_{\text{mul}}) - \partial_i \mathcal{K}_{\text{mul}} \circ ds_{\text{mul}} = F_i \circ j_1 s_{\text{mul}}\,,$$

which, if F_{mul} is conservative, is equivalent to the system of Lagrange's equations

$$D(\dot\partial_i \mathcal{L}_{\text{mul}} \circ ds_{\text{mul}}) - \partial_i \mathcal{L}_{\text{mul}} \circ ds_{\text{mul}} = 0\,,$$

3.2. **Constrained particles.**

According to our programme, the analysis of the geometry of the constrained space for a system of n particles can be carried out by analogy with the case of one-particle.

We assume the *multi–configuration space* of a constrained system of n particles to be an embedded submanifold $\boldsymbol{P}_{\text{mul con}} \subset \boldsymbol{P}_{\text{mul}}$.

In general, it is not possible to write $\boldsymbol{P}_{\text{mul con}} = \boldsymbol{P}_{\text{con } 1} \times \ldots \times \boldsymbol{P}_{\text{con } n}$, with $\boldsymbol{P}_{\text{con } i} \subset \boldsymbol{P}_i$. for each $i = 1, \ldots, n$. In the particular case when this holds, we say that the constraint is *without interference between particles*.

Moreover, in general, it is not possible to write $\boldsymbol{P}_{\text{mul con}} = \boldsymbol{P}_{\text{cen con}} \times \boldsymbol{S}_{\text{rel con}}$, with $\boldsymbol{P}_{\text{cen con}} \subset \boldsymbol{P}_{\text{cen}}$ and $\boldsymbol{S}_{\text{rel con}} \subset \boldsymbol{S}_{\text{rel}}$. In the particular case when this holds, we say that the constraint is *without interference between center of mass and relative positions*. In this case, the intrinsic metric G_{con} splits into the sum of the metrics $G_{\text{cen con}}$ and $G_{\text{rel con}}$ (according to Corollary 3.2), with interesting consequences in dynamics.

We leave to the reader the task to formulate the kinematics and dynamics of a constrained system of n particles according to our scheme.

4. Rigid systems

Now, we specialise the theory of constrained systems of n particles to the case of a rigid constraint.

We devote emphasis to the geometric structure of the rigid configuration space. In particular, we show that this is the true source of the classical formulas of the velocity of rigid systems.

Throughout the section, we suppose that $n \geq 2$.

4.1. **Geometry of the configuration space.**
Let us define the scaled functions, for $1 \le i, j \le n$,

$$r_{ij} : \boldsymbol{P}_{\text{mul}} \to \mathbb{L} \otimes I\!R : p_{\text{mul}} \equiv (p_1, \dots, p_n) \mapsto \|p_i - p_j\|_g .$$

A *rigid configuration space* is defined to be a subset of the type

$$i_{\text{rig}} : \boldsymbol{P}_{\text{rig}} := \{p_{\text{mul}} \in \boldsymbol{P}_{\text{mul}} \mid r_{ij}(p_{\text{mul}}) = l_{ij} , 1 \le i < j \le n\} \subset \boldsymbol{P}_{\text{mul}} ,$$

where $l_{ij} \in \mathbb{L}$ fulfill

$$\begin{aligned} l_{ij} &= l_{ji} , && 1 \le i, j \le n , && i \ne j , \\ l_{ik} &\le l_{ij} + l_{jk} , && 1 \le i, j, k \le n , && i \ne j, j \ne k, k \ne i . \end{aligned}$$

Note that we have excluded the case in which the positions of different particles coincide.

From now on, let us consider a given rigid configuration space $\boldsymbol{P}_{\text{rig}}$.

We define the *rotational space* to be the subset

$$i_{\text{rot}} : \boldsymbol{S}_{\text{rot}} := \big\{v_{\text{rel}} \in \boldsymbol{S}_{\text{rel}} \mid \|v_i - v_j\| = l_{ij} , 1 \le i < j \le n\big\} \subset \boldsymbol{S}_{\text{rel}} .$$

A typical notation for the elements of $\boldsymbol{S}_{\text{rot}}$ will be $r_{\text{rot}} = (r_1, \dots, r_n) \in \boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rel}}$.

Due to the equality $\|p_i - p_j\| = \|\pi_{\text{rel}}(p_{\text{mul}})_i - \pi_{\text{rel}}(p_{\text{mul}})_j\| = l_{ij}$, the rigid constraint does not involve the center of mass but only relative positions. Then, the restrictions of the projections $\pi_{\text{cen}} : \boldsymbol{P}_{\text{mul}} \to \boldsymbol{P}_{\text{cen}}$ and $\pi_{\text{rel}} : \boldsymbol{P}_{\text{mul}} \to \boldsymbol{S}_{\text{rel}}$ to the subset $\boldsymbol{P}_{\text{rig}} \subset \boldsymbol{P}_{\text{mul}}$ yield the bijection

$$(\pi_{\text{cen}} , \pi_{\text{rot}}) : \boldsymbol{P}_{\text{rig}} \to \boldsymbol{P}_{\text{cen}} \times \boldsymbol{S}_{\text{rot}} .$$

Next, we classify the rigid constraints.

For each $v_{\text{rel}} \in \boldsymbol{S}_{\text{rel}}$, we define the vector subspace

$$\langle v_{\text{rel}} \rangle := \operatorname{span}\{v_i - v_j \mid 1 \le i, j \le n\} \subset \boldsymbol{S} .$$

If $r_{\text{rot}} \in \boldsymbol{S}_{\text{rot}}$, then we call $\langle r_{\text{rot}} \rangle$ the *characteristic space* of $\langle r_{\text{rot}} \rangle$. It can be proved [5, p. 257] that the dimension of the characteristic spaces does not depend on elements in $\boldsymbol{S}_{\text{rot}}$, but only on $\boldsymbol{S}_{\text{rot}}$. More precisely, for each r_{rot}, $r'_{\text{rot}} \in \boldsymbol{S}_{\text{rot}}$ there exists an isometry $\phi : \boldsymbol{S} \to \boldsymbol{S}$ sigh that $\phi(r_i) = r'_i$ for $i = 1, \dots, n$.

We define the *characteristic* of $\boldsymbol{P}_{\text{rig}}$ to be the integer number $C_{\boldsymbol{P}_{\text{rig}}} := \dim \langle r_{\text{rot}} \rangle$, where $r_{\text{rot}} \in \boldsymbol{S}_{\text{rot}}$. Obviously, we have $1 \le C_{\boldsymbol{P}_{\text{rig}}} \le 3$, and we can classify the rigid configuration space in terms of $C_{\boldsymbol{P}_{\text{rig}}}$. We say $\boldsymbol{P}_{\text{rig}}$ to be

– *strongly non degenerate* if $C_{\boldsymbol{P}_{\text{rig}}} = 3$,
– *weakly non degenerate* if $C_{\boldsymbol{P}_{\text{rig}}} = 2$,
– *degenerate* if $C_{\boldsymbol{P}_{\text{rig}}} = 1$.

Of course, if $n = 2$, then $\boldsymbol{P}_{\text{rig}}$ is degenerate; if $n = 3$, then $\boldsymbol{P}_{\text{rig}}$ can be degenerate or weakly non degenerate.

Thus, by considering all particles as assuming positions in the same space $\boldsymbol{P}$, the above cases correspond respectively to the case when the minimal affine subspace containing all particles is a line, or a plane, or the whole $\boldsymbol{P}$. As a consequence of the above result, the case occurring for a given rigid system does not change during the motion.

Let us denote by $O(\boldsymbol{S}, g)$ the group of orthogonal transformations of $\boldsymbol{S}$ with respect to g. We want to study the topological subspace $\boldsymbol{S}_{\rm rot} \subset \boldsymbol{S}_{\rm rel}$ through the natural action of the Lie group $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$. More precisely, we can easily prove that the map

$$O(\boldsymbol{S}, g) \times \boldsymbol{S}_{\rm rot} \to \boldsymbol{S}_{\rm rot} : (\phi, r_{\rm rot}) \mapsto \big(\phi(r_1), \dots, \phi(r_n)\big)$$

is well–defined and yields a continuous action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$. Such an action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$ is transitive because of the above discussion. Let us denote the isotropy group at $r_{\rm rot}$ by $H(r_{\rm rot}) \subset O(\boldsymbol{S}, g)$. Then:

– in the *strongly non degenerate case* the isotropy subgroup $H[r_{\rm rot}]$ is the trivial subgroup $\{1\}$;

– in the *weakly non degenerate case* the isotropy subgroup $H[r_{\rm rot}]$ is the discrete subgroup of reflections with respect to $\langle r_{\rm rot} \rangle$;

– in the *degenerate case* the isotropy subgroup $H[r_{\rm rot}]$ is the 1 dimensional subgroup of rotations whose axis is $\langle r_{\rm rot} \rangle$; we stress that this subgroup is not normal.

4.1. Proposition. The following facts hold:

– $\boldsymbol{S}_{\rm rot}$ is *strongly non degenerate* if and only if the action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$ is free; in this case $\boldsymbol{S}_{\rm rot}$ is an affine space associated with the group $O(\boldsymbol{S}, g)$;

– $\boldsymbol{S}_{\rm rot}$ is *weakly non degenerate* if and only if the action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$ is not free, but the action of the subgroup $SO(\boldsymbol{S}, g) \subset O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$ is free; in this case $\boldsymbol{S}_{\rm rot}$ is an affine space associated with the group $SO(\boldsymbol{S}, g)$;

– $\boldsymbol{S}_{\rm rot}$ is *degenerate* if and only if the action of $SO(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$ is not free; in this case $\boldsymbol{S}_{\rm rot}$ is a homogeneous space (i.e. the quotient of a Lie group with respect to a closed subgroup) with two possible distinguished diffeomorphisms (depending on a chosen orientation on the straight line of the rigid system) with the unit sphere $\mathbf{S}^2 \subset \mathbb{L}^* \otimes \boldsymbol{S}$, with respect to the metric g.

In particular, in all cases $\boldsymbol{S}_{\rm rot}$ turns out to be a manifold.

In the non degenerate case, the choice of a configuration $r_{\rm rot} \in \boldsymbol{S}_{\rm rot}$ and of a scaled orthonormal basis in $\mathbb{L}^* \otimes \boldsymbol{S}$ yield the diffeomorphisms (via the action of $O(\boldsymbol{S}, g)$ on $\boldsymbol{S}_{\rm rot}$)

$$\begin{aligned} \boldsymbol{S}_{\rm rot} &\simeq O(\boldsymbol{S}, g) \quad \simeq O(3)\,, && \text{in the strongly non degenerate case;} \\ \boldsymbol{S}_{\rm rot} &\simeq SO(\boldsymbol{S}, g) \simeq SO(3)\,, && \text{in the weakly non degenerate case.} \end{aligned}$$

In the degenerate case, the continuous choice of an orientation on the straight lines $\langle r_{\rm rot} \rangle \subset \boldsymbol{S}$ generated by each configuration $r_{\rm rot} \in \boldsymbol{S}_{\rm rot}$ and of a scaled orthonormal basis in $\mathbb{L}^* \otimes \boldsymbol{S}$ yields the diffeomorphisms

$$\boldsymbol{S}_{\rm rot} \simeq \mathbf{S}^2(\mathbb{L}^* \otimes \boldsymbol{S}, g) \simeq \mathbf{S}^2(3)\,.$$

From now on, in the non degenerate case, we shall refer only to one of the two connected components of $\boldsymbol{S}_{\rm rot}$, for the sake of simplicity and for physical reasons of continuity. Accordingly, we shall just refer to the non degenerate case (without specification of strongly or weakly non degenerate), or to the degenerate case.

4.2. **Tangent space of rotational space.**
The rotational space $\boldsymbol{S}_{\text{rot}}$ is embedded into the environmental relative vector space $\boldsymbol{S}_{\text{rel}}$. Hence, the tangent vectors of $\boldsymbol{S}_{\text{rot}}$ can be regarded as multi–vectors of $\boldsymbol{S}_{\text{rel}}$, which respect the rigid constraint. Here, we describe in a geometric way the tangent vectors of $\boldsymbol{S}_{\text{rot}}$, and obtain a geometric interpretation of classical formulas of the mechanics of rigid systems.

4.2.1. *Non degenerate case.*
We recall that the Lie algebra $so(\boldsymbol{S}, g)$ of $O(\boldsymbol{S}, g)$ can be identified with the subspace $so(\boldsymbol{S}, g) \subset \boldsymbol{S}^* \otimes \boldsymbol{S}$, consisting of the tensors which are antisymmetric with respect to g.

4.2. Proposition. We have the natural parallelising linear isomorphism

$$\tau_{\text{rot}} \colon T\boldsymbol{S}_{\text{rot}} \to \boldsymbol{S}_{\text{rot}} \times so(\boldsymbol{S}, g) \tag{4.11}$$

and the associated projection $\rho_{\text{rot}} \colon T\boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rot}} \times \boldsymbol{S}_{\text{rel}} \to so(\boldsymbol{S}, g) \colon (r_{\text{rot}}\,,\ v_{\text{rot}}) \mapsto \omega$.

The expression of the inverse isomorphism τ_{rot}^{-1} is

$$\tau_{\text{rot}}^{-1} \colon \boldsymbol{S}_{\text{rot}} \times so(\boldsymbol{S}, g) \to T\boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rot}} \times \boldsymbol{S}_{\text{rel}} \colon (r_{\text{rot}},\, \omega) \mapsto \big(r_{\text{rot}},\, \big(\omega(r_1), \dots, \omega(r_n)\big)\big)\,.$$

Thus, for each $(r_{\text{rot}}, v_{\text{rot}}) \in T\boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rot}} \times \boldsymbol{S}_{\text{rel}}$, there is a unique $\omega \in so(\boldsymbol{S}, g)$, such that $v_i = \omega(r_i)$, for $1 \le i \le n$.

Proof. The first statement comes from the fact that $\boldsymbol{S}_{\text{rot}}$ is an affine space associated with the Lie group $O(\boldsymbol{S}, g)$. Next, in order to compute τ_{rot}^{-1}, let us consider an element $\omega \in so(\boldsymbol{S}, g)$. Then, there exists a map $\tilde{\omega} \colon I\!\!R \to SO(\boldsymbol{S}, g)$ such that $\tilde{\omega}(0) = \text{id} \in SO(\boldsymbol{S}, g)$ and $(D\tilde{\omega})(0) = \omega$. Hence, for each $r_{\text{rot}} \in \boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rel}}$, the curve $c \colon I\!\!R \to \boldsymbol{S}_{\text{rel}} \colon \lambda \mapsto \big(\tilde{\omega}(\lambda)(r_1), \dots, \tilde{\omega}(\lambda)(r_n)\big)$ is valued in $\boldsymbol{S}_{\text{rot}}$ because

$$\|\tilde{\omega}(\lambda)(r_i) - \tilde{\omega}(\lambda)(r_j)\| = \|\tilde{\omega}(\lambda)(r_i - r_j)\| = \|r_i - r_j\|\,, \qquad \forall \lambda \in I\!\!R\,,\ \forall 1 \le i, j \le n\,.$$

Hence, the tangent map $Dc(0) \in \boldsymbol{S}_{\text{rel}}$ is valued in $T_{r_{\text{rot}}}\boldsymbol{S}_{\text{rot}} \in \boldsymbol{S}_{\text{rel}}$. On the other hand, we have

$$Dc(0) = \big(\omega(r_1), \dots, \omega(r_n)\big)\,. \qquad \square$$

Later, we shall give an explicit expression of the parallelisation τ_{rot}, via the "inertia isomorphism" (Corollary 4.8). We can read the parallelisation τ_{rot} in a further interesting way, by means of an algebraic re–interpretation of $so(\boldsymbol{S}, g)$. For this purpose, we recall the cross products $\times$ of $\boldsymbol{S}$ and of $\boldsymbol{S}^*$, defined by

$$u \times v := *(v \wedge w) := g^\sharp(i(u \wedge v)\,\eta) = i\big(g^\flat(u) \wedge g^\flat(v)\big)\,\bar{\eta}\,, \qquad \forall\, u, v \in \boldsymbol{S}\,,$$
$$\alpha \times \beta := *(\alpha \wedge \beta) := g^\flat(i(\alpha \wedge \beta)\,\bar{\eta}) = i\big(g^\sharp(\alpha) \wedge g^\sharp(\beta)\big)\,\eta\,, \qquad \forall\, \alpha, \beta \in \boldsymbol{S}^*\,.$$

The cross product commutes with the metric isomorphisms, i.e. we have

$$g^\flat(u \times v) = g^\flat(u) \times g^\flat(v) \qquad \text{and} \qquad g^\sharp(\alpha \times \beta) = g^\sharp(\alpha) \times g^\sharp(\beta)\,.$$

For short, we set

$$u \times_{\mathrm{mul}} v_{\mathrm{mul}} := (u \times v_1, \ldots, u \times v_n)\,, \qquad \forall\, u \in \boldsymbol{S}\,, \quad \forall\, v_{\mathrm{mul}} \in \boldsymbol{S}_{\mathrm{mul}}\,,$$
$$\alpha \times_{\mathrm{mul}} \beta_{\mathrm{mul}} := (\alpha \times \beta_1, \ldots, \alpha \times \beta_n)\,, \qquad \forall\, \alpha \in \boldsymbol{S}^*\,, \quad \forall\, \beta_{\mathrm{mul}} \in \boldsymbol{S}^*_{\mathrm{mul}}\,.$$

Moreover, we introduce the scaled vector space

$$\boldsymbol{V}_{\mathrm{ang}} := \mathbb{L}^{-1} \otimes \boldsymbol{S}\,. \tag{4.12}$$

The metric isomorphism $g^\flat \colon \boldsymbol{S}^* \otimes \boldsymbol{S} \to \mathbb{L}^2 \otimes \boldsymbol{S}^* \otimes \boldsymbol{S}^* \colon \alpha \otimes v \mapsto \alpha \otimes g(v, \cdot)$ and the Hodge isomorphism $* \colon \mathbb{L}^2 \otimes \Lambda^2 \boldsymbol{S}^* \to \mathbb{L}^{-1} \otimes \boldsymbol{S} \colon \omega \mapsto i(\omega)\, \bar{\eta}$ yield the linear isomorphism

$$* \circ g^\flat \colon so(\boldsymbol{S}, g) \to \boldsymbol{V}_{\mathrm{ang}} \colon \omega \mapsto i\big(g^\flat(\omega)\big)\, \bar{\eta}\,. \tag{4.13}$$

The following corollary is a straightforward analogue of Proposition 4.2.

4.3. Corollary. We have the natural parallelising isomorphism

$$\tau_{\mathrm{ang}} := * \circ g^\flat \circ \tau_{\mathrm{rot}} \colon T\boldsymbol{S}_{\mathrm{rot}} \to \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}} \tag{4.14}$$

and the associated projection $\rho_{\mathrm{ang}} \colon T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{rel}} \to \boldsymbol{V}_{\mathrm{ang}} \colon (r_{\mathrm{rot}},\ v_{\mathrm{rot}}) \mapsto \Omega$. The expression of the inverse isomorphism τ_{ang}^{-1} is

$$\tau_{\mathrm{ang}}^{-1} \colon \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}} \to T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{rel}} \colon (r_{\mathrm{rot}}\,,\ \Omega) \mapsto \big(r_{\mathrm{rot}}\,,\ (\Omega \times r_1, \ldots, \Omega \times r_n)\big)\,.$$

Thus, for each $(r_{\mathrm{rot}}, v_{\mathrm{rot}}) \in T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{rel}}$, there is a unique $\Omega \in \boldsymbol{V}_{\mathrm{ang}}$, such that $v_i = \Omega \times r_i$, for $1 \le i \le n$.

Later, we shall give an explicit expression of the parallelisation τ_{ang}, via the "inertia isomorphism" (Theorem 4.6). The above Corollary 4.3 is just a geometric formulation of the well–known formula expressing the relative velocity of the particles of a rigid system through the angular velocity.

4.4. Corollary. The transpose $(\tau_{\mathrm{ang}}^{-1})^*$ of the isomorphism τ_{ang}^{-1} has the expression

$$(\tau_{\mathrm{ang}}^{-1})^* \colon T^*\boldsymbol{S}_{\mathrm{rot}} \to \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}^*_{\mathrm{ang}} \colon (r_{\mathrm{rot}}, \alpha) \mapsto \Big(r_{\mathrm{rot}},\ \sum_i g^\flat(r_i) \times \alpha_i\Big)\,. \tag{4.15}$$

Proof. The expression of τ_{ang}^{-1} and cyclic permutations yield

$$\begin{aligned}
(\tau_{\mathrm{ang}}^{-1})^*(r_{\mathrm{rot}}, \alpha)\,(\Omega) &:= \alpha\big(\tau_{\mathrm{ang}}(r_{\mathrm{rot}}, \Omega)\big) = \alpha\,(\Omega \times_{\mathrm{mul}} r_{\mathrm{rot}}) = \sum_i \alpha_i\,(\Omega \times r_i) \\
&= \sum_i g\left(g^\sharp(\alpha_i)\,, (\Omega \times r_i)\right) = \sum_i g\left(r_i\,,\ ({g^\sharp}_i(\alpha_i) \times \Omega)\right) \\
&= \sum_i g\left(\Omega\,,\ (r_i \times {g^\sharp}_i(\alpha_i))\right) = \sum_i \left(({g^\flat}_i(r_i) \times \alpha_i\right)(\Omega)\,. \qquad \square
\end{aligned}$$

The cross product $\times$ is equivariant with respect to the action of $SO(\boldsymbol{S}, g)$. Hence, the isomorphism τ_{ang} turns out to be equivariant with respect to this group.

4.2.2. *Degenerate case.*
Let us consider the quotient vector bundle $[\boldsymbol{S}_{\mathrm{rot}} \times so(\boldsymbol{S}, g)]$ over $\boldsymbol{S}_{\mathrm{rot}}$ of the trivial vector bundle $\boldsymbol{S}_{\mathrm{rot}} \times so(\boldsymbol{S}, g)$, with respect to the vector subbundle $h[\boldsymbol{S}_{\mathrm{rot}}]$ consisting, for each $r_{\mathrm{rot}} \in \boldsymbol{S}_{\mathrm{rot}}$, of the isotropy Lie subalgebra $h[r_{\mathrm{rot}}] \subset so(\boldsymbol{S}, g)$ of $\langle r_{\mathrm{rot}} \rangle \subset \boldsymbol{S}$ (see, for instance, [25]).

We can rephrase in the degenerate case the results concerning the non degenerate case by a quotient procedure.

For each $r_{\mathrm{rot}} \in \boldsymbol{S}_{\mathrm{rot}}$, the isotropy Lie subalgebra associated with r_{rot} consists of antisymmetric endomorphisms $\phi \in \boldsymbol{S}^* \otimes \boldsymbol{S}$ which preserve the 1–dimensional vector subspace $\langle r_{\mathrm{rot}} \rangle \subset \boldsymbol{S}$ generated by r_{rot}. We have the natural linear fibred isomorphism $[\tau_{\mathrm{rot}}] \colon T\boldsymbol{S}_{\mathrm{rot}} \to [\boldsymbol{S}_{\mathrm{rot}} \times so(\boldsymbol{S}, g)]$.

Let us consider the quotient vector bundle $[\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}}]$ over $\boldsymbol{S}_{\mathrm{rot}}$ of the vector bundle $\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}}$ with respect to the vector subbundle $a[\boldsymbol{S}_{\mathrm{rot}}]$ consisting, for each $r_{\mathrm{rot}} \in \boldsymbol{S}_{\mathrm{rot}}$, by the 1–dimensional vector subspace $\langle r_{\mathrm{rot}} \rangle \subset \boldsymbol{V}_{\mathrm{ang}}$ generated by r_{rot}.

4.5. Proposition. We have the linear fibred isomorphism

$$[\tau_{\mathrm{ang}}] \colon T\boldsymbol{S}_{\mathrm{rot}} \to [\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}}] \,.$$

The expression of the inverse isomorphism $[\tau_{\mathrm{ang}}^{-1}]$ is

$$\begin{aligned}[\tau_{\mathrm{ang}}]^{-1} : [\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}}] \to T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{mul}} &: (r_{\mathrm{rot}} \,,\, [r_{\mathrm{rot}}, \Omega]) \\ &\mapsto \big(r_{\mathrm{rot}} \,,\, (\Omega \times r_1, \ldots, \Omega \times r_n)\big) \,,\end{aligned}$$

where the cross products $\Omega \times r_i$ turn out to be independent of the choice of representative for the class $[r_{\mathrm{rot}}, \Omega]$. Thus, for each $(r_{\mathrm{rot}}, v_{\mathrm{rot}}) \in T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{rel}}$, there is a unique $[r_{\mathrm{rot}}, \Omega] \in [\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}}]_{r_{\mathrm{rot}}}$, such that $v_i = \Omega \times r_i$, for $1 \le i \le n$.

It follows that a continuous choice of an orientation of the straight lines $\langle r_{\mathrm{rot}} \rangle \subset \boldsymbol{S}$ generated by the configurations $r_{\mathrm{rot}} \in \boldsymbol{S}_{\mathrm{rot}}$ yields the linear isomorphism

$$T\boldsymbol{S}_{\mathrm{rot}} \,\simeq\, T\boldsymbol{S}^2(\mathbb{L}^* \otimes \boldsymbol{S}, g) \,.$$

4.3. **Rigid system metrics.**
The multi–dynamical metric of $\boldsymbol{S}_{\mathrm{mul}}$ induces a metric on $\boldsymbol{S}_{\mathrm{rot}}$, which can be regarded also in another useful way through the isomorphism τ_{ang}, and will be interpreted as the inertia tensor. Moreover, the standard pattern metric of $\boldsymbol{V}_{\mathrm{ang}}$ induces a further metric on $\boldsymbol{S}_{\mathrm{rot}}$.

We will consider only the non degenerate case. We leave to the reader the task to consider the degenerate case.

The inclusion $i_{\mathrm{rig}} \colon \boldsymbol{P}_{\mathrm{rig}} \hookrightarrow \boldsymbol{P}_{\mathrm{mul}}$ yields the geometrical and weighted scaled Riemannian metrics

$$g_{\mathrm{rig}} := i^*_{\mathrm{rig}}\, g_{\mathrm{mul}} \,, \quad G_{\mathrm{rig}} := i^*_{\mathrm{rig}}\, G_{\mathrm{mul}} \,.$$

The splitting $\boldsymbol{P}_{\mathrm{rig}} = \boldsymbol{P}_{\mathrm{dia}} \oplus \boldsymbol{S}_{\mathrm{rot}}$ is orthogonal with respect to the metric G_{rig}.

The inclusion $i_{\mathrm{rot}} \colon \boldsymbol{S}_{\mathrm{rot}} \hookrightarrow \boldsymbol{S}_{\mathrm{rel}}$ yields the geometrical and weighted scaled Riemannian metrics

$$g_{\mathrm{rot}} := i^*_{\mathrm{rot}}\, g_{\mathrm{rel}}\,, \quad G_{\mathrm{rot}} := i^*_{\mathrm{rot}}\, G_{\mathrm{rel}}\,.$$

For each $(\Omega \times_{\mathrm{mul}} r_{\mathrm{rot}})\,,\ (\Omega' \times_{\mathrm{mul}} r_{\mathrm{rot}}) \in T_{r_{\mathrm{rot}}}\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rel}}$, in virtue of standard properties of the cross product, we obtain

$$g_{\mathrm{rot}}(r_{\mathrm{rot}})\,(\Omega \times_{\mathrm{mul}} r_{\mathrm{rot}}\,,\, \Omega' \times_{\mathrm{mul}} r_{\mathrm{rot}}) = \sum_i \big(g(r_i, r_i)\, g(\Omega, \Omega') - g(r_i, \Omega)\, g(r_i, \Omega')\big)\,,$$

$$G_{\mathrm{rot}}(r_{\mathrm{rot}})\,(\Omega \times_{\mathrm{mul}} r_{\mathrm{rot}}\,,\, \Omega' \times_{\mathrm{mul}} r_{\mathrm{rot}}) = \sum_i \mu_i \big(g(r_i, r_i)\, g(\Omega, \Omega') - g(r_i, \Omega)\, g(r_i, \Omega')\big).$$

We can regard the metrics g_{rot} and G_{rot} in another interesting way, via τ_{ang}; we have the two scaled metrics

$$\sigma := (\tau^{-1}_{\mathrm{ang}})^*\, g_{\mathrm{rot}} : \boldsymbol{S}_{\mathrm{rot}} \to \mathbb{L}^2 \otimes (\boldsymbol{V}^*_{\mathrm{ang}} \otimes \boldsymbol{V}^*_{\mathrm{ang}})\,, \tag{4.16}$$

$$\Sigma := (\tau^{-1}_{\mathrm{ang}})^*\, G_{\mathrm{rot}} : \boldsymbol{S}_{\mathrm{rot}} \to \mathbb{L}^2 \otimes (\boldsymbol{V}^*_{\mathrm{ang}} \otimes \boldsymbol{V}^*_{\mathrm{ang}})\,, \tag{4.17}$$

with expressions

$$\sigma(r_{\mathrm{rot}})(\Omega, \Omega') = \sum_i \big(g(r_i, r_i)\, g(\Omega, \Omega') - g(r_i, \Omega)\, g(r_i, \Omega')\big)\,,$$

$$\Sigma(r_{\mathrm{rot}})(\Omega, \Omega') = \sum_i \mu_i \, \big(g(r_i, r_i)\, g(\Omega, \Omega') - g(r_i, \Omega)\, g(r_i, \Omega')\big)\,.$$

We have a further natural metric of $\boldsymbol{S}_{\mathrm{rot}}$. For this purpose, let us consider the metric $g \in \boldsymbol{V}^*_{\mathrm{ang}} \otimes \boldsymbol{V}^*_{\mathrm{ang}}$ of $\boldsymbol{V}_{\mathrm{ang}}$ naturally induced by the pattern metric g of $\boldsymbol{S}$. We can make the natural identification $O(\boldsymbol{V}_{\mathrm{ang}}, g) \simeq O(\boldsymbol{S}, g)$. We obtain the unscaled Riemannian metric

$$g_{\mathrm{ang}} := \tau^*_{\mathrm{ang}}\, g : \boldsymbol{S}_{\mathrm{rot}} \to T^*\boldsymbol{S}_{\mathrm{rot}} \otimes T^*\boldsymbol{S}_{\mathrm{rot}}\,.$$

For each $(r_{\mathrm{rot}}\,,\ \Omega \times_{\mathrm{mul}} r_{\mathrm{rot}})\,,\ (r_{\mathrm{rot}}\,,\ \Omega' \times_{\mathrm{mul}} r_{\mathrm{rot}}) \in T\boldsymbol{S}_{\mathrm{rot}} \subset \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{S}_{\mathrm{rel}}$, we have the expression $g_{\mathrm{ang}}(r_{\mathrm{rot}})\,(\Omega \times_{\mathrm{mul}} r_{\mathrm{rot}}\,,\, \Omega' \times_{\mathrm{mul}} r_{\mathrm{rot}}) = g(\Omega, \Omega')$.

All metrics of $\boldsymbol{S}_{\mathrm{rot}}$ considered above are invariant with respect to the left action of $O(\boldsymbol{S}, g)$.

The choice of a configuration $r_{\mathrm{rot}} \in \boldsymbol{S}_{\mathrm{rot}}$ and of an orthonormal basis in $\boldsymbol{V}_{\mathrm{ang}}$, respectively, yields the following diffeomorphisms (via the action of $SO(\boldsymbol{V}_{\mathrm{ang}}, g)$ on $\boldsymbol{S}_{\mathrm{rot}}$)

$$\boldsymbol{S}_{\mathrm{rot}} \simeq SO(\boldsymbol{V}_{\mathrm{ang}}, g) \simeq SO(3)\,,$$

which turn out to be isometries with respect to the Riemannian metrics g_{ang}, $-\frac{1}{2}\, k_{\mathrm{ang}}$ and $-\frac{1}{2}\, k_3$, of $\boldsymbol{S}_{\mathrm{rot}}$, $\boldsymbol{V}_{\mathrm{ang}}$ and $SO(3)$, where k_{ang} and k_3 are the Killing forms. In fact, the above diffeomorphisms yield the linear fibred isomorphisms $T_{r_{\mathrm{rot}}}\boldsymbol{S}_{\mathrm{rot}} \simeq so(\boldsymbol{V}_{\mathrm{ang}}, g) \simeq so(3)$. The first isomorphism is metric because it comes from the natural isomorphism $so(\boldsymbol{V}_{\mathrm{ang}}, g) \to \boldsymbol{V}_{\mathrm{ang}}$, induced by $g^\flat$ and $*$, which is metric. Moreover, the metric g of $\boldsymbol{V}_{\mathrm{ang}}$ turns out to coincide with the metric $-\frac{1}{2}\, k_{\mathrm{ang}}$ of $so(\boldsymbol{V}_{\mathrm{ang}}, g)$. In fact, we have $g(\omega, \omega') = -\frac{1}{2}\ \mathrm{tr}\,\big((\Omega\times) \circ (\Omega'\times)\big)$. It is easy to realise that the isomorphism $so(\boldsymbol{V}_{\mathrm{ang}}, g) \simeq so(3)$ is metric.

The unscaled metric g of $\boldsymbol{V}_{\text{ang}}$ allows us to regard the metrics σ and Σ as scaled symmetric fibred automorphisms

$$\begin{aligned}\hat{\sigma} &:= g^\sharp \circ \sigma^\flat : \boldsymbol{S}_{\text{rot}} \to \mathbb{L}^2 \otimes (\boldsymbol{V}^*_{\text{ang}} \otimes \boldsymbol{V}_{\text{ang}})\,,\\ \hat{\Sigma} &:= g^\sharp \circ \Sigma^\flat : \boldsymbol{S}_{\text{rot}} \to \mathbb{L}^2 \otimes (\boldsymbol{V}^*_{\text{ang}} \otimes \boldsymbol{V}_{\text{ang}})\,.\end{aligned}$$

We have the expressions

$$\begin{aligned}\hat{\sigma}(r_{\text{rot}})(\Omega) &= \sum_i r_i \times (\Omega \times r_i) &&= \sum_i \big(g(r_i, r_i)\,\Omega - g(r_i, \Omega)\, r_i\big)\\ \hat{\Sigma}(r_{\text{rot}})(\Omega) &= \sum_i \mu_i\, r_i \times (\Omega \times r_i) &&= \sum_i \mu_i \,\big(g(r_i, r_i)\,\Omega - g(r_i, \Omega)\, r_i\big)\,.\end{aligned}$$

We have

$$\begin{aligned}\hat{\sigma}^{-1} &= \sigma^\sharp \circ g^\flat \qquad \text{and} \qquad &\hat{\Sigma}^{-1} &= \Sigma^\sharp \circ g^\flat\,,\\ (\hat{\sigma}^{-1})^* &= g^\flat \circ \sigma^\sharp \qquad \text{and} \qquad &(\hat{\Sigma}^{-1})^* &= g^\flat \circ \Sigma^\sharp\,.\end{aligned}$$

The automorphisms $\hat{\sigma}$ and $\hat{\Sigma}$ yield the following explicit expressions of the map τ_{ang}.

4.6. Theorem. *The isomorphism τ_{ang} has the expression*

$$\tau_{\text{ang}} : T\boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rot}} \times \boldsymbol{S}_{\text{rel}} \to \boldsymbol{S}_{\text{rot}} \times \boldsymbol{V}_{\text{ang}} : (r_{\text{rot}}\,,\, v_{\text{rot}}) \mapsto (r_{\text{rot}}\,,\, \Omega)\,,$$

where

$$\Omega = \hat{\sigma}^{-1}(r_{\text{rot}})\Big(\sum_i r_i \times v_i\Big) = \hat{\Sigma}^{-1}(r_{\text{rot}})\Big(\sum_i \mu_i\, r_i \times v_i\Big)\,. \tag{4.18}$$

Proof. Let $r_{\text{rot}} \in \boldsymbol{S}_{\text{rot}}\,,\;\; v_{\text{rot}} \equiv (v_1, \dots, v_n) \in T_{r_{\text{rot}}}\boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rel}}$ and set $\Omega := \rho_{\text{ang}}(r_{\text{rot}}, v_{\text{rot}}) \in \boldsymbol{V}_{\text{ang}}$.

The definitions of σ and of g_{rot} yield, respectively, the following equalities, for each $\Omega' \in \boldsymbol{V}_{\text{ang}}$,

$$\begin{aligned}g_{\text{rot}}(r_{\text{rot}})\,(v_{\text{rot}}\,, \Omega' \times_{\text{mul}} r_{\text{rot}}) &:= \sigma(r_{\text{rot}})\,(\Omega\,,\,\Omega')\\ g_{\text{rot}}(r_{\text{rot}})\,(v_{\text{rot}}\,, \Omega' \times_{\text{mul}} r_{\text{rot}}) &:= \sum_i g(v_i\,,\,\Omega' \times r_i) = g\Big(\Omega'\,,\,\sum_i r_i \times v_i\Big)\\ &= g\Big(\sum_i r_i \times v_i\,,\,\Omega'\Big)\,.\end{aligned}$$

Then, by comparison of the above equalities, we obtain $\sigma^\flat{}_{\text{rot}}(r_{\text{rot}})(\Omega) = g^\flat\big(\sum_i r_i \times v_i\big)$, hence $\hat{\sigma}(r_{\text{rot}})(\Omega) := (g^\sharp \circ \sigma^\flat{}_{\text{rot}})(r_{\text{rot}})(\Omega) = \sum_i r_i \times v_i$, which yields $\Omega = \hat{\sigma}^{-1}(r_{\text{rot}})\big(\sum_i r_i \times v_i\big)$.

We can prove the 2nd expression of Ω in analogous way, by replacing g_{rot} with G_{rot}. □

In the classical literature, Ω is computed by means of the Poisson's formulas, in terms of a basis. The above Theorem provides an intrinsic expression of Ω, which plays an essential role in next sections.

4.7. Note. The map τ_{ang} is a geometric object, which has nothing to do with masses and weights, because the rigid constraint does not involve the masses. Accordingly, the 1st formula in the above Theorem is natural, while the 2nd one sounds quite strange. Indeed, the 2nd formula is true for any arbitrary choice of the weights. We have added the 2nd formula for the sake of completeness. Actually, in the 2nd formula, the weights appear both in the expressions of the sum and of $\hat{\Sigma}$; eventually, the contribution of the weights disappear. To realize that, we remark that it is possible to prove directly that the 1st formula implies the 2nd one. □

4.8. Corollary. In the non degenerate case, the isomorphism τ_{rot} has the equivalent expression

$$\tau_{\text{rot}} \colon T\boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rot}} \times \boldsymbol{S}_{\text{rel}} \to \boldsymbol{S}_{\text{rot}} \times (\boldsymbol{S}^* \otimes \boldsymbol{S}) : (r_{\text{rot}}\,,\, v_{\text{rot}}) \mapsto \big(r_{\text{rot}}\,,\, g^\sharp\big(i(\Omega)\eta\big)\big)\,,$$

where

$$\Omega = \hat{\sigma}^{-1}(r_{\text{rot}})\Big(\sum_i r_i \times v_i\Big) = \hat{\Sigma}^{-1}(r_{\text{rot}})\Big(\sum_i \mu_i\, r_i \times v_i\Big)\,.$$

The eigenvalues of $\hat{\Sigma}$ turn out to be constant with respect to $\boldsymbol{S}_{\text{rot}}$, in virtue of the invariance of Σ with respect to $SO(\boldsymbol{S}, g)$.

In the non degenerate case, we have three eigenvalues. Then, three cases may occur:

$$\begin{aligned}
&\lambda := \lambda_1 = \lambda_2 = \lambda_3\,, && \textit{spherical case}\,,\\
&\lambda := \lambda_1 = \lambda_2 \neq \lambda_3\,, && \textit{symmetric case}\,,\\
&\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1\,, && \textit{asymmetric case.}
\end{aligned}$$

In the degenerate case, we have two coinciding eigenvalues

$$\lambda := \lambda_1 = \lambda_2 = \sum_i \mu_i\, g(r_i, r_i)\,.$$

Analogous results hold for $\hat{\sigma}$.

We have studied the diagonalisation of Σ with respect to g. In an analogous way, we can diagonalise G_{rot} with respect to G_{ang}. Indeed, in this way we obtain the same eigenvalues and the same classification, because the two diagonalisations are related by the isomorphism τ_{ang}.

The scaled metric $m_0\,\Sigma$, or the scaled automorphism $m_0\,\hat{\Sigma}$, are called the *inertia tensor* and the scaled eigenvalues $I_i = m_0\,\lambda_i \colon \boldsymbol{S}_{\text{rot}} \to (\mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!\!R$ of the inertia tensor are called *principal inertia momenta*.

4.3.1. *Continuous interpretation.*

We can interpret the above results concerning the parallelisation of $\boldsymbol{S}_{\text{rot}}$ also in terms of continuous transformations. Here, in order to keep the thread of our reasoning, we adopt a purely geometric approach which does not involve time, but this section can be easily rephrased in a true kinematical way, by replacing $I\!\!R$ with $\boldsymbol{T}$, or $\mathbb{T} \otimes I\!\!R$, as appropriate.

We define a *continuous transformation* as a map

$$C \colon I\!\!R \times (I\!\!R \times \boldsymbol{P}) \to \boldsymbol{P}\,,$$

such that, for each $\tau, \tau', t \in \mathbb{R}\,,\ p \in \boldsymbol{P}$,

$$C(0,t,p) = p \qquad \text{and} \qquad C\big(\tau', t+\tau, C(\tau,t,p)\big) = C(\tau+\tau', t, p)\,.$$

A continuous transformation is said to be *rigid* if, for each $\tau, t \in \mathbb{R}\,,\ p, q \in \boldsymbol{P}$,

$$\|C(\tau,t,q) - C(\tau,t,p)\| = \|q - p\|\,.$$

We can prove that a continuous transformation C is rigid if and only if its expression is of the type

$$C(\tau,t,p) = c(t) + \Phi_{(\tau,t)}\,(p-o)\,, \qquad \forall\,\tau, t \in \mathbb{R}\,,\ p \in \boldsymbol{P}\,,$$

where $o \in \boldsymbol{P}$, $c \colon \mathbb{R} \to \boldsymbol{P}$ and $\Phi \colon \mathbb{R} \times \mathbb{R} \to SO(\boldsymbol{S}, g)$.

Let us suppose that C be rigid. The partial derivative of Φ with respect to time, at $\tau = 0$, turns out to be an antisymmetric endomorphism $\delta\Phi \colon \mathbb{R} \to so(\boldsymbol{S}, g) \subset \boldsymbol{S}^* \otimes \boldsymbol{S}$. Hence, the *velocity* of the continuous transformation $\nu \colon \boldsymbol{T} \times \boldsymbol{P} \to \boldsymbol{S}$ is given by $\nu(t,p) = Dc(t) + \delta\Phi(t)\,(p-o)$, for all $t \in \boldsymbol{T}$, $p \in \boldsymbol{P}$. On the other hand, we obtain the map $\Omega := (* \circ g^\flat)(\delta\Phi) : \mathbb{R} \to \mathbb{L}^* \otimes \boldsymbol{S}$, Therefore, we can express the velocity of the continuous transformation by the classical formula

$$\nu(t,p) = Dc(t) + \Omega(t) \times (p-o)\,, \qquad \forall\, t \in \mathbb{R}\,,\ p \in \boldsymbol{P}\,.$$

4.9. Note. Let $\boldsymbol{P}_{\mathrm{rig}} \subset \boldsymbol{P}_1 \times \ldots \boldsymbol{P}_n$ be a non degenerate rigid configuration space and $s_{\mathrm{rig}} \colon \mathbb{R} \to \boldsymbol{P}_{\mathrm{rig}}$ be a map. Then, there is a unique continuous rigid transformation such that the particles of the continuous transformation, which coincide with the particles of the discrete rigid system at a certain time, move as the particles of the discrete rigid system. In other words, there is a unique rigid continuous transformation $C \colon \mathbb{R} \times (\mathbb{R} \times \boldsymbol{P}) \to \boldsymbol{P}$ such that, for each $p = (p_1, \ldots, p_n) \in \boldsymbol{P}_{\mathrm{rig}}$, $C(\tau, t, p_i) = s_i(t+\tau)$ for all $\tau, t \in \mathbb{R}$. Then, for each $p = (p_1, \ldots, p_n) \in \boldsymbol{P}_{\mathrm{rig}}$ and $t \in \mathbb{R}$, we have

$$\nu(t, p_i) = ds_i(t) = ds_{\mathrm{cen}}(t) + \Omega(t) \times r_i\,.$$

Indeed, the rotational components of the velocity of the continuous and discrete rigid maps coincide. □

4.4. **Splitting of the tangent and cotangent multi–space.**

We exhibit a natural orthogonal splitting of the environmental tangent and cotangent spaces into three components: the component of the center of mass, the angular component and the component orthogonal to the rigid configuration space. This splitting will have a fundamental role in mechanics of rigid systems.

4.4.1. *Splitting of the tangent multi–space.*

Let us consider the space $T\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} = \boldsymbol{P}_{\mathrm{rig}} \times \boldsymbol{S}_{\mathrm{rel}}$.

4.10. Theorem. *We have the orthogonal splitting, with respect to* G_{mul}

$$T\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} = T\boldsymbol{P}_{\mathrm{rig}} \underset{\boldsymbol{P}_{\mathrm{rig}}}{\oplus} T^\perp \boldsymbol{P}_{\mathrm{rig}} = (T\boldsymbol{P}_{\mathrm{cen}} \times T\boldsymbol{S}_{\mathrm{rot}}) \underset{\boldsymbol{P}_{\mathrm{rig}}}{\oplus} T^\perp \boldsymbol{P}_{\mathrm{rig}}\,,$$

where $T^\perp \boldsymbol{P}_{\mathrm{rig}}$ *is the orthogonal complement of* $T\boldsymbol{P}_{\mathrm{rig}}$ *in* $T\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}}$.

The subspace $T^{\perp}\boldsymbol{P}_{\text{rig}}$ *is characterised by the following equality*

$$T^{\perp}\boldsymbol{P}_{\text{rig}} = \Big\{(p_{\text{rig}}\,,\, v_{\text{mul}}) \in \boldsymbol{P}_{\text{rig}} \times \boldsymbol{S}_{\text{mul}} \mid \sum_i \mu_i\, v_i = 0\,,\ \sum_i \mu_i\, r_i \times v_i = 0\Big\}$$
$$\subset T\boldsymbol{P}_{\text{mul}}|_{\boldsymbol{P}_{\text{rig}}}.$$

Moreover, the expressions of the projections associated with the splitting are

$$T\pi_{\text{cen}} : T\boldsymbol{P}_{\text{mul}}|_{\boldsymbol{P}_{\text{rig}}} \to T\boldsymbol{P}_{\text{cen}} \ : (p_{\text{rig}}\,,\, v_{\text{mul}}) \mapsto (p_{\text{cen}}\,,\, v_{\text{cen}})\,, \tag{4.19}$$

$$T\pi_{\text{rot}} : T\boldsymbol{P}_{\text{mul}}|_{\boldsymbol{P}_{\text{rig}}} \to T\boldsymbol{S}_{\text{rot}} \ : (p_{\text{rig}}\,,\ v_{\text{mul}}) \mapsto (r_{\text{rot}}\,,\, \Omega \times_{\text{mul}} r_{\text{rot}})\,, \tag{4.20}$$

$$\begin{aligned}\pi^{\perp}_{\text{rig}} : T\boldsymbol{P}_{\text{mul}}|_{\boldsymbol{P}_{\text{rig}}} \to T^{\perp}\boldsymbol{P}_{\text{rig}} &: (p_{\text{rig}}\,,\ v_{\text{mul}}) \\ &\mapsto (p_{\text{rig}}\,,\ v_{\text{mul}} - v_{\text{dia}} - \Omega \times_{\text{mul}} r_{\text{rot}})\,,\end{aligned} \tag{4.21}$$

where

$$p_{\text{cen}} = o + \sum_i \mu_i\,(p_i - o)\,, \qquad p_{\text{dia}} := (p_{\text{cen}}, \dots, p_{\text{cen}})\,, \qquad r_{\text{rot}} = p_{\text{rig}} - p_{\text{dia}}\,,$$
$$v_{\text{cen}} = \sum_i \mu_i\, v_i\,, \qquad v_{\text{dia}} := (v_{\text{cen}}, \dots, v_{\text{cen}})\,, \qquad v_{\text{rot}} = v_{\text{mul}} - v_{\text{dia}}\,,$$
$$\Omega := \hat{\Sigma}^{-1}(r_{\text{rot}})\Big(\sum_i \mu_i\, r_i \times v_i\Big)\,.$$

Proof. The expression of the 1st projection is obvious.

Let us prove the expression of the 2nd projection. For each $r_{\text{rot}} \in \boldsymbol{S}_{\text{rot}}\,,\ v_{\text{rel}} \in \boldsymbol{S}_{\text{rel}}$ and $\Omega' \in \boldsymbol{V}_{\text{ang}}$, in virtue of the definitions of Σ and of G_{rel}, and by a cyclic permutation, we obtain the equalities

$$\begin{aligned}G_{\text{rel}}\,(v_{\text{rel}}\,,\ \Omega' \times_{\text{mul}} r_{\text{rot}}) &= G_{\text{rel}}\,\big(T\pi_{\text{rot}}(r_{\text{rot}}, v_{\text{rel}})\,,\ \Omega' \times_{\text{mul}} r_{\text{rot}}\big) \\ &= G_{\text{rel}}\,\big(\Omega \times_{\text{mul}} r_{\text{rot}}\,,\ \Omega' \times_{\text{mul}} r_{\text{rot}}\big) = \Sigma(r_{\text{rot}})\,\big(\Omega, \Omega'\big)\,,\end{aligned}$$

$$\begin{aligned}G_{\text{rel}}\,(v_{\text{rel}}\,, \Omega' \times_{\text{mul}} r_{\text{rot}}) &= \sum_i \mu_i\, g(v_i\,,\ \Omega' \times r_i) \\ &= g\Big(\Omega'\,,\ \sum_i \mu_i\, r_i \times v_i\Big) = g\Big(\sum_i \mu_i\, r_i \times v_i\,,\ \Omega'\Big)\,.\end{aligned}$$

A comparison of the above equalities yields $\Sigma^{\flat}(r_{\text{rot}})\,(\Omega) = g^{\flat}\big(\sum_i \mu_i\, r_i \times v_i\big)$, hence

$$\Omega = (\Sigma^{\sharp}(r_{\text{rot}}) \circ g^{\flat})\Big(\sum_i \mu_i\, r_i \times v_i\Big) = \hat{\Sigma}^{-1}(r_{\text{rot}})\Big(\sum_i \mu_i\, r_i \times v_i\Big)\,.$$

Then, the characterization of $T^{\perp}\boldsymbol{P}_{\text{rig}}$ is easily obtained by considering the multivectors whose previous projections vanish and by recalling that τ_{ang} and $\hat{\Sigma}$ are isomorphisms.

Eventually, the 3rd projection is obtained by subtracting the previous projections. □

We observe that the expression of $T\pi_{\text{rot}}$ is similar to the 2nd formula of Proposition 4.6. However, we stress that the multivector v_{mul} in the above Theorem needs not to be tangent to $\boldsymbol{S}_{\text{rot}}$ and its projection on $\boldsymbol{S}_{\text{rot}}$ involves the weights. In the particular case when the multivector v_{mul} is tangent to $\boldsymbol{S}_{\mathrm{rot}}$, the expression of $T\pi_{\mathrm{rot}}$ reduces to the 2nd formula of Proposition 4.6.

4.4.2. *Splitting of the cotangent space.*
Let us consider the space $T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} := \boldsymbol{P}_{\mathrm{rig}} \times \boldsymbol{S}^*_{\mathrm{rel}}$.

4.11. Theorem. *We have the orthogonal splitting, with respect to* $\bar{G}_{\mathrm{mul}}$

$$T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} = T^*\boldsymbol{S}_{\mathrm{rig}} \underset{\boldsymbol{P}_{\mathrm{rig}}}{\oplus} T^*_{\perp}\boldsymbol{P}_{\mathrm{rig}} = (T^*\boldsymbol{P}_{\mathrm{cen}} \times T^*\boldsymbol{S}_{\mathrm{rot}}) \underset{\boldsymbol{P}_{\mathrm{rig}}}{\oplus} T^*_{\perp}\boldsymbol{P}_{\mathrm{rig}}\,,$$

where $T^*_{\perp}\boldsymbol{P}_{\mathrm{rig}}$ *is the orthogonal complement of* $T^*\boldsymbol{P}_{\mathrm{rig}}$ *in* $T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}}$ *(i.e., the space of annihilators of* $T\boldsymbol{P}_{\mathrm{rig}}$*).*

The subspace $T^*_{\perp}\boldsymbol{P}_{\mathrm{rig}}$ *is characterised by the following equality*

$$\begin{aligned} T^*_{\perp}\boldsymbol{P}_{\mathrm{rig}} &= \Big\{(p_{\mathrm{rig}}\,,\, \alpha_{\mathrm{mul}}) \in \boldsymbol{P}_{\mathrm{rig}} \times \boldsymbol{S}^*_{\mathrm{mul}} \mid \sum_i \alpha_i = 0\,,\ \sum_i g^{\flat}(v_i) \times \alpha_i = 0\Big\} \\ &\subset T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}}\,. \end{aligned}$$

Moreover, the expressions of the projections associated with the splitting are

$$T^*\pi_{\mathrm{cen}} \colon T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} \to T^*\boldsymbol{P}_{\mathrm{cen}} \colon (p_{\mathrm{rig}}\,,\, \alpha_{\mathrm{mul}}) \mapsto (p_{\mathrm{cen}}\,,\, \alpha_{\mathrm{cen}}) \tag{4.22}$$

$$T^*\pi_{\mathrm{rot}} \colon T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} \to T^*\boldsymbol{S}_{\mathrm{rot}} \colon (p_{\mathrm{rig}}\,,\, \alpha_{\mathrm{mul}}) \mapsto (r_{\mathrm{rot}}\,,\, \alpha_{\mathrm{rot}}) \tag{4.23}$$

$$\begin{aligned} \pi_{\perp} \colon T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} &\to T^*_{\perp}\boldsymbol{P}_{\mathrm{rig}} \colon (p_{\mathrm{rig}}\,,\, \alpha_{\mathrm{mul}}) \\ &\mapsto (r_{\mathrm{rot}}\,,\, \alpha_{\mathrm{mul}} - \alpha_{\mathrm{dia}} - \alpha_{\mathrm{rot}})\,, \end{aligned} \tag{4.24}$$

where

$$p_{\mathrm{cen}} = o + \sum_i \mu_i\,(p_i - o)\,, \qquad p_{\mathrm{dia}} := (p_{\mathrm{cen}}, \dots, p_{\mathrm{cen}})\,, \qquad r_{\mathrm{rot}} = p_{\mathrm{rig}} - p_{\mathrm{dia}}\,,$$

$$\alpha_{\mathrm{cen}} = \sum_i \alpha_i\,, \qquad \alpha_{\mathrm{dia}} = (\mu_1\,\alpha_{\mathrm{cen}}, \dots, \mu_n\,\alpha_{\mathrm{cen}})\,,$$

$$\alpha_{\mathrm{rot}} = \Big((\hat{\Sigma}^{-1})^*(r_{\mathrm{rot}})\big(\sum_i g^{\flat}(r_i) \times \alpha_i\big)\Big) \times_{\mathrm{mul}} G^{\flat}_{\mathrm{mul}}(r_{\mathrm{rot}})\,. \tag{4.25}$$

Proof. The commutative diagram

$$\begin{tikzcd} \mathbb{L}^2 \otimes T\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} \arrow[r, "T\pi_{\mathrm{rot}}"] \arrow[d, "G^{\flat}{}_{\mathrm{mul}}"'] & \mathbb{L}^2 \otimes T\boldsymbol{S}_{\mathrm{rot}} \arrow[d, "G^{\flat}{}_{\mathrm{rot}}"] \\ T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} \arrow[r, "T^*\pi_{\mathrm{rot}}"] & T^*\boldsymbol{S}_{\mathrm{rot}} \end{tikzcd}$$

Theorem 4.10 and the definition of $\hat{\Sigma}$ give

$$\begin{aligned} T^*\pi_{\mathrm{rot}}(p_{\mathrm{rig}}\,,\,\alpha_{\mathrm{mul}}) &= (G^\flat_{\mathrm{rot}} \circ T\pi_{\mathrm{rot}} \circ G^\sharp_{\mathrm{mul}})(p_{\mathrm{rig}}\,,\,\alpha_{\mathrm{mul}}) \\ &= (G^\flat_{\mathrm{rot}} \circ T\pi_{\mathrm{rot}})\big(p_{\mathrm{rig}}\,,\,\tfrac{1}{\mu_1}\,g^\sharp(\alpha_1), \dots, \tfrac{1}{\mu_n}\,g^\sharp(\alpha_n)\big) \\ &= G^\flat_{\mathrm{rot}}\Big(r_{\mathrm{rot}}\,,\,\hat{\Sigma}^{-1}(r_{\mathrm{rot}})\big(\sum_i r_i \times g^\sharp(\alpha_i)\big) \times_{\mathrm{mul}} r_{\mathrm{rot}}\Big) \\ &= \Big(r_{\mathrm{rot}}\,,\,\big((\hat{\Sigma}^{-1})^*(r_{\mathrm{rot}})(\sum_i g^\flat(r_i) \times \alpha_i)\big) \times_{\mathrm{mul}} G^\flat_{\mathrm{mul}}(r_{\mathrm{rot}})\Big)\,. \end{aligned}$$

□

The projection $T^*\pi_{\mathrm{rot}}$ can be expressed in terms of $\boldsymbol{V}_{\mathrm{ang}}$. In this way, we recover the classical formula of the "total momentum" of a multi–form. Here, this formula arises naturally from our geometric interpretation of $T^*\boldsymbol{S}_{\mathrm{rot}}$.

4.12. Corollary. We have the projection

$$\begin{aligned} \mathcal{S}_{\mathrm{ang}} &:= (\tau^{-1}_{\mathrm{ang}})^* \circ T^*\pi_{\mathrm{rot}} : T^*\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}} \\ &\to \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}^*_{\mathrm{ang}} : (p_{\mathrm{rig}},\,\alpha_{\mathrm{mul}}) \mapsto \big(r_{\mathrm{rot}},\,\sum_i g^\flat(r_i) \times \alpha_i\big)\,, \end{aligned} \tag{4.26}$$

where $r_{\mathrm{rot}} := \pi_{\mathrm{rot}}(p_{\mathrm{rig}})$.

Proof. By recalling the expressions of $T^*\pi_{\mathrm{rot}}$, $(\hat{\Sigma}^{-1})^*$, $(\tau^{-1}_{\mathrm{ang}})^*$ and $\hat{\Sigma}$, we obtain

$$\begin{aligned} \mathcal{S}_{\mathrm{ang}}(p_{\mathrm{rig}},\,\alpha_{\mathrm{mul}}) &= \big((\tau^{-1}_{\mathrm{ang}})^* \circ T^*\pi_{\mathrm{rot}}\big)(p_{\mathrm{rig}},\,\alpha_{\mathrm{mul}}) \\ &= (\tau^{-1}_{\mathrm{ang}})^*\Big(r_{\mathrm{rot}},\,\big((\hat{\Sigma}^{-1})^*(r_{\mathrm{rot}})(\sum_j g^\flat(r_j) \times \alpha_j)\big) \times_{\mathrm{mul}} G^\flat_{\mathrm{mul}}(r_{\mathrm{rot}})\Big) \\ &= \Big(r_{\mathrm{rot}},\,\sum_i g^\flat(r_i) \times \big(\big((\hat{\Sigma}^{-1})^*(r_{\mathrm{rot}})(\sum_j g^\flat(r_j) \times \alpha_j)\big) \times \mu_i\, g^\flat(r_i)\big)\Big) \\ &= \Big(r_{\mathrm{rot}},\,\sum_i \Big(g^\flat(r_i) \times G^\flat_{\mathrm{mul}}\big(\hat{\Sigma}^{-1}(r_{\mathrm{rot}})(\sum_j r_j \times g^\sharp(\alpha_j)) \times r_{\mathrm{rot}}\big)_i\Big)\Big) \\ &= \Big(r_{\mathrm{rot}},\,g^\flat\big(\sum_i \mu_i\, r_i \times \big(\hat{\Sigma}^{-1}(r_{\mathrm{rot}})(\sum_j r_j \times g^\sharp(\alpha_i)) \times r_i\big)\big)\Big) \\ &= \big(r_{\mathrm{rot}},\,\sum_i g^\flat(r_i) \times \alpha_i\big)\,. \end{aligned}$$

□

4.5. **Kinetic energy and momentum of the rigid system.**
According to our scheme, we define the *rigid kinetic energy*, the *rigid kinetic momentum*, the *rotational kinetic energy* and the *rotational kinetic momentum* as

$$\mathcal{K}_{\mathrm{rig}} := i^*_{\mathrm{rig}}\mathcal{K}_{\mathrm{mul}} : \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{rig}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!\!R : v_{\mathrm{rig}} \mapsto \tfrac{1}{2}\,m_0\,G_{\mathrm{rig}}(v_{\mathrm{rig}}, v_{\mathrm{rig}})\,,$$

$$\mathcal{P}_{\mathrm{rig}} := i^*_{\mathrm{rig}}\mathcal{P}_{\mathrm{mul}} : \mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{rig}} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\mathrm{rig}} : v_{\mathrm{rig}} \mapsto m_0\,G^\flat_{\mathrm{rig}}(v_{\mathrm{rig}})\,,$$

$$\mathcal{K}_{\mathrm{rot}} := i^*_{\mathrm{rot}}\mathcal{K}_{\mathrm{mul}} : \mathbb{T}^{-1} \otimes T\boldsymbol{S}_{\mathrm{rot}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!\!R : v_{\mathrm{rot}} \mapsto \tfrac{1}{2}\,m_0\,G_{\mathrm{rot}}(v_{\mathrm{rot}}, v_{\mathrm{rot}})\,,$$

$$\mathcal{P}_{\mathrm{rot}} := i^*_{\mathrm{rot}}\mathcal{P}_{\mathrm{mul}} : \mathbb{T}^{-1} \otimes T\boldsymbol{S}_{\mathrm{rot}} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{S}_{\mathrm{rot}} : v_{\mathrm{rot}} \mapsto m_0\,G^\flat_{\mathrm{rot}}(v_{\mathrm{rot}})\,.$$

Then, by taking into account the angular parallelisation $\tau_{\mathrm{ang}} : T\boldsymbol{S}_{\mathrm{rot}} \to \boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}}$, we obtain the *angular kinetic energy* and the *angular kinetic momentum*

$$\begin{aligned}\mathcal{K}_{\mathrm{ang}} &:= \tau^*_{\mathrm{ang}}\, \mathcal{K}_{\mathrm{rot}} : \mathbb{T}^{-1} \otimes T\boldsymbol{S}_{\mathrm{rot}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R : (r_{\mathrm{rot}}, v_{\mathrm{rot}}) \\ &\mapsto \tfrac{1}{2}\, m_0\, \Sigma(r_{\mathrm{rot}})(\Omega, \Omega) = \tfrac{1}{2}\, m_0 \sum_i \mu_i \left(g(r_i, r_i)\, g(\Omega, \Omega) - g(r_i, \Omega)\, g(r_i, \Omega)\right),\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{\mathrm{ang}} &:= \tau^*_{\mathrm{ang}}\, \mathcal{P}_{\mathrm{rot}} : \mathbb{T}^{-1} \otimes T\boldsymbol{S}_{\mathrm{rot}} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes \boldsymbol{V}^*_{\mathrm{ang}} : (r_{\mathrm{rot}}, v_{\mathrm{rot}}) \\ &\mapsto m_0\, \Sigma^\flat(r_{\mathrm{rot}})(\Omega) = m_0 \sum_i \mu_i \left(g(r_i, r_i)\, g^\flat(\Omega) - g(r_i, \Omega)\, g^\flat(r_i)\right),\end{aligned}$$

where $\Omega := \rho_{\mathrm{ang}}(v_{\mathrm{rot}})$.

According to the splittings $T\boldsymbol{P}_{\mathrm{rig}} = T\boldsymbol{P}_{\mathrm{cen}} \times T\boldsymbol{S}_{\mathrm{rot}}$ and $T\boldsymbol{P}_{\mathrm{rig}} = T\boldsymbol{P}_{\mathrm{cen}} \times (\boldsymbol{S}_{\mathrm{rot}} \times \boldsymbol{V}_{\mathrm{ang}})$, we have, respectively, the splittings

$$\begin{aligned}\mathcal{K}_{\mathrm{rig}} &= \mathcal{K}_{\mathrm{cen}} + \mathcal{K}_{\mathrm{rot}}, \qquad & \mathcal{P}_{\mathrm{rig}} &= (\mathcal{P}_{\mathrm{cen}}, \mathcal{P}_{\mathrm{rot}}), \\ \mathcal{K}_{\mathrm{rig}} &= \mathcal{K}_{\mathrm{cen}} + \mathcal{K}_{\mathrm{ang}}, \qquad & \mathcal{P}_{\mathrm{rig}} &= (\mathcal{P}_{\mathrm{cen}}, \mathcal{P}_{\mathrm{ang}}).\end{aligned}$$

4.6. **Connections induced on the rigid system.**
First of all, in the non degenerate case, the generalised affine structure of $\boldsymbol{P}_{\mathrm{rig}}$ induces a flat connection ∇_{aff} (see the Preliminaries).

Moreover, according to the Gauss' Theorem (see, for instance, [7]), the geometric metric g_{rig} and the weighted metric G_{rig} induce two distinct connections ∇_{rig} and ∇_{Rig}, respectively, on $\boldsymbol{P}_{\mathrm{rig}}$. Actually, we shall be mainly concerned with ∇_{Rig}, which is the most important of the two, because of its role in dynamics.

4.13. Proposition. The connection ∇_{Rig} splits into the cartesian product of the connections ∇_{cen} and ∇_{rot} of $\boldsymbol{P}_{\mathrm{cen}}$ and $\boldsymbol{S}_{\mathrm{rot}}$.

4.7. **Kinematics of rigid systems.**
In this section, we apply the splitting of $T\boldsymbol{P}_{\mathrm{mul}}|_{\boldsymbol{P}_{\mathrm{rig}}}$ to the velocity and the acceleration of a rigid system. Namely, the velocity splits into the two components of the center of mass and the velocity relative to the center of mass. On the other hand, the acceleration splits into the three components of the center of mass, relative to the center of mass and the term given by the 2nd fundamental form of the rigid configuration space.

Let us consider a rigid motion $s_{\mathrm{rig}} : \boldsymbol{T} \to \boldsymbol{P}_{\mathrm{rig}}$.

4.14. Proposition. According to Corollary 4.3 and Theorem 4.6, we obtain the splittings

$$\begin{aligned}s_{\mathrm{rig}} &= (s_{\mathrm{cen}}, s_{\mathrm{rot}}) & &: \boldsymbol{T} \to \boldsymbol{P}_{\mathrm{cen}} \times \boldsymbol{S}_{\mathrm{rot}} \\ ds_{\mathrm{rig}} &= \big(ds_{\mathrm{cen}}, (s_{\mathrm{rot}}, \Omega)\big) & &: \boldsymbol{T} \to (\mathbb{T}^{-1} \otimes T\boldsymbol{P}_{\mathrm{cen}}) \times \big(\boldsymbol{S}_{\mathrm{rot}} \times (\mathbb{T}^{-1} \otimes \boldsymbol{V}_{\mathrm{ang}})\big),\end{aligned}$$

where

$$s_{\mathrm{cen}} := \pi_{\mathrm{cen}} \circ s_{\mathrm{rig}}, \quad s_{\mathrm{rot}} := \pi_{\mathrm{rot}} \circ s_{\mathrm{rig}}, \quad \Omega := \hat{\Sigma}^{-1}\Big(\sum_i \mu_i\, (s_{\mathrm{rot}})_i \times d(s_{\mathrm{rot}})_i\Big).$$

The map $\Omega := \rho_{\mathrm{ang}} \circ T\tau_{\mathrm{ang}} \circ ds_{\mathrm{rig}} : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{V}_{\mathrm{ang}}$ is called the *angular velocity* of the rigid motion.

4.15. Theorem. *The acceleration splits into the three components as*

$$\nabla_{\mathrm{mul}} ds_{\mathrm{rig}} = \nabla_{\mathrm{cen}} ds_{\mathrm{cen}} + \nabla_{\mathrm{rot}} ds_{\mathrm{rot}} + N \circ ds_{\mathrm{rot}}\,,$$

where $N : T\boldsymbol{P}_{\mathrm{rig}} \to T^{\perp}\boldsymbol{P}_{\mathrm{rig}}$ *is the 2nd fundamental form of the connection* ∇_{mul}*, with respect to the metric* G_{mul}*. We have the expressions*

$$\rho_{\mathrm{ang}}(\nabla_{\mathrm{rot}} ds_{\mathrm{rot}}) = d\Omega + \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big)\,, \tag{4.27}$$

$$\begin{aligned} N(s_{\mathrm{rig}})(\Omega) &= \Omega \times_{\mathrm{mul}} (\Omega \times_{\mathrm{mul}} s_{\mathrm{rot}}) \\ &\quad - \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big) \times_{\mathrm{mul}} s_{\mathrm{rot}}\,. \end{aligned} \tag{4.28}$$

Proof. The proof is analogous to that for the case of one constrained particle.

In virtue of Theorem 4.6, Corollary 4.3 and the Leibnitz identity for the cross product, we obtain

$$\begin{aligned} \rho_{\mathrm{ang}}(\nabla_{\mathrm{rot}} ds_{\mathrm{rot}}) &= (\rho_{\mathrm{ang}} \circ T\pi_{\mathrm{rot}})(\nabla_{\mathrm{mul}} ds_{\mathrm{rot}}) \\ &= \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\Big(\sum_i \mu_i (s_{\mathrm{rot}})_i \times \nabla d(s_{\mathrm{rot}})_i\Big) \\ &= \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\Big(\sum_i \mu_i (s_{\mathrm{rot}})_i \times \nabla\big(\Omega \times (s_{\mathrm{rot}})_i\big)\Big) \\ &= \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\Big(\sum_i \mu_i (s_{\mathrm{rot}})_i \times \big(d\Omega \times (s_{\mathrm{rot}})_i\big)\Big) \\ &\quad + \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\Big(\sum_i \mu_i (s_{\mathrm{rot}})_i \times \big(\Omega \times (\Omega \times (s_{\mathrm{rot}})_i)\big)\Big) \\ &= \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\Big(\sum_i \mu_i (s_{\mathrm{rot}})_i \times \big(d\Omega \times (s_{\mathrm{rot}})_i\big)\Big) \\ &\quad + \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\Big(\sum_i \mu_i \Omega \times \big((s_{\mathrm{rot}})_i \times (\Omega \times (s_{\mathrm{rot}})_i)\big)\Big) \\ &= d\Omega + \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big)\,. \end{aligned}$$

Moreover, we have

$$\begin{aligned} N(ds_{\mathrm{rot}}) &= \nabla_{\mathrm{mul}} ds_{\mathrm{rot}} - \nabla_{\mathrm{rot}} ds_{\mathrm{rot}} \\ &= d\Omega \times_{\mathrm{mul}} s_{\mathrm{rot}} + \Omega \times_{\mathrm{mul}} (\Omega \times_{\mathrm{mul}} s_{\mathrm{rot}}) - d\Omega \times_{\mathrm{mul}} s_{\mathrm{rot}} \\ &\quad - \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big) \times_{\mathrm{mul}} s_{\mathrm{rot}} \\ &= \Omega \times (\Omega \times_{\mathrm{mul}} s_{\mathrm{rot}}) - \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big) \times_{\mathrm{mul}} s_{\mathrm{rot}}\,. \end{aligned}$$ □

4.16. Note. The map $d\Omega$ is the covariant derivative of Ω with respect to the natural connection ∇_{aff} of $\boldsymbol{S}_{\mathrm{rot}}$ induced by τ_{ang}. Hence, the map $\hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big)$ expresses the Christoffel symbol of the connection ∇_{rot} with respect to the parallelisation τ_{ang}. □

4.8. **Dynamics of rigid systems.**
In this section we study the equation of motion for a rigid system.

According to the results of the previous section, we show that the equation of motion in the environmental space splits into three components: the equation of motion for the center of mass (related to the linear momentum), the equation of motion for the relative motion (related to the angular momentum) and equation for the reaction force (related to the 2nd fundamental form of the rigid configuration space).

4.8.1. *Splitting of multi–forces.*
According to the scheme discussed for a constrained system, we assume a multi–force F_{mul} and consider its restriction to the phase space of the rigid system

$$F := \tilde{F}_{\text{mul}}|_{J_1\boldsymbol{P}_{\text{rig}}} : J_1\boldsymbol{P}_{\text{rig}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\text{mul}} .$$

4.17. Proposition. According to the splitting of $T^*\boldsymbol{P}_{\text{mul}}|\boldsymbol{P}_{\text{rig}}$, we can write

$$F = F_{\text{rig}} + F_{\text{rig}\,\perp} ,$$

where $F_{\text{rig}} = i^*_{\text{rig}}F : J_1\boldsymbol{P}_{\text{rig}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}$. Moreover, according to the splitting of $T^*\boldsymbol{P}_{\text{rig}}$ (see Theorem 4.11), we can write

$$F_{\text{rig}} = (F_{\text{cen}}, F_{\text{rot}}) = (F_{\text{cen}}, F_{\text{ang}}) ,$$

where F_{cen} and F_{ang} turn out to be, respectively, the *total force* and the *total momentum of the force*. We have the following expressions

$$F_{\text{cen}} = T^*\pi_{\text{cen}} \circ F_{\text{rig}} = \sum_i F_i , \qquad F_{\text{ang}} = \mathcal{S}_{\text{ang}} \circ F_{\text{rig}} = \sum_i \underline{r}_i \times F_i ,$$
$$F_{\text{rot}} = ((\Sigma^{-1})^*(F_{\text{ang}})) \times_{\text{mul}} \underline{r}_{\text{rig}} , \qquad F_{\text{dia}} = (\mu_1 F_{\text{cen}}, \ldots, \mu_n F_{\text{cen}}) ,$$
$$F_\perp = F_{\text{mul}} - F_{\text{dia}} - F_{\text{rot}} ,$$

where we have set $\underline{r}_{\text{rig}} : \boldsymbol{S}_{\text{rot}} \subset \boldsymbol{S}_{\text{rel}} \to \mathbb{L}^2 \otimes \boldsymbol{S}^*_{\text{rel}} : r_{\text{rot}} \mapsto G^\flat_{\text{mul}}(r_{\text{rot}})$.

4.18. Note. If the multi–force F_{mul} fulfills the 3rd Newton's principle then its component tangent to the constraint F_{rig} vanishes.

Moreover, we assume a reaction force

$$R : J_1\boldsymbol{P}_{\text{rig}} \to \mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M} \otimes T^*\boldsymbol{P}_{\text{mul}} ,$$

which splits analogously to the force. The above Note holds also for the reaction. So, we assume $R_{\text{rig}} = 0$, i.e. $R = R_\perp$.

4.8.2. *Splitting of the equation of motion.*
Eventually, we are ready to split the equation of motion into three components. We follow the scheme developed for one constrained particle with the additional results arising from the present framework.

The following statement is a consequence of Theorem 4.15.

4.19. Corollary. The rigid motion $s_{\mathrm{rig}} \colon \boldsymbol{T} \to \boldsymbol{P}_{\mathrm{rig}}$ and the reaction force R fulfill the Newton's law of motion

$$m_0\, G^{\flat}_{\mathrm{mul}}(\nabla_{\mathrm{mul}} d s_{\mathrm{rig}}) = (F_{\mathrm{rig}} + R) \circ j_1 s_{\mathrm{rig}}$$

if and only if

$$\begin{aligned} m_0\, G^{\flat}_{\mathrm{cen}}(\nabla_{\mathrm{cen}} d s_{\mathrm{cen}}) &= F_{\mathrm{cen}} \circ j_1 s_{\mathrm{rig}}\,, \\ m_0\, \hat{\Sigma}\big(\tau_{\mathrm{ang}}(\nabla_{\mathrm{rot}} d s_{\mathrm{rot}})\big) &= F_{\mathrm{ang}} \circ j_1 s_{\mathrm{rig}}\,, \end{aligned} \tag{4.29}$$

$$m_0\, G^{\flat}_{\mathrm{mul}}(N \circ d s_{\mathrm{rot}}) = (F_{\perp} + R_{\perp}) \circ j_1 s_{\mathrm{rig}}\,, \tag{4.30}$$

where $\rho_{\mathrm{ang}}(\nabla_{\mathrm{rot}} d s_{\mathrm{rot}}) = d\Omega + \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big)$.

The angular component of the above equation of motion is referred to as *Euler's equation*.

4.20. Note. Contrary to the general constrained case, here the equation on the constrained space can be split into the center of mass and rotational components, due to the fact that the rigid constraint is without interference between particles and the center of mass.

On the other hand, as in the case of a system of free particles, we cannot solve the first two equations independently, unless the total force factors through the projection π_{cen} on $\boldsymbol{P}_{\mathrm{cen}}$. □

4.21. Corollary. The reaction $R_{\perp}$ is given by

$$\begin{aligned} R_{\perp} = \Omega \times \big(\underline{\Omega} \times \underline{r}_{\mathrm{rig}}\big) - \Sigma^{-1*}(s_{\mathrm{rot}})\big(\big(\underline{\Omega} \times \Sigma^{\flat}(s_{\mathrm{rot}})(\Omega)\big) \times_{\mathrm{mul}} \underline{r}_{\mathrm{rig}}\big) \\ - F_{\mathrm{mul}} + (\mu_1\, F_{\mathrm{cen}}, \ldots, \mu_n\, F_{\mathrm{cen}}) + \big((\Sigma^{-1})^{*}(F_{\mathrm{ang}})\big) \times_{\mathrm{mul}} \underline{r}_{\mathrm{rig}}\,. \end{aligned} \tag{4.31}$$

where we set $\underline{\Omega} := g^{\flat}(\Omega)$.

Proof. The reaction $R_{\perp}$ is determined by on solutions of the equations of motion by the following equalities

$$\begin{aligned} R_{\perp} &= G^{\flat}_{\mathrm{mul}}(N \circ d s_{\mathrm{rot}}) - F_{\perp} \circ j_1 s_{\mathrm{rig}} \\ &= G^{\flat}_{\mathrm{mul}}(N \circ d s_{\mathrm{rot}}) - (F_{\mathrm{mul}} - F_{\mathrm{dia}} - F_{\mathrm{rot}}) \circ j_1 s_{\mathrm{rig}} \\ &= G^{\flat}_{\mathrm{mul}}\big(s_{\mathrm{rig}}, \Omega \times_{\mathrm{mul}} (\Omega \times_{\mathrm{mul}} s_{\mathrm{rot}}) - \hat{\Sigma}^{-1}(s_{\mathrm{rot}})\big(\Omega \times \hat{\Sigma}(s_{\mathrm{rot}})(\Omega)\big) \times_{\mathrm{mul}} s_{\mathrm{rot}}\big) \\ &\quad - F_{\mathrm{mul}} \circ j_1 s_{\mathrm{rig}} + (\mu_1\, F_{\mathrm{cen}}, \ldots, \mu_n\, F_{\mathrm{cen}}) \circ j_1 s_{\mathrm{rig}} \\ &\quad + \Big(g^{\flat}\big(\Sigma^{\sharp}(F_{\mathrm{ang}})\big)\Big) \circ j_1 s_{\mathrm{rig}} \times_{\mathrm{mul}} \underline{s}_{\mathrm{rig}} \\ &= \big(s_{\mathrm{rig}},\ \Omega \times (\underline{\Omega} \times_{\mathrm{mul}} \underline{s}_{\mathrm{rig}}) - g^{\flat} \circ \Sigma^{\sharp}(s_{\mathrm{rot}})(\underline{\Omega} \times \Sigma^{\flat}(s_{\mathrm{rot}})(\Omega)) \times_{\mathrm{mul}} \underline{s}_{\mathrm{rig}}\big) \\ &\quad - F_{\mathrm{mul}} \circ j_1 s_{\mathrm{rig}} + (\mu_1\, F_{\mathrm{cen}}, \ldots, \mu_n\, F_{\mathrm{cen}}) \circ j_1 s_{\mathrm{rig}} \\ &\quad + \Big(g^{\flat}\big(\Sigma^{\sharp}(F_{\mathrm{ang}})\big)\Big) \circ j_1 s_{\mathrm{rig}} \times_{\mathrm{mul}} \underline{s}_{\mathrm{rig}}\,. \end{aligned}$$

□

Now, we express the Newton's law in Lagrangian form, in our special case of rigid systems. To this aim, we introduce an appropriate chart on $\boldsymbol{P}_{\mathrm{rig}}$. We refer to a chart (x^i) on $\boldsymbol{P}_{\text{cen}}$ and to a chart (for instance, the *Euler's angles*) (α^j) on $\boldsymbol{S}_{\text{rot}}$. Then, the induced chart on $T\boldsymbol{Q}$ is $(x^i, \alpha^j, \dot{x}^i, \dot{\alpha}^j)$.

Suppose that $F_{\text{rig}} \colon \boldsymbol{P}_{\text{rig}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{P}_{\text{rig}}$ be a conservative positional force, with potential $\mathcal{U}_{\text{rig}} \colon \boldsymbol{P}_{\text{rig}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R$, and that $R_{\text{rig}} = 0$. Then, the induced Lagrangian function turns out to be the map

$$\mathcal{L}_{\text{rig}} := \mathcal{K}_{\text{rig}} - \mathcal{U}_{\text{rig}} \colon T\boldsymbol{P}_{\text{rig}} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes I\!R\,.$$

4.22. Corollary. Let $s_{\text{rig}} \colon \boldsymbol{T} \to \boldsymbol{P}_{\text{rig}}$ be a motion. Then, s_{rig} and the reaction force R fulfill the Newton's law of motion if and only if the following equations hold

$$D\left(\frac{\partial \mathcal{L}_{\text{rig}}}{\partial \dot{x}^i} \circ ds_{\text{rig}}\right) - \frac{\partial \mathcal{L}_{\text{rig}}}{\partial x^i} \circ ds_{\text{rig}} = 0\,, \quad D\left(\frac{\partial \mathcal{L}_{\text{rig}}}{\partial \dot{\alpha}^i} \circ ds_{\text{rig}}\right) - \frac{\partial \mathcal{L}_{\text{rig}}}{\partial \alpha^i} \circ ds_{\text{rig}} = 0\,,$$
$$G^{\flat}_{\text{mul}}(N \circ ds_{\text{rig}}) = (F_{\perp} + R_{\perp}) \circ j_1 s_{\text{rig}}\,.$$

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