

**NON-DEGENERATE HYPERSURFACES
OF A SEMI-RIEMANNIAN MANIFOLD
WITH A SEMI-SYMMETRIC METRIC CONNECTION**

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ABSTRACT. We derive the equations of Gauss and Weingarten for a non-degenerate hypersurface of a semi-Riemannian manifold admitting a semi-symmetric metric connection, and give some corollaries of these equations. In addition, we obtain the equations of Gauss curvature and Codazzi-Mainardi for this non-degenerate hypersurface and give a relation between the Ricci and the scalar curvatures of a semi-Riemannian manifold and of its a non-degenerate hypersurface with respect to a semi-symmetric metric connection. Eventually, we establish conformal equations of Gauss curvature and Codazzi-Mainardi.

1. INTRODUCTION

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced for the first time by Friedmann and Schouten [4] in 1924. In 1932, Hayden [5] introduced a semi-symmetric metric connection on a Riemannian manifold. Yano [10], in 1970, proved the theorem: *In order that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat.* Some topics relative to this theorem were studied by Imai [7] in 1972. Imai [6] gave basic properties of a hypersurface of a Riemannian manifold with the semi-symmetric metric connection and got the conformal equations of Gauss curvature and Codazzi-Mainardi.

In 1986, Duggal and Sharma [3] studied semi-symmetric metric connection in a semi-Riemannian manifold. In this work, they gave some properties of Ricci tensor, affine conformal motions, geodesics and group manifolds with respect to a semi-symmetric metric connection.

In 2001, A. Konar and B. Biswas [8] considered a semi-symmetric metric connection on a Lorentz manifold. They showed that the perfect fluid spacetime with a non-vanishing constant scalar curvature admits a semi-symmetric metric connection whose Ricci tensor vanishes and that it has vanishing speed vector.

In the present paper, we defined a semi-symmetric metric connection on a non-degenerate hypersurface of a semi-Riemannian manifold similar to the hypersurface

2000 *Mathematics Subject Classification*: Primary: 53B15; Secondary: 53B30, 53C05, 53C50.

Key words and phrases: semi-symmetric metric connection, Levi-Civita connection, mean curvature, Ricci tensor, conformally flat.

Received September 9, 2007. Editor P. W. Michor.

of a Riemannian manifold (see [9] for the terminologies of semi-Riemannian manifolds). And we gave the equations of Gauss and Weingarten for a non-degenerate hypersurface of a semi-Riemannian manifold admitting a semi-symmetric metric connection. After having stated these, we derived the equations of Gauss curvature and Codazzi-Mainardi. We obtained a relation between the Ricci and the scalar curvatures of a semi-Riemannian manifold and of its a non-degenerate hypersurface. Then we had a condition under which the Ricci tensor of a non-degenerate hypersurface with respect to the semi-symmetric metric connection is symmetric. Finally, we established the conformal equations of Gauss curvature and Codazzi-Mainardi for this type of a hypersurface.

The semi-symmetric metric connection is one of the three basic types of metric connections, as already described by E. Cartan in [2], and this connection is also called a metric connection with vectorial torsion. Connections with vectorial torsion on spin manifolds may also play a role in superstring theory (see [1]), but this aspect was not discussed in the present paper.

2. PRELIMINARIES

Let \widetilde{M} be an $(n + 1)$ -dimensional differentiable manifold of class C^∞ and M an n -dimensional differentiable manifold immersed in \widetilde{M} by a differentiable immersion

$$i: M \rightarrow \widetilde{M}.$$

$i(M)$, identical to M , is said to be a hypersurface of \widetilde{M} . The differential di of the immersion i will be denoted by B so that a vector field X in M corresponds to a vector field BX in \widetilde{M} . We now suppose that the manifold \widetilde{M} is a semi-Riemannian manifold with the semi-Riemannian metric \widetilde{g} of index $0 \leq \nu \leq n + 1$. Hence the index of \widetilde{M} is the ν and we will denote with $\text{ind } \widetilde{M} = \nu$. If the induced metric tensor $g = \widetilde{g}|_M$ defined by

$$g(X, Y) = \widetilde{g}(BX, BY), \quad \forall X, Y \in \chi(M)$$

is non-degenerate, the hypersurface M is called a *non-degenerate hypersurface*. Also, M is a semi-Riemannian manifold with the induced semi-Riemannian metric g (see [9]). If the semi-Riemannian manifolds \widetilde{M} and M are both orientable, we can choose a unit vector field N defined along M such that

$$\widetilde{g}(BX, N) = 0, \quad \widetilde{g}(N, N) = \varepsilon = \begin{cases} +1, & \text{for spacelike } N \\ -1, & \text{for timelike } N \end{cases}$$

for $\forall X \in \chi(M)$, which is called the unit normal vector field to M , and it should be noted that $\text{ind } M = \text{ind } \widetilde{M}$ if $\varepsilon = 1$, but $\text{ind } M = \text{ind } \widetilde{M} - 1$ if $\varepsilon = -1$.

3. SEMI-SYMMETRIC METRIC CONNECTION

Let \widetilde{M} be an $(n + 1)$ -dimensional differentiable manifold of class C^∞ and $\widetilde{\nabla}$ a linear connection in \widetilde{M} . Then the torsion tensor \widetilde{T} of $\widetilde{\nabla}$ is given by

$$\widetilde{T}(\widetilde{X}, \widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}} \widetilde{X} - [\widetilde{X}, \widetilde{Y}], \quad \forall \widetilde{X}, \widetilde{Y} \in \chi(\widetilde{M})$$

and is of type (1, 2). When the torsion tensor \tilde{T} satisfies

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{\pi}(\tilde{X})\tilde{Y}$$

for a 1-form $\tilde{\pi}$, the connection $\tilde{\nabla}$ is said to be semi-symmetric (see [10]).

Let there be given a semi-Riemannian metric \tilde{g} of index ν with $0 \leq \nu \leq n + 1$ in \tilde{M} and $\tilde{\nabla}$ satisfy

$$\tilde{\nabla}\tilde{g} = 0.$$

Such a linear connection is called a metric connection (see [9]).

We now suppose that the semi-Riemannian manifold \tilde{M} admits a semi-symmetric metric connection given by

$$(3.1) \quad \tilde{\nabla}_{\tilde{X}}\tilde{Y} = \overset{\circ}{\tilde{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Y})\tilde{P}$$

for arbitrary vector fields \tilde{X} and \tilde{Y} of \tilde{M} , where $\overset{\circ}{\tilde{\nabla}}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric \tilde{g} , $\tilde{\pi}$ a 1-form and \tilde{P} the vector field defined by

$$\tilde{g}(\tilde{P}, \tilde{X}) = \tilde{\pi}(\tilde{X})$$

for an arbitrary vector field \tilde{X} of \tilde{M} (see [3]). Since M is a non-degenerate hypersurface, we have

$$\chi(\tilde{M}) = \chi(M) \oplus \chi(M)^\perp.$$

Hence we can write

$$(3.2) \quad \tilde{P} = BP + \lambda N,$$

where P is a vector field and λ a function in M .

Denoting by $\overset{\circ}{\nabla}$ the Levi-Civita connection induced on the non-degenerate hypersurface from $\overset{\circ}{\tilde{\nabla}}$ with respect to the unit spacelike or timelike normal vector field N , from [10] we have

$$(3.3) \quad \overset{\circ}{\nabla}_{BX}BY = B(\overset{\circ}{\nabla}_X Y) + \overset{\circ}{h}(X, Y)N$$

for arbitrary vector fields X and Y of M , where $\overset{\circ}{h}$ is the second fundamental form of the non-degenerate hypersurface M . Denoting by ∇ the connection induced on the non-degenerate hypersurface from $\tilde{\nabla}$ with respect to the unit spacelike or timelike normal vector field N , we have

$$(3.4) \quad \tilde{\nabla}_{BX}BY = B(\nabla_X Y) + h(X, Y)N$$

for arbitrary vector fields X and Y of M , where h is the second fundamental form of the non-degenerate hypersurface M and we call (3.4) the *equation of Gauss* with respect to the induced connection ∇ .

From (3.1), we obtain

$$\tilde{\nabla}_{BX}BY = \overset{\circ}{\tilde{\nabla}}_{BX}BY + \tilde{\pi}(BY)BX - \tilde{g}(BX, BY)\tilde{P},$$

and hence, using (3.3) and (3.4), we have

$$(3.5) \quad \begin{aligned} B(\nabla_X Y) + h(X, Y)N &= B(\overset{\circ}{\nabla}_X Y) + \overset{\circ}{h}(X, Y)N \\ &+ \tilde{\pi}(BY)BX - \tilde{g}(BX, BY)\tilde{P}. \end{aligned}$$

Substituting (3.2) into (3.5), we get

$$B(\nabla_X Y) + h(X, Y)N = B(\overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X, Y)P) + \{\overset{\circ}{h}(X, Y) - \lambda g(X, Y)\}N,$$

from which

$$(3.6) \quad \nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X, Y)P,$$

where $\pi(X) = \tilde{\pi}(BX)$ and

$$(3.7) \quad h(X, Y) = \overset{\circ}{h}(X, Y) - \lambda g(X, Y).$$

Taking account of (3.6), we find

$$\nabla_X(g(Y, Z)) = (\nabla_X g)(Y, Z) + \overset{\circ}{\nabla}_X(g(Y, Z)),$$

from which

$$(3.8) \quad (\nabla_X g)(Y, Z) = 0.$$

We also have from (3.6)

$$(3.9) \quad T(X, Y) = \pi(Y)X - \pi(X)Y.$$

From (3.8) and (3.9), we have the following theorem:

Theorem 3.1. *The connection induced on a non-degenerate hypersurface of a semi-Riemannian manifold with a semi-symmetric metric connection with respect to the unit spacelike or timelike normal vector field is also a semi-symmetric metric connection.*

Now, the equation of Weingarten with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$ is

$$(3.10) \quad \overset{\circ}{\nabla}_{BX}N = -B(\overset{\circ}{A}_N X)$$

for any vector field X in M , where $\overset{\circ}{A}_N$ is a tensor field of type (1, 1) of M defined by

$$g(\overset{\circ}{A}_N X, Y) = \varepsilon \overset{\circ}{h}(X, Y)$$

(see [9]). On the other hand, using (3.1), we get

$$\tilde{\nabla}_{BX}N = \overset{\circ}{\nabla}_{BX}N + \varepsilon \lambda BX$$

since

$$\tilde{\pi}(N) = \tilde{g}(\tilde{P}, N) = \tilde{g}(BP + \lambda N, N) = \lambda \tilde{g}(N, N) = \varepsilon \lambda.$$

Thus using (3.10), we find the *equation of Weingarten* with respect to the semi-symmetric metric connection as

$$(3.11) \quad \tilde{\nabla}_{BX}N = -B(\overset{\circ}{A}_N - \varepsilon\lambda I)X, \quad \varepsilon = \mp 1,$$

where I is the unit tensor. Defining A_N by

$$(3.12) \quad A_N = \overset{\circ}{A}_N - \varepsilon\lambda I,$$

then (3.11) can be written as

$$(3.13) \quad \tilde{\nabla}_{BX}N = -B(A_N X)$$

for any vector field X in M . Then, we have the following corollary:

Corollary 3.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold \tilde{M} . Then*

- i) *If M has a spacelike normal vector field, the shape operator A_N with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is*

$$A_N = \overset{\circ}{A}_N - \lambda I,$$

- ii) *If M has a timelike normal vector field, the shape operator A_N with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is*

$$A_N = \overset{\circ}{A}_N + \lambda I.$$

Now, let $E_1, E_2, \dots, E_\nu, E_{\nu+1}, \dots, E_n$ be principal vector fields corresponding to unit spacelike or timelike normal vector field N with respect to $\overset{\circ}{\nabla}$. Then, by using (3.12), we have

$$(3.14) \quad A_N(E_i) = \overset{\circ}{A}_N(E_i) - \varepsilon\lambda E_i = \overset{\circ}{k}_i E_i - \varepsilon\lambda E_i = (\overset{\circ}{k}_i - \varepsilon\lambda)E_i, \quad 1 \leq i \leq n,$$

where $\overset{\circ}{k}_i$, $1 \leq i \leq n$, are the principal curvatures corresponding to the unit spacelike or timelike normal vector field N with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$. If we write

$$(3.15) \quad k_i = \overset{\circ}{k}_i - \varepsilon\lambda, \quad 1 \leq i \leq n,$$

we deduce that

$$(3.16) \quad A_N(E_i) = k_i E_i, \quad 1 \leq i \leq n,$$

where k_i , $1 \leq i \leq n$, are the principal curvatures corresponding to the normal vector field N (spacelike or timelike) with respect to the semi-symmetric metric connection $\tilde{\nabla}$. Hence, it yields the following:

Corollary 3.3. *Let M be a non-degenerate hypersurface of the semi-Riemannian manifold \tilde{M} . Then*

- i) If M has a spacelike normal vector field, the principal curvatures corresponding to unit spacelike normal N with respect to the semi-symmetric metric connection $\tilde{\nabla}$ are $k_i = \overset{\circ}{k}_i - \lambda$, $1 \leq i \leq n$,
- ii) If M has a timelike normal vector field, the principal curvatures corresponding to unit timelike normal N with respect to the semi-symmetric metric connection $\tilde{\nabla}$ are $k_i = \overset{\circ}{k}_i + \lambda$, $1 \leq i \leq n$.

The function $\frac{1}{n} \sum_{i=1}^n \varepsilon_i \overset{\circ}{h}(E_i, E_i)$ is the mean curvature of M with respect to $\overset{\circ}{\nabla}$ and $\frac{1}{n} \sum_{i=1}^n \varepsilon_i h(E_i, E_i)$ is called the mean curvature of M with respect to ∇ , where

$$\varepsilon_i = \begin{cases} -1, & \text{for timelike } E_i \\ +1, & \text{for spacelike } E_i \end{cases}$$

If $\overset{\circ}{h}$ vanishes, then M is *totally geodesic* with respect to $\overset{\circ}{\nabla}$, and if $\overset{\circ}{h}$ is proportional to g , then M is *totally umbilical* with respect to $\overset{\circ}{\nabla}$ (see [9]). Similarly, if h vanishes, then M is said to be *totally geodesic* with respect to ∇ . If h is proportional to g , then M is said to be *totally umbilical* with respect to ∇ .

From (3.7), we have the following propositions:

Proposition 3.4. *In order that the mean curvature of M with respect to $\overset{\circ}{\nabla}$ coincides with that of M with respect to ∇ , it is necessary and sufficient that the vector field \tilde{P} is tangent to M .*

Proposition 3.5. *A non-degenerate hypersurface is totally umbilical with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$ if and only if it is totally umbilical with respect to the semi-symmetric metric connection ∇ .*

4. EQUATIONS OF GAUSS CURVATURE AND CODAZZI-MAINARDI

We denote by

$$\overset{\circ}{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \overset{\circ}{\nabla}_{\tilde{X}}\overset{\circ}{\nabla}_{\tilde{Y}}\tilde{Z} - \overset{\circ}{\nabla}_{\tilde{Y}}\overset{\circ}{\nabla}_{\tilde{X}}\tilde{Z} - \overset{\circ}{\nabla}_{[\tilde{X}, \tilde{Y}]}\tilde{Z}$$

the curvature tensor of \tilde{M} with respect to $\overset{\circ}{\nabla}$ and by

$$\overset{\circ}{R}(X, Y)Z = \overset{\circ}{\nabla}_X\overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Y\overset{\circ}{\nabla}_X Z - \overset{\circ}{\nabla}_{[X, Y]}Z$$

that of M with respect to $\overset{\circ}{\nabla}$. Then the equation of Gauss curvature is given by

$$\overset{\circ}{R}(X, Y, Z, U) = \overset{\circ}{\tilde{R}}(BX, BY, BZ, BU) + \varepsilon \{ \overset{\circ}{h}(X, U)\overset{\circ}{h}(Y, Z) - \overset{\circ}{h}(Y, U)\overset{\circ}{h}(X, Z) \},$$

where

$$\begin{aligned} \overset{\circ}{\tilde{R}}(BX, BY, BZ, BU) &= \tilde{g}(\overset{\circ}{\tilde{R}}(BX, BY)BZ, BU), \\ \overset{\circ}{R}(X, Y, Z, U) &= g(\overset{\circ}{R}(X, Y)Z, U), \end{aligned}$$

and the equation of Codazzi-Mainardi is given by

$$\overset{\circ}{\tilde{R}}(BX, BY, BZ, N) = \varepsilon\{(\overset{\circ}{\nabla}_X h)(Y, Z) - (\overset{\circ}{\nabla}_Y h)(X, Z)\}$$

(see [9]).

Now, we shall find the equation of Gauss curvature and Codazzi-Mainardi with respect to the semi-symmetric metric connection. The curvature tensor of the semi-symmetric metric connection $\tilde{\nabla}$ of \tilde{M} is, by definition,

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}\tilde{Z}.$$

Putting $\tilde{X} = BX, \tilde{Y} = BY, \tilde{Z} = BZ$, we get

$$\tilde{R}(BX, BY)BZ = \tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ - \tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ - \tilde{\nabla}_{B[X, Y]}BZ.$$

Thus, using (3.4) and (3.13), we have

$$\begin{aligned} \tilde{R}(BX, BY)BZ &= B(R(X, Y)Z + h(X, Z)A_N Y - h(Y, Z)A_N X) \\ &\quad + \{(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ (4.1) \quad &\quad + h(\pi(Y)X - \pi(X)Y, Z)\}N, \end{aligned}$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$$

is the curvature tensor of the semi-symmetric metric connection ∇ . Putting now

$$\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = \tilde{g}(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{U}), \quad R(X, Y, Z, U) = g(R(X, Y)Z, U),$$

we obtain, from (4.1),

$$\begin{aligned} \tilde{R}(BX, BY, BZ, BU) &= R(X, Y, Z, U) \\ (4.2) \quad &\quad + \varepsilon\{h(X, Z)h(Y, U) - h(Y, Z)h(X, U)\}, \end{aligned}$$

and

$$\begin{aligned} \tilde{R}(BX, BY, BZ, N) &= \varepsilon\{(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ (4.3) \quad &\quad + h(\pi(Y)X - \pi(X)Y, Z)\}. \end{aligned}$$

Equations (4.2) and (4.3) are called respectively *the equations of Gauss curvature and Codazzi-Mainardi* with respect to the semi-symmetric metric connection.

5. THE RICCI AND SCALAR CURVATURES

We denote by R the Riemannian curvature tensor of a non-degenerate hypersurface M with respect to the semi-symmetric metric connection ∇ and by $\overset{\circ}{R}$ that of M with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$. Then, by a straightforward computation, we find

$$(5.1) \quad \begin{aligned} R(X, Y)Z &= \overset{\circ}{R}(X, Y)Z - \alpha(Y, Z)X + \alpha(X, Z)Y \\ &\quad - g(Y, Z)\gamma(X) + g(X, Z)\gamma(Y), \end{aligned}$$

where

$$(5.2) \quad \alpha(Y, Z) = (\overset{\circ}{\nabla}_Y \pi)Z - \pi(Y)\pi(Z) + \frac{1}{2}g(Y, Z)\pi(P)$$

and

$$(5.3) \quad \gamma(Y) = \overset{\circ}{\nabla}_Y P - \pi(Y)P + \frac{1}{2}\pi(P)Y$$

such that

$$g(\gamma(Y), Z) = \alpha(Y, Z).$$

Theorem 5.1. *The Ricci tensor of a non-degenerate hypersurface M with respect to the semi-symmetric metric connection is symmetric if and only if π is closed.*

Proof. The Ricci tensor of a non-degenerate hypersurface M with respect to semi-symmetric metric connection is given by

$$\text{Ric}(X, Y) = \sum_{i=1}^n \varepsilon_i g(R(E_i, X)Y, E_i).$$

Then, from (5.1) we get

$$\text{Ric}(Y, Z) = \overset{\circ}{\text{Ric}}(Y, Z) - (n-2)\alpha(Y, Z) + ag(Y, Z)$$

where $\overset{\circ}{\text{Ric}}$ denotes the Ricci tensor of M with respect to the Levi-Civita connection and $a = \text{trace of } \gamma$ given by (5.3). Since $\overset{\circ}{\text{Ric}}$ is symmetric, we obtain

$$(5.4) \quad \begin{aligned} \text{Ric}(Y, Z) - \text{Ric}(Z, Y) &= (n-2)\{\alpha(Z, Y) - \alpha(Y, Z)\} \\ &= 2(n-2)d\pi(Y, Z). \end{aligned}$$

Hence, from (5.4) we find that the Ricci tensor of M with respect to the semi-symmetric connection is symmetric if and only if $d\pi = 0$, where d denotes exterior differentiation. That is, π is closed. \square

Theorem 5.2. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold \widetilde{M} . If $\widetilde{\text{Ric}}$ and Ric are the Ricci tensors of \widetilde{M} and M with respect to the*

semi-symmetric metric connection, respectively, then for $\forall X, Y \in \chi(M)$

$$(5.5) \quad \begin{aligned} \widetilde{\text{Ric}}(BX, BY) &= \text{Ric}(X, Y) - f h(X, Y) \\ &+ \varepsilon \left\{ \sum_{i=1}^n \varepsilon_i k_i^2 g(X, E_i) g(Y, E_i) + \widetilde{g}(\widetilde{R}(N, BX)BY, N) \right\} \end{aligned}$$

where $\varepsilon_i = g(E_i, E_i)$, $\varepsilon_i = 1$, if E_i is spacelike or $\varepsilon_i = -1$, if E_i is timelike, and $f = \text{trace of } A_N$.

Proof. Suppose that $\{BE_1, \dots, BE_\nu, BE_{\nu+1}, \dots, BE_n, N\}$ is an orthonormal basis of $\chi(\widetilde{M})$, then the Ricci curvature of \widetilde{M} with respect to the semi-symmetric metric connection is

$$(5.6) \quad \widetilde{\text{Ric}}(BX, BY) = \sum_{i=1}^n \varepsilon_i \widetilde{g}(\widetilde{R}(BE_i, BX)BY, BE_i) + \varepsilon \widetilde{g}(\widetilde{R}(N, BX)BY, N)$$

for all $X, Y \in \chi(M)$. By using the equation of Gauss curvature (4.2) and (3.16), and considering the symmetry of shape operator we get

$$(5.7) \quad \begin{aligned} g(\widetilde{R}(BE_i, BX)BY, BE_i) &= g(R(E_i, X)Y, E_i) \\ &+ \varepsilon g(A_N E_i, Y) g(A_N E_i, X) - h(X, Y) g(A_N E_i, E_i). \end{aligned}$$

Hence, inserting (5.7) into (5.6) yields to (5.5). □

Theorem 5.3. *Let M be a non-degenerate hypersurface of a semi-Riemannian manifold \widetilde{M} . If $\widetilde{\rho}$ and ρ are the scalar curvatures of \widetilde{M} and M with respect to the semi-symmetric metric connection, respectively, then*

$$(5.8) \quad \widetilde{\rho} = \rho - \varepsilon f^2 + f^* + 2\varepsilon \widetilde{\text{Ric}}(N, N)$$

where $f = \text{trace of } A_N$ and $f^* = \text{trace of } A_N^2$.

Proof. Assume that $\{BE_1, \dots, BE_\nu, BE_{\nu+1}, \dots, BE_n, N\}$ is an orthonormal basis of $\chi(\widetilde{M})$, then the scalar curvature of \widetilde{M} with respect to the semi-symmetric metric connection is

$$(5.9) \quad \widetilde{\rho} = \sum_{i=1}^n \varepsilon_i \widetilde{\text{Ric}}(E_i, E_i) + \varepsilon \widetilde{\text{Ric}}(N, N).$$

As (5.5) is considered, we get

$$\widetilde{\text{Ric}}(E_i, E_i) = \text{Ric}(E_i, E_i) + \varepsilon \{g(\widetilde{R}(N, e_i)e_i, N) + 2\varepsilon_i k_i^2\}$$

Hence, we obtain

$$\widetilde{\rho} = \rho - \varepsilon f^2 + f^* + 2\varepsilon \widetilde{\text{Ric}}(N, N).$$

□

6. THE CONFORMAL EQUATIONS OF GAUSS CURVATURE AND CODAZZI-MAINARDI

Denoting the conformal curvature tensors of type $(0, 4)$ of the semi-symmetric metric connections $\tilde{\nabla}$ and ∇ , respectively, by \tilde{C} and C we have

$$(6.1) \quad \begin{aligned} \tilde{C}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) &= \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) + \tilde{g}(\tilde{X}, \tilde{U})\tilde{L}(\tilde{Y}, \tilde{Z}) - \tilde{g}(\tilde{Y}, \tilde{U})\tilde{L}(\tilde{X}, \tilde{Z}) \\ &+ \tilde{g}(\tilde{Y}, \tilde{Z})\tilde{L}(\tilde{X}, \tilde{U}) - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{L}(\tilde{Y}, \tilde{U}), \end{aligned}$$

where

$$\tilde{L}(\tilde{Y}, \tilde{Z}) = -\frac{1}{n-1}\widetilde{\text{Ric}}(\tilde{Y}, \tilde{Z}) + \frac{\tilde{\rho}}{2n(n-1)}\tilde{g}(\tilde{Y}, \tilde{Z})$$

and $\widetilde{\text{Ric}}$ is the Ricci tensor and $\tilde{\rho}$ is the scalar curvature of \tilde{M} with respect to the connection $\tilde{\nabla}$. Similarly, we get

$$(6.2) \quad \begin{aligned} C(X, Y, Z, U) &= R(X, Y, Z, U) + g(X, U)L(Y, Z) - g(Y, U)L(X, Z) \\ &+ g(Y, Z)L(X, U) - g(X, Z)L(Y, U), \end{aligned}$$

where

$$L(Y, Z) = -\frac{1}{n-2}\text{Ric}(Y, Z) + \frac{\rho}{2(n-1)(n-2)}g(Y, Z)$$

and Ric is the Ricci tensor and ρ is the scalar curvature of M with respect to the connection ∇ . From (4.2), we have

$$(6.3) \quad \widetilde{\text{Ric}}(BY, BZ) - \varepsilon\tilde{R}(N, BY, BZ, N) = \text{Ric}(Y, Z) - \varepsilon fh(Y, Z) + h(A_N Y, Z),$$

where $f = \text{trace of } A_N$. On the other hand, from (6.1), we find

$$(6.4) \quad \begin{aligned} \tilde{C}(N, BY, BZ, N) &= \tilde{R}(N, BY, BZ, N) + \varepsilon\frac{\tilde{\rho}}{n(n-1)}g(Y, Z) \\ &- \frac{1}{n-1}\{\varepsilon\widetilde{\text{Ric}}(BY, BZ) + \widetilde{\text{Ric}}(N, N)g(Y, Z)\}. \end{aligned}$$

Substituting (6.4) into (6.3), we get

$$(6.5) \quad \begin{aligned} \text{Ric}(Y, Z) &= \frac{n-2}{n-1}\widetilde{\text{Ric}}(BY, BZ) - \varepsilon\tilde{C}(N, BY, BZ, N) \\ &- \left\{ \frac{1}{n-1}\varepsilon\widetilde{\text{Ric}}(N, N) - \frac{1}{n(n-1)}\tilde{\rho} \right\} g(Y, Z) \\ &+ \varepsilon fh(Y, Z) - h(A_N Y, Z). \end{aligned}$$

From (6.5) and (5.8), we have

$$(6.6) \quad \begin{aligned} L(Y, Z) &= \tilde{L}(BY, BZ) + \frac{1}{n-2}\{\varepsilon\tilde{C}(N, BY, BZ, N) - \varepsilon fh(Y, Z) + h(A_N Y, Z)\} \\ &+ \frac{1}{2(n-1)(n-2)}(\varepsilon f^2 - f^*)g(Y, Z), \end{aligned}$$

where $f^* = \text{trace of } A_N^2$. Thus, from (6.1), we obtain

$$\begin{aligned}
 \tilde{C}(BX, BY, BZ, BU) &= \tilde{R}(BX, BY, BZ, BU) + g(X, U)\tilde{L}(BY, BZ) \\
 &\quad - g(Y, U)\tilde{L}(BX, BZ) + g(Y, Z)\tilde{L}(BX, BU) \\
 (6.7) \qquad \qquad \qquad &\quad - g(X, Z)\tilde{L}(BY, BU).
 \end{aligned}$$

Using (6.2), (6.6), (6.7) and (4.2), we get

$$\begin{aligned}
 C(X, Y, Z, U) &= \tilde{C}(BX, BY, BZ, BU) + \varepsilon\{h(Y, Z)h(X, U) - h(X, Z)h(Y, U)\} \\
 &\quad + \frac{\varepsilon}{n-2}\{\tilde{C}(N, BY, BZ, N)g(X, U) - \tilde{C}(N, BX, BZ, N)g(Y, U) \\
 &\quad + \tilde{C}(N, BX, BU, N)g(Y, Z) - \tilde{C}(N, BY, BU, N)g(X, Z)\} \\
 &\quad - \frac{1}{n-2}\{(\varepsilon fh(Y, Z) - h(A_N Y, Z))g(X, U) - (\varepsilon fh(X, Z) \\
 &\quad - h(A_N X, Z))g(Y, U) + (\varepsilon fh(X, U) - h(A_N X, U))g(Y, Z) \\
 &\quad - (\varepsilon fh(Y, U) - h(A_N Y, U))g(X, Z)\} \\
 (6.8) \qquad \qquad \qquad &\quad + \frac{(\varepsilon f^2 - f^*)}{(n-1)(n-2)}\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}.
 \end{aligned}$$

Equation (6.8) is the conformal equation of Gauss curvature. Hence, from (6.1), we have

$$\begin{aligned}
 \tilde{C}(BX, BY, BZ, N) &= \tilde{R}(BX, BY, BZ, N) \\
 (6.9) \qquad \qquad \qquad &\quad - \frac{1}{n-1}\{g(Y, Z)\widetilde{\text{Ric}}(BX, N) - g(X, Z)\widetilde{\text{Ric}}(BY, N)\}.
 \end{aligned}$$

Taking into consideration equation (4.3), we obtain

$$\begin{aligned}
 \tilde{C}(BX, BY, BZ, N) &= \varepsilon\{(\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z) + h(\pi(Y)X - \pi(X)Y, Z)\} \\
 (6.10) \qquad \qquad \qquad &\quad - \frac{1}{n-1}\{g(Y, Z)\widetilde{\text{Ric}}(BX, N) - g(X, Z)\widetilde{\text{Ric}}(BY, N)\}.
 \end{aligned}$$

Equation (6.10) is the conformal equation of Codazzi-Mainardi.

We suppose that the semi-Riemannian manifold \tilde{M} is conformally flat ($\tilde{C} = 0$) and that the $(n > 3)$ -dimensional non-degenerate hypersurface M is totally umbilical, then we have $\tilde{R} = 0$ (see [3]) and we also have $h = cg$, since M is totally umbilical with respect to ∇ by Proposition 3.5. Then from (6.8) we get the following theorem:

Theorem 6.1. *A totally umbilical non-degenerate hypersurface in a conformally flat semi-Riemannian manifold with a semi-symmetric metric connection is conformally flat.*

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