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ON THE PRIME GRAPHS OF THE AUTOMORPHISM GROUPS OF SPORADIC SIMPLE GROUPS

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Abstract. In this paper as the main result, we determine finite groups with the same prime graph as the automorphism group of a sporadic simple group, except J_2.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then the set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of order elements of G is denoted by $\pi_e(G)$. We construct the prime graph of G as follows: *The prime graph* $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G)$, $\pi_2(G)$, …, $\pi_{t(G)}(G)$ be the connected components of $\Gamma(G)$. We use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. The author in [11] determined finite groups with the same prime graph as $PSL(2,q)$. For some simple groups S, finite groups with the same prime graph as $\Gamma(S)$ are determined (see the references of [11]). Hagie in [7] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. As the main result of this paper, we determine finite groups with the same prime graph as the automorphism group of a sporadic simple group, except J_2. The structure of the automorphism groups of sporadic simple groups are described in [3].

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [3].

2. Preliminary results

First we give an easy remark:

Remark 2.1. Let N be a normal subgroup of G and $p \sim q$ in $\Gamma(G/N)$. Then $p \sim q$ in $\Gamma(G)$. In fact if $xN \in G/N$ has order pq, then there is a power of x which has order pq.

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Definition 2.1 ([6]). A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.1 ([2, Lemma 5]). *Let G be a finite group with disconnected prime graph. Then we have two possibilities.*

(i) *G is a Frobenius group or a 2-Frobenius group;*

(ii) *G has a chain $G \supseteq M \supseteq N \supseteq 1$ of normal subgroups such that N is a nilpotent π-group, M/N is a non-abelian simple group and G/M is a solvable π-group where π is the connected component of $\Gamma(G)$ containing 2.*

By the above lemma it follows that if G is a solvable group with $t(G) \geq 2$, then G is a Frobenius group or a 2-Frobenius group, and $t(G) = 2$.

Lemma 2.2 ([12]). *Let G be a finite group, N a normal subgroup of G, and G/N a Frobenius group with Frobenius kernel F and cyclic complement C. If $\big(|F|, |N|\big) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.*

Lemma 2.3 ([14]). *Let G be a finite group and N a nontrivial normal p-subgroup, for some prime p, and set $K = G/N$. Suppose that K contains an element x of order m coprime to p such that $\langle \varphi|_{\langle x\rangle}, 1|_{\langle x\rangle}\rangle > 0$ for every Brauer character φ of (an absolutely irreducible representation of) K in characteristic p. Then G contains elements of order pm.*

Definition 2.2. Let p be a prime number. A group G is called a C_{pp} group if the centralizers in G of its elements of order p are p-groups.

Lemma 2.4 ([1]).

(i) *The $C_{13,13}$-simple groups are: A_{13}, A_{14}, A_{15};* Suz, Fi_{22}*; $L_2(q)$, $q = 3^3$, 5^2, 13^n or $2 \times 13^n - 1$ which is a prime, $n \geq 1$; $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(q)$, $q = 2^2$, 3; $F_4(2)$,$U_3(q)$, $q = 2^2$, 23; $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$.*

(ii) *The $C_{19,19}$-simple groups are: A_{19}, A_{20}, A_{21}; J_1, J_3, $O'N$, Th, HN; $L_2(q)$, $q = 19^n$, $2 \times 19^n - 1$ which is a prime, $(n \geq 1)$; $L_3(7)$, $U_3(2^3)$, $R(3^3)$, ${}^2E_6(2)$.*

Lemma 2.5 (Zsigmondy's Theorem [16]). *Let p be a prime and n be a positive integer. Then one of the following holds:*

(i) *p is a Mersenne prime and $n = 2$;*

(ii) *$p = 2$, $n = 1$ or 6;*

(iii) *there is a primitive prime p' for $p^n - 1$, that is, $p'|(p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$.*

Lemma 2.6 ([4]).

(i) *With the exceptions of the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation $p^m - 2q^n = \pm 1$; where p and q are prime and $m, n > 1$; has exponents $m = n = 2$.*

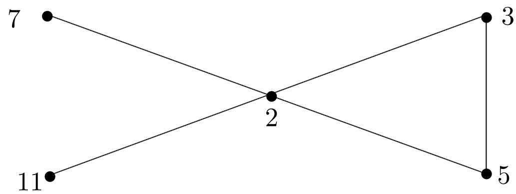
(ii) *The only solution of the equation* $p^m - q^n = 1$*; p, q prime; and* $m, n > 1$ *is* $3^2 - 2^3 = 1$.

3. Main results

We note that for some of the sporadic simple groups we have $\mathrm{Aut}(S) = S$. Also if S is one of the following groups: M_{12}, He, Fi_{22} or HN, then $\mathrm{Aut}(S) \neq S$ but $\Gamma(S) = \Gamma(\mathrm{Aut}(S))$. These cases were considered by Hagie [7]. Therefore we consider the case $A = \mathrm{Aut}(S)$, where S is one of the following groups: M_{22}, J_3, HS, Suz, $O'N$, Fi'_{24} or McL. First we consider $\mathrm{Aut}(McL)$, since its prime graph is connected.

Theorem 3.1. *Let G be a finite group such that $\Gamma(G) = \Gamma(\mathrm{Aut}(McL))$. Then $G/O_2(G)$ is isomorphic to HS, $\mathrm{Aut}(HS)$, McL, $\mathrm{Aut}(McL)$, $U_6(2)$ or $U_6(2):2$,*

Proof. We note that the prime graph of $\mathrm{Aut}(McL)$ is connected and $\Gamma(G)$ is as follows:



If G is a solvable group, then consider a Hall $\{5,7,11\}$-subgroup H of G. Then H is solvable and $t(H) = 3$, which is a contradiction. Therefore G is a non-solvable group.

Let N be a maximal normal solvable subgroup of G. It is obvious that $N \neq G$. Let $\overline{G} = G/N$ and $S = \mathrm{Socle}(\overline{G})$. We know that $C_{\overline{G}}(S) = 1$ and $N_{\overline{G}}(S) = \overline{G}$, which implies that $S \leq \overline{G} \leq \mathrm{Aut}(S)$. The socle of a group is a direct product of minimal normal subgroups and so $S = M_1 \times M_2 \times \cdots \times M_r$, where M_i, $1 \leq i \leq r$, are minimal normal subgroups. Also every minimal normal subgroup is characteristically simple and so is a product of isomorphic simple groups. Hence $S = P_1 \times \cdots \times P_k$, where P_i, $1 \leq i \leq k$, are non-abelian simple groups.

Step 1. *If $A = \pi(N) \cap \{5,7,11\}$, then A has at most one element.*

If $|A| = 3$, then similar to the above argument we get a contradiction. If $|A| = 2$, then let $A = \{p_1, p_2\}$, $p \in \{5,7,11\} \setminus A$ and H be a Hall A-subgroup of N. Now N is a normal subgroup of G and H is a Hall subgroup of N. Therefore $G = NN_G(H)$, by the Frattini argument. Since $p \notin \pi(N)$, it follows that $p \in \pi\big(N_G(H)\big)$ and so there is an element $y \in N_G(H)$ of order p. It is obvious that y acts fixed point freely on H and $o(y) = p$. Therefore H is nilpotent by Thompson's Theorem [5, Theorem 10.2.1], which implies that $p_1 \sim p_2$, a contradiction. Similarly we can prove that $\pi(N) \cap \{3,7,11\}$ has at most one element.

As a consequence of this result we conclude that $\pi(\overline{G}) \cap A$ has at least two elements and so there exists $p \in \{7,11\}$ such that $p \in \pi(\overline{G})$.

Step 2. *The subgroup S is a nonabelian simple group.*

As we mentioned above, $S = P_1 \times \cdots \times P_k$, where every P_i, $1 \le i \le k$, is a non-abelian simple group. Also note that $\pi(S) \subseteq \pi(G) = \{2,3,5,7,11\}$ and so $\pi(P_i) \subseteq \{2,3,5,7,11\}$, for every $1 \le i \le k$. There exist only finitely many nonabelian simple groups P such that $\pi(P) \subseteq \{2,3,5,7,11\}$ and if P is a nonabelian simple group such that $\pi(P) \subseteq \{2,3,5,7,11\}$, then we can see that $2,3 \in \pi(P)$ and $\pi\big(\mathrm{Out}(P)\big) \subseteq \{2,3\}$ (see [13]).

We claim that $k = 1$. Let $k \ge 2$. Then $7, 11 \notin \pi(S)$, since $3 \in \pi(P_i)$, for every $1 \le i \le k$, and $3 \nsim 7$ and $3 \nsim 11$ in $\Gamma(G)$. Hence $\pi(P_i) \subseteq \{2,3,5\}$ and by using [13] we see that for every $1 \le i \le k$, P_i is isomorphic to A_5, A_6 or $U_4(2)$. On the other hand, $7, 11 \in \pi\big(\mathrm{Out}(S)\big)$, since $Z(S) = 1$. We note that $\{7,11\} \cap \pi(N)$ has at most one element. So let $p \in \{7,11\} \cap \pi(\overline{G})$ and let $\varphi \in \overline{G}$ be an element of order p. Obviously $\varphi \in \mathrm{Aut}(S)$. Let $Q = P_1^{\varphi}$ and $f_i \colon Q \to P_i$, $1 \le i \le k$, be the natural projection of Q to P_i. Also P_1 is a normal subgroup of S and so Q is a normal subgroup of S. Therefore $Im\ f_i \trianglelefteq P_i$ and P_i is a simple group, which implies that $Im\ f_i = 1$ or $Im\ f_i = P_i$, for every $1 \le i \le k$. On the other hand, P_1 is a simple group, and so Q is a simple group. Therefore $\ker f_i = 1$ or $\ker f_i = Q$. If $\ker f_i = 1$, then $\mathrm{Im}\ f_i = P_i$, which implies that $Q \cong P_i$. Also if $\ker f_i = Q$, then $\mathrm{Im}\ f_i = 1$. Hence there exists a unique j, $1 \le j \le k$, such that $P_1^{\varphi} = P_j$. Now if $j \ne 1$, then there exists a φ-orbit of length p. Without loss of generality let $\{P_1, \ldots, P_p\}$ be a φ-orbit. As we mentioned above $3 \in \pi(P_1)$. Let $g_1 \in P_1$ be an element of order 3 and let $g_{i+1} = g_i^{\varphi}$, where $1 \le i \le p-1$. Now let x be the element of S whose projections x_i to P_i are defined as follows: $x_i = g_i$ for $i = 1, \ldots, p$ and $x_i = 1$ otherwise. Obviously x is of order 3 and so $x\varphi \in \overline{G}$ is of order $3p$, which is a contradiction since $3 \nsim p$ in $\Gamma(G)$. Therefore for every $1 \le i \le k$, we have $P_i^{\varphi} = P_i$. Since $\varphi \ne 1$, there exists $1 \le i \le k$ such that φ acts nontrivially on P_i. Therefore φ induces an outer automorphism of P_i of order p. Hence p is a divisor of $\big|\mathrm{Out}(P_i)\big|$, which is a contradiction. Therefore $k = 1$ and S is a nonabelian simple group.

Step 3. *The subgroup S is isomorphic to McL, HS or $U_6(2)$.*

Up to now we prove that there is a nonabelian simple group S such that $S \le G/N \le \mathrm{Aut}(S)$. Also we know that $\pi(S) \subseteq \{2,3,5,7,11\}$. Now we consider each possibility for S, separately.

If $S \cong A_5$, then $\pi(S) = \pi\big(\mathrm{Aut}(S)\big) = \{2,3,5\}$ and so $\{7,11\} \subseteq \pi(N)$, which is a contradiction by Step 1. Similarly it follows that S is not isomorphic to $L_2(7)$, $L_2(8)$, $A_6 \cong L_2(9)$, $U_3(3)$, $U_4(2)$.

If $S \cong L_2(11)$, then $\pi(S) = \{2,3,5,11\}$ and so $7 \in \pi(N)$. Also $S \le G/N$ contains a Frobenius subgroup $11 : 5$ of order 55. Now by using Lemma 2.2, G contains an element of order 35, which is a contradiction. Similarly if $S \cong M_{11}$, M_{12}, $U_5(2)$, then $L_2(11) < S$ and $7 \in \pi(N)$. Therefore similarly follows that $5 \sim 7$ in $\Gamma(G)$, which is a contradiction.

If $S \cong A_7$, $A_8 \cong L_4(2)$, $L_3(4)$, $L_2(49)$, $U_3(5)$, A_9, J_2, $S_6(2)$, $U_4(3)$, $O_8^+(2)$, then $L_2(7) < S$ and $\pi(S) = \{2,3,5,7\}$. Therefore $11 \in \pi(N)$ and also $L_2(7)$ contains a Frobenius subgroup $7 : 3$ of order 21. Now Lemma 2.4 implies that G contains an element of order 33 and so $3 \sim 11$ in $\Gamma(G)$, which is a contradiction.

If $S \cong A_{10}$, A_{11}, A_{12}, $S_4(7)$, then $3 \sim 7$ in $\Gamma(S)$, which is a contradiction by Remark 2.1, since $3 \nsim 7$ in $\Gamma(G)$.

If $S \cong M_{22}$, then since $3 \nsim 5$ in $\Gamma(S)$ it follows that $3 \in \pi(N)$ or $5 \in \pi(N)$.

Let $5 \in \pi(N)$. Let $x \in G/N$, $X = \langle x \rangle$ and $o(x) = 11$. Now by using [9] about the irreducible characters of $M_{22} \pmod 5$, we can see that

$$\begin{aligned}
\langle 1_G|_X, 1|_X \rangle &= 1; \\
\langle 21|_X, 1|_X \rangle &= \frac{1}{11}(21 + (-1) \times 10) = 1; \\
\langle 45_1|_X, 1|_X \rangle = \langle 45_2|_X, 1|_X \rangle &= \frac{1}{11}(45 + 10) = 5; \\
\langle 55|_X, 1|_X \rangle &= \frac{1}{11}(55 + 0) = 5; \\
\langle 98|_X, 1|_X \rangle &= \frac{1}{11}(98 + (-1) \times 10) = 8; \\
\langle 133|_X, 1|_X \rangle &= \frac{1}{11}(133 + 10) = 13; \\
\langle 210|_X, 1|_X \rangle &= \frac{1}{11}(210 + 10) = 20; \\
\langle 385|_X, 1|_X \rangle &= \frac{1}{11}(385 + 0) = 35;
\end{aligned}$$

$$\begin{aligned}
\langle 280_1|_X, 1|_X \rangle = \langle 280_2|_X, 1|_X \rangle &= \frac{1}{11}(280 + 5(b_{11} + \overline{b_{11}})) \\
&= \frac{1}{11}(280 + 5(\frac{-1 + i\sqrt{11}}{2} + \frac{-1 - i\sqrt{11}}{2})) = 25\,.
\end{aligned}$$

Therefore for every irreducible character φ of $M_{22} \pmod 5$ we show that

$$\langle \varphi|_X, 1|_X \rangle = \frac{1}{|X|} \sum_{x \in X} \varphi(x) > 0\,.$$

Now by using Lemma 2.3, it follows that $55 \in \pi_e(G)$, which is a contradiction. Therefore $5 \notin \pi(N)$. Similarly we can prove that $3 \notin \pi(N)$ and so $S \ncong M_{22}$.

If $S \cong HS$, then $HS \leq G/N \leq \mathrm{Aut}(HS)$. Therefore $G/N \cong HS$ or $G/N \cong \mathrm{Aut}(HS)$. In each case there exists a subgroup H of G such that $H/N \cong HS$. If $\{3, 5, 11\} \cap \pi(N) \neq \emptyset$, then let $p \in \{3, 5, 11\} \cap \pi(N)$, x be an element of order 7 in H/N and $X = \langle x \rangle$. Similar to the last case by using [9] we can see that for every irreducible character φ of $HS \pmod p$ we have

$$\langle \varphi|_X, 1|_X \rangle = \frac{1}{|X|} \sum_{x \in X} \varphi(x) > 0\,,$$

and so G has an element of order $7p$, by Lemma 2.3, which is a contradiction. Similarly it follows that $7 \notin \pi(N)$. Therefore N is a 2-group.

Similar to the above discussion it follows that $G/O_2(G) \cong McL$. With the same method we conclude that $G/O_2(G) \cong \mathrm{Aut}(McL)$, $G/O_2(G) \cong U_6(2)$ or $G/O_2(G) \cong U_6(2) : 2$. We omit the details of the proof for convenience. Now the proof of this theorem is completed. $\square$

We note that if k is a natural number, then obviously

$$\begin{aligned}\Gamma\big(\operatorname{Aut}(McL)\big) &= \Gamma(\mathbb{Z}_{2^k} \times \operatorname{Aut}(HS)) = \Gamma(\mathbb{Z}_{2^k} \times HS) = \Gamma(\mathbb{Z}_{2^k} \times McL) \\ &= \Gamma\big(\mathbb{Z}_{2^k} \times \operatorname{Aut}(McL)\big) = \Gamma\big(\mathbb{Z}_{2^k} \times U_6(2)\big) = \Gamma\big(\mathbb{Z}_{2^k} \times U_6(2):2\big).\end{aligned}$$

Now we discuss about the automorphism group of M_{22}, J_3, HS, Suz, $O'N$ and Fi'_{24}. We note that the prime graphs of the automorphism groups of these groups are disconnected. Now by using Lemma 2.1 we have the following result.

Lemma 3.1. *Let G be a finite group and let A be the automorphism group of M_{22}, J_3, HS,* Suz*, $O'N$ or Fi'_{24}. If $\Gamma(G) = \Gamma(A)$, then one of the following holds:*
(a) *G is a Frobenius or a 2-Frobenius group;*
(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group with $t(K/H) \geq 2$ and $G/K \leq \operatorname{Out}(K/H)$. Also $\pi_2(A) = \pi_i(K/H)$ for some $i \geq 2$ and $\pi_2(A) \subseteq \pi(K/H) \subseteq \pi(S)$.*

Lemma 3.2. *Let M be a simple group of Lie type over $GF(q)$, where $q = p_0^\alpha$ and p_0 is a prime number.*

- **(a)** *If $p_0 \in \{2,3,5,7\}$, and M is a $C_{11,11}$-group, then M is one of the following simple groups: $L_2(11)$, $L_5(3)$, $L_6(3)$, $U_5(2)$, $U_6(2)$, $O_{11}(3)$, $S_{10}(3)$ or $O^+_{10}(3)$.*
- **(b)** *If $p_0 \in \{2,3,5,7,11,13,17,19,23\}$ and M is a $C_{29,29}$-group, then $M = L_2(29)$.*
- **(c)** *If $p_0 \in \{2,3,5,7,11,19\}$ and M is a $C_{31,31}$-group, then M is $L_5(2)$, $L_3(5)$, $L_6(2)$, $L_4(5)$, $O^+_{10}(2)$, $O^+_{12}(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or* $\mathrm{Sz}(32)$.

Proof. The odd order components of finite non-abelian simple groups are listed in Table 1 in [8]. The odd order components of some non-abelian simple groups of Lie type are of the form $(q^p \pm 1)/\big((q \pm 1)(p, q \pm 1)\big)$. Therefore we consider the following diophantine equations:

$$\text{(i)} \quad \frac{q^p - 1}{q - 1} = y^n, \qquad \text{(ii)} \quad \frac{q^p - 1}{(q-1)(p, q-1)} = y^n,$$

$$\text{(iii)} \quad \frac{q^p + 1}{q + 1} = y^n, \qquad \text{(iv)} \quad \frac{q^p + 1}{(q+1)(p, q+1)} = y^n,$$

where $p \geq 3$ is a prime number. Now by solving these diophantine equations we get the result.
(a) If M is a $C_{11,11}$ simple group and the odd order component of M is of the form (i)–(iv), then in the corresponding diophantine equation we have $y = 11$. We will show that $(p, q, n) = (5, 3, 2)$ is the only solution of (i) and (ii). If $(q^p - 1)/(q-1) = 11^n$ or $(q^p - 1)/((q-1)(p, q-1)) = 11^n$, then 11 is a primitive prime for $p_0^{\alpha p} - 1$. Therefore $\operatorname{ord}_{11}(p_0) = \alpha p$, by the definition of primitive prime (see Lemma 2.5). Now by using the Fermat theorem, αp is a divisor of 10. Hence $p = 5$ and so $1 \leq \alpha \leq 2$. Now by checking the possibilities for q it follows that $(p, q, n) = (5, 3, 2)$ is the only solution of the diophantine equations (i) and (ii). Similar to the above discussion, by considering the diophantine equations (iii) and (iv) for $y = 11$, we

conclude that 11 is a divisor of $p_0^{2\alpha p}-1$ and in a similar manner it follows that $p=5$ and $\alpha=1$. Therefore the only solution of these diophantine equations is $(p,q,n)=(5,2,1)$. Now by using this result and by using Table 1 in [8], we can determine $C_{11,11}$ simple groups. We omit the details of the proof for convenience.

For the proof of (b) and (c), similarly we can prove that if $p_0 \in \{2,3,5,7,11,13,17,19,23\}$ and $y=29$, then the diophantine equations (i)–(iv) have no solution. Also we can show that if $y=31$ and $p_0 \in \{2,3,5,7,11,19\}$, then $(p,q,n)=(5,2,1)$ and $(3,5,1)$ are the only solutions of (i) and (ii). Also (iii) and (iv) have no solution in this case. For convenience we omit the proof. □

We recall a definition from graph theory. A nonempty subset I of $\pi(G)$ is called an *independent subset* if there exists no edge between elements of I in $\Gamma(G)$.

Lemma 3.3. *Let G be a finite group such that G has an independent subset I such that $|I|=3$. Also let there exist two nonadjacent primes p_1 and p_2 such that $J=\{p_1,p_2\} \subseteq \pi(G)\setminus\{2,3,5\}$ and each p_i $(1\le i\le 2)$ is nonadjacent to at least one element of $\{2,3,5\}$ in $\Gamma(G)$. Then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. First we prove that G is not solvable. If G is a solvable group, then let H be a Hall I-subgroup of G. Since H is solvable it follows that $t(H)\le 2$, which is a contradiction, since there exists no edge between elements I in $\Gamma(G)$. Thus G is not solvable, and so G is not a 2-Frobenius group.

If G is a non-solvable Frobenius group, then G has a Frobenius kernel K and a Frobenius complement H. By using Lemma 2.3 in [11], it follows that H has a normal subgroup $H_0=SL(2,5)\times Z$, where $|H:H_0|\le 2$ and $\big(|Z|,30\big)=1$. Since each p_i $(1\le i\le 2)$ is not adjacent to at least one element of $\{2,3,5\}$ in $\Gamma(G)$, we conclude that $\{p_1,p_2\}\subseteq \pi(K)$. Now since the kernel of every Frobenius group is nilpotent, it follows that $p_1\sim p_2$ in $\Gamma(G)$, which is a contradiction. Therefore G is not a Frobenius group or a 2-Frobenius group. □

Theorem 3.2. *Let G be a finite group.*

(a) *If $\Gamma(G)=\Gamma(\mathrm{Aut}(M_{22}))$, then $G/O_2(G)\cong M_{22}$ and $O_2(G)\neq 1$ or $G/O_2(G)\cong \mathrm{Aut}(M_{22})$.*

(b) *If $\Gamma(G)=\Gamma\big(\mathrm{Aut}(\mathrm{Fi}_{24}')\big)$, then $G/O_\pi(G)\cong \mathrm{Fi}_{24}'$, where $2\in\pi$, $\pi\subseteq\{2,3\}$ and $O_\pi(G)\neq 1$ or $G/O_{\{2,3\}}(G)\cong \mathrm{Aut}(\mathrm{Fi}_{24}')$.*

Proof. (a) We can see that $\{3,5,7\}$ is an independent subset of $\Gamma(G)$. Also $5\nsim 7$ and $3\nsim 11$ in $\Gamma(G)$ and since $7\nsim 11$ in $\Gamma(G)$, by using Lemma 3.3, we conclude that G is not a Frobenius group nor a 2-Frobenius group. Now by using Lemma 3.1 it follows that G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that K/H is a $C_{11,11}$-simple group. If K/H is an alternating group or a sporadic simple group which is a $C_{11,11}$-group, then K/H is: A_{11}, A_{12}, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, McL, HS, Sz, $O'N$, Co_2 or J_1. Also $\Gamma(K/H)$ is a subgraph of $\Gamma(G)$, by Remark 2.1. Therefore $3\nsim 5$ in $\Gamma(K/H)$ and $\pi(K/H)\subseteq\{2,3,5,7,11\}$, which implies that the only possibilities for K/H are $L_2(11)$, M_{11}, M_{12} and M_{22}. If $K/H\cong M_{11}$, M_{12} or $L_2(11)$, then K/H has a $11:5$ subgroup by [3]. Also in these cases $7\notin\pi(K/H)$ and hence $7 \in \pi(H)$. Now consider the $\{5, 7, 11\}$ subgroup T of G which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore $K/H \cong M_{22}$ and since $Out(M_{22}) \cong \mathbb{Z}_2$ it follows that $G/H \cong M_{22}$ or $M_{22} \cdot 2$. Also H is a nilpotent π_1-group and so $\pi(H) \subseteq \{2, 3, 5, 7\}$. By using [3] we know that M_{22} has a $11 : 5$ subgroup. If $3 \in \pi(H)$, then let T be a $\{3, 5, 11\}$ subgroup of G which is solvable and hence $t(T) \leq 2$, which is a contradiction, since there exists any edge between 3, 5 and 11 in $\Gamma(G)$. Therefore $3 \notin \pi(H)$. Similarly it follows that $7 \notin \pi(H)$. Let $5 \in \pi(H)$ and $Q \in \mathrm{Syl}_5(H)$. Also let $P \in Syl_3(K)$. We know that H is nilpotent and hence Q char H. Since $H \lhd K$ it follows that $Q \lhd K$. Therefore P acts by conjugation on Q and since $3 \nsim 5$ in $\Gamma(G)$ it follows that P acts fixed point freely on Q. Hence QP is a Frobenius group with Frobenius kernel Q and Frobenius complement P. Now by using Lemma 2.9 it follows that P is a cyclic group which implies that a Sylow 3-subgroup of M_{22} is cyclic. But this is a contradiction since a 3-Sylow subgroup of M_{22} are elementary abelian by [3]. Therefore H is a 2-group. Then $G/O_2(G) \cong M_{22}$ where $O_2(G) \neq 1$ or $G/O_\pi(G) \cong \mathrm{Aut}(M_{22})$, where $\pi \subseteq \{2\}$.

(b) Let $I = \{7, 17, 23\}$ and $J = \{11, 13\}$. Now using Lemmas 3.1 and 3.3 we conclude that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{29,29}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore K/H is $L_2(29)$, Ru or Fi'_{24}. If $K/H \cong L_2(29)$ or Ru, then $\{17, 23\} \subseteq \pi(H)$, which is a contradiction, since H is nilpotent and $17 \nsim 23$ in $\Gamma(G)$. Therefore $K/H \cong \mathrm{Fi}'_{24}$ and so $G/H \cong \mathrm{Fi}'_{24}$ or $\mathrm{Aut}(\mathrm{Fi}'_{24})$. By using [3], we know that Fi'_{24} has a $23 : 11$ subgroup. Therefore $\pi(H) \cap \{5, 7, 13, 17\} = \emptyset$. Also Fi'_{24} has a $29 : 7$ subgroup, and hence $\pi(H) \cap \{11, 13\} = \emptyset$. Therefore $\pi(H) \subseteq \{2, 3\}$ and so $G/O_\pi(G) \cong \mathrm{Fi}'_{24}$ where $2 \in \pi$, $\pi \subseteq \{2, 3\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong \mathrm{Aut}(\mathrm{Fi}'_{24})$ where $\pi \subseteq \{2, 3\}$. □

Theorem 3.3. *Let G be a finite group satisfying $\Gamma(G) = \Gamma\big(\mathrm{Aut}(J_3)\big)$. Then $G/O_\pi(G) \cong J_3$, where $2 \in \pi$, $\pi \subseteq \{2, 3, 5\}$ and $O_\pi(G) \neq 1$ or $G/O_{\{2,3,5\}}(G) \cong \mathrm{Aut}(J_3)$.*

Proof. Let $I = \{5, 17, 19\}$, $J = \{17, 19\}$. Now by using Lemmas 3.1 and 3.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{19,19}$ simple group. By using Lemma 2.4, K/H is A_{19}, A_{20}, A_{21}, J_1, J_3, $O'N$, Th, HN, $L_3(7)$, $U_3(8)$, $R(27)$, ${}^2E_6(2)$, $L_2(q)$ where $q = 19^n$ or $L_2(q)$ where $q = 2 \times 19^n - 1$ $(n \geq 1)$ is a prime number. But $\pi(K/H) \subseteq \pi(J_3)$ and $\pi(J_3) \cap \{7, 11, 13, 31\} = \emptyset$. Also $q = 2 \times 19^n - 1 > 19$ and hence the only possibilities for K/H are J_3 and $L_2(19^n)$, where $n \geq 1$. The orders of maximal tori of $A_m(q) = \mathrm{PSL}(m + 1, q)$ are

$$\frac{\prod_{i=1}^{k}(q^{r_i} - 1)}{(q - 1)(m + 1, q - 1)}; \qquad (r_1, \ldots, r_k) \in \mathrm{Par}(m + 1).$$

Therefore every element of $\pi_e(\mathrm{PSL}(2, q))$ is a divisor of q, $(q + 1)/d$ or $(q - 1)/d$ where $d = (2, q - 1)$. If $q = 19^n$, then $3 \mid (19^n - 1)/2$ and since $3 \sim 5$ and $3 \nsim 17$ in $\Gamma(G)$, it follows that if 5 divides $|G|$, then $5 \mid (19^n - 1)$ and if 17 is a divisor of $|G|$, then $17 \mid (19^n + 1)$. Note that $\pi(19 - 1) = \{2, 3\}$, $\pi(19^2 - 1) = \{2, 3, 5\}$ and $17 \mid (19^4 + 1)$. Now by using the Zsigmondy's Theorem, Lemma 2.9 it follows that the only possibility is $n = 1$.

Now we consider these possibilities for K/H, separately. First let $K/H \cong J_3$. We note that $\mathrm{Out}(J_3) \cong \mathbb{Z}_2$ and hence G/H is isomorphic to J_3 or $J_3 \cdot 2$. Also H is a nilpotent π_1-group. Hence $\pi(H) \subseteq \{2,3,5,17\}$. If $17 \in \pi(H)$, then let T be a $\{3,17,19\}$ subgroup of G, since J_3 has a $19:9$ subgroup. Obviously T is solvable and hence $t(T) \leq 2$, which is a contradiction. Therefore $\pi = \pi(H) \subseteq \{2,3,5\}$ and $G/O_\pi(G) \cong J_3$ or $G/O_\pi(G) \cong \mathrm{Aut}(J_3)$. If $G/O_\pi(G) \cong J_3$, then $O_\pi(G) \neq 1$ and $2 \in \pi$, since $2 \nsim 17$ in $\Gamma(J_3)$.

Now let $K/H \cong L_2(19)$. Since $\mathrm{Out}(L_2(19)) \cong \mathbb{Z}_2$, it follows that $G/H \cong L_2(19)$ or $L_2(19) \cdot 2$. But in this case $\pi(K/H) = \{2,3,5,19\}$ and so $17 \mid |H|$. We know that $L_2(19)$ contains a $19:9$ subgroup and hence G has a $\{3,17,19\}$-subgroup T which is solvable and so $t(T) \leq 2$. But this is a contradiction, since $t(T) = 3$. Therefore $K/H \ncong L_2(19)$. □

Theorem 3.4. *Let G be a finite group satisfying $\Gamma(G) = \Gamma\big(\mathrm{Aut}(HS)\big)$. Then $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$ or $G/O_{\{2,3,5\}}(G) \cong \mathrm{Aut}(HS)$, $U_6(2) \cdot 2$ or McL.*

Proof. Let $I = \{3,7,11\}$ and $J = \{7,11\}$. Now by using Lemmas 3.1 and 3.3 we conclude that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is one of the following groups: M_{11}, M_{12}, M_{22}, McL, HS, $U_5(2)$, $U_6(2)$ and $L_2(11)$.

Case 1. Let $K/H \cong M_{11}$, M_{12}, $U_5(2)$ or $L_2(11)$.
By using [3] we know that $|Out(K/H)|$ is a divisor of 2. Therefore $7 \notin \pi(G/H)$, and hence $7 \in \pi(H)$. Since in each case, K/H has a $11:5$ subgroup it follows that G has a $\{5,7,11\}$ subgroup T, which is solvable and hence $t(T) \leq 2$. But this is a contradiction and so this case is impossible.

Case 2. Let $K/H \cong M_{22}$.
We note that $\mathrm{out}(M_{22}) \cong \mathbb{Z}_2$. Hence $G/H \cong M_{22}$ or $\mathrm{Aut}(M_{22})$. First let $G/H \cong M_{22}$, where H is a π_1-group and $\pi_1 = \{2,3,5,7\}$. We know that M_{22} has a $11:5$ subgroup (see [3]). If $2 \in \pi(H)$, then G has a $\{2,5,11\}$ subgroup T which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore $2 \notin \pi(H)$. If $3 \in \pi(H)$ or $7 \in \pi(H)$, then let T be a $\{3,5,11\}$ or $\{5,7,11\}$ subgroup of G, respectively. Then $t(T) \leq 2$, which is a contradiction. If $5 \in \pi(H)$, then let P be a Sylow 5-subgroup of H. If $Q \in \mathrm{Syl}_3(G)$, then Q acts fixed point freely on P, since $3 \nsim 5$ in $\Gamma(G)$. Therefore PQ is a Frobenius group which implies that Q be a cyclic group and it is a contradiction. Hence $H = 1$ and so $G = M_{22}$. But $\Gamma(M_{22}) \neq \Gamma(\mathrm{Aut}(HS))$, since $2 \nsim 5$ in $\Gamma(M_{22})$. Therefore this case is impossible.

Now let $G/H \cong \mathrm{Aut}(M_{22})$. By using [3], M_{22} has a $11:5$ subgroup. Similar to the above discussion we conclude that $\{3,5,7\} \cap \pi(H) = \emptyset$, and hence H is a 2-group. But in this case 3 and 5 are not joined which is a contradiction. Therefore Case 2 is impossible, too.

Case 3. Let $K/H \cong U_6(2)$.
By using [3], it follows that $\mathrm{Out}(K/H) \cong S_3$. We know that $U_6(2) \cdot 3$ has an element of order 21. Therefore $G/H \cong U_6(2)$ or $U_6(2) \cdot 2$. Also $7 \notin \pi(H)$, since $U_6(2)$ has a $11:5$ subgroup. Therefore if $G/H \cong U_6(2)$, then $2 \in \pi$, $\pi \subseteq \{2,3,5\}$

and $G/O_\pi(G) \cong U_6(2)$, where $O_\pi(G) \neq 1$. Similarly if $G/H \cong U_6(2) \cdot 2$, then $G/O_\pi(G) \cong U_6(2) \cdot 2$ where $\pi \subseteq \{2,3,5\}$.

Case 4. Let $K/H \cong McL$.
Note that $\mathrm{Out}(McL) = 2$, but $G/H \ncong \mathrm{Aut}(McL)$, since $\mathrm{Aut}(McL)$ has an element of order 22. Similar to the above proof it follows that $G/O_\pi(G) \cong McL$ and $\pi \subseteq \{2,3,5\}$, since McL has a $11:5$ subgroup.

Case 5. Let $K/H \cong HS$.
There exists a $11:5$ subgroup in HS. Similar to Case 3, it follows that $G/O_\pi(G) \cong HS$, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$, or $G/O_\pi(G) \cong \mathrm{Aut}(HS)$ where $\pi \subseteq \{2,3,5\}$. □

Theorem 3.5. *Let G be a finite group.*

(a) *If $\Gamma(G) = \Gamma\big(\mathrm{Aut}(O'N)\big)$, then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$ or $G/O_2(G) \cong \mathrm{Aut}(O'N)$.*

(b) *If $\Gamma(G) = \Gamma\big(\mathrm{Aut}(\mathrm{Suz})\big)$, then $G/O_\pi(G) \cong \mathrm{Suz}$, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$ or $G/O_{\{2,3,5\}}(G) \cong \mathrm{Aut}(\mathrm{Suz})$.*

Proof. (a) Let $I = \{3,11,31\}$ and $J = \{7,11\}$. Now by using Lemmas 3.1 and 3.3 we conclude that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{31,31}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Hence K/H is $L_3(5)$, $L_5(2)$, $L_6(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $O'N$. If $K/H \cong L_2(5)$, $L_6(2)$, $L_2(31)$ or $G_2(5)$, then $11, 19 \in \pi(H)$, which is a contradiction, since $209 \notin \pi_e(G)$ and H is nilpotent. If $K/H \cong L_3(5)$ or $L_2(32)$, then $\{7,19\} \subseteq \pi(H)$, which is a contradiction, since $7 \nsim 19$ in $\Gamma(G)$. Therefore $K/H \cong O'N$ and $\mathrm{Out}(O'N) = 2$, which implies that $G/H \cong O'N$ or $O'N.2$. We know that $O'N$ has a $11:5$ subgroup by [3] and if we consider $\{5,11,p\}$-subgroup of G, where $p \in \{7,19,31\}$, it follows that $\pi(H) \cap \{7,19,31\} = \emptyset$. Therefore $\pi(H) \subseteq \{2,3,5,11\}$. Also $O'N$ has a $19:3$ subgroup, which implies that $\pi(H) \cap \{11\} = \emptyset$. Let $p \in \{3,5\}$. If $p \in \pi(H)$, then let P be the p-Sylow subgroup of H. If $Q \in \mathrm{Syl}_7(G)$, then Q acts fixed point freely on P, since $7 \nsim 3$ and $7 \nsim 5$ in $\Gamma(G)$. Therefore PQ is a Frobenius group and hence Q is a cyclic group. But this is a contradiction since Sylow 7-subgroups of $O'N$ are elementary abelian by [3]. Therefore $\pi(H) \cap \{3,5\} = \emptyset$. Hence $\pi(H)$ is a 2-group. Then $G/O_2(G) \cong O'N$, where $O_2(G) \neq 1$; or $G/O_\pi(G) \cong \mathrm{Aut}(O'N)$ where $\pi \subseteq \{2\}$.

(b) Let $I = \{7,11,13\}$ and $J = \{11,13\}$. Now by using Lemmas 3.1 and 3.3 we conclude that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a $C_{13,13}$ simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore K/H is $\mathrm{Sz}(8)$, $U_3(4)$, ${}^3D_4(2)$, Suz, Fi_{22}, ${}^2F_4(2)'$, $L_2(27)$, $L_2(25)$, $L_2(13)$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $O_8^+(3)$, $S_6(3)$, $G_2(4)$, $S_4(5)$ or $G_2(3)$.

If $K/H \cong {}^2F_4(2)'$, $U_3(4)$, $L_2(25)$, $L_4(3)$, $S_4(5)$ or $G_2(3)$, then $\{7,11\} \subseteq \pi(H)$, which implies that $7 \sim 11$, since H is nilpotent. But this is a contradiction. If $K/H \cong {}^3D_4(2)$, $L_2(27)$, $L_2(13)$ or $L_3(3)$, then $\{5,11\} \subseteq \pi(H)$ and we get a contradiction similarly, since $5 \nsim 11$.

If $K/H \cong G_2(4)$, $S_6(3)$, $O_7(3)$ or $O_8^+(3)$, then $11 \in \pi(H)$ and K/H has a $13:3$ subgroup by [3]. Let T be a $\{3,11,13\}$-subgroup of G. It follows that $t(T)=3$, which is a contradiction since T is solvable.

If $K/H \cong \mathrm{Fi}_{22}$, then $G/H \cong \mathrm{Fi}_{22}$ or $\mathrm{Fi}_{22} \cdot 2$, where $\pi(H) \subseteq \{2,3,5,7,11\}$. Since Fi_{22} has $11:5$ and $13:3$ subgroups it follows that $\{7,11\} \cap \pi(H) = \emptyset$. Therefore $G/O_\pi(G) \cong \mathrm{Fi}_{22}$ or $\mathrm{Aut}(\mathrm{Fi}_{22})$, where $\pi \subseteq \{2,3,5\}$.

Let $K/H \cong Sz(8)$. It is known that $\mathrm{Out}(\mathrm{Sz}(8)) \cong \mathbb{Z}_3$ and so $G/H \cong \mathrm{Sz}(8)$ or $\mathrm{Sz}(8)\cdot 3$. If $G/H \cong \mathrm{Sz}(8)$, then $\{3,11\} \subseteq \pi(H)$ which is a contradiction, since $3 \nsim 11$. If $G/H \cong \mathrm{Sz}(8)\cdot 3$, then let T be $\{3,7,11\}$-subgroup of G, since $Sz(8)$ has a $7:6$ subgroup. Then $t(T)=3$, which is a contradiction.

If $K/H \cong \mathrm{Suz}$, then $G/H \cong \mathrm{Suz}$ or $\mathrm{Aut}(\mathrm{Suz})$. If $G/K \cong \mathrm{Suz}$, then $\pi(H) \subseteq \{2,3,5,7,11\}$. Since Suz has a $11:5$ and $13:3$ subgroups it follows that $7, 11 \notin \pi(H)$. Therefore $G/O_\pi(G) \cong \mathrm{Suz}$, where $2 \in \pi$ and $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$. If $G/H \cong \mathrm{Suz}\cdot 2$, then it follows that $G/O_\pi(G) \cong \mathrm{Aut}(\mathrm{Suz})$, where $\pi \subseteq \{2,3,5\}$. $\square$

Remark. W. Shi and J. Bi in [15] put forward the following conjecture:
Conjecture. Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if (i) $|G| = |M|$, (ii) $\pi_e(G) = \pi_e(M)$.

This conjecture is valid for sporadic simple groups, alternating groups and some simple groups of Lie type. As a consequence of the main results, we prove the validity of this conjecture for the groups under discussion.

Theorem 3.6. *Let G be a finite group and A be the automorphism group of a sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If $|G| = |A|$ and $\pi_e(G) = \pi_e(A)$, then $G \cong A$.*

We note that Theorem 3.6 was proved in [10] by using the characterization of almost sporadic simple groups with their order components. Now we give a new proof for this theorem. In fact we prove the following result which is a generalization of Shi-Bi Conjecture and so Theorem 3.6 is an immediate consequence of Theorem 3.7.

Theorem 3.7. *Let A be the automorphism group of a sporadic simple group, except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(McL)$. *If G is a finite group satisfying $|G| = |A|$ and $\Gamma(G) = \Gamma(A)$, then $G \cong A$.*

Proof. First let $A = \mathrm{Aut}(M_{22})$. By using Theorem 3.3, it follows that $G/O_2(G) \cong M_{22}$ or $G/O_\pi(G) \cong \mathrm{Aut}(M_{22})$, where $\pi \subseteq \{2\}$. If $G/O_2(G) \cong M_{22}$, then $\left|O_2(G)\right| = 2$ and hence $O_2(G) \subseteq Z(G)$ which is a contradiction, since G has more than one component and hence $Z(G) = 1$. Therefore $G/O_\pi(G) \cong M_{22}\cdot 2(M_{22})$, where $2 \in \pi$, which implies that $O_\pi(G) = 1$ and hence $G \cong \mathrm{Aut}(M_{22})$

Let $A = \mathrm{Aut}(HS)$. By using Theorem 3.4, it follows that $G/O_\pi(G) \cong U_6(2)$ or HS, where $2 \in \pi$, $\pi \subseteq \{2,3,5\}$ and $O_\pi(G) \neq 1$; or $G/O_\pi(G) \cong U_6(2)\cdot 2$, McL or $\mathrm{Aut}(HS)$, where $\pi \subseteq \{2,3,5\}$.

By using [3], it follows that 3^6 divides the orders of $U_6(2)$, $U_6(2)\cdot 2$ and McL, but $3^6 \nmid |G|$.

Therefore $G/O_\pi(G) \cong HS$ or $\mathrm{Aut}(HS)$. Now we get the result similarly to the last case.

For convenience we omit the details of the proof of other cases. □

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