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STABILITY PROBLEMS FOR LINEAR DIFFERENTIAL AND DIFFERENCE SYSTEMS

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Abstract. In this paper, there are derived sufficient conditions for exponential and asymptotic stability of differential and difference systems.

1. Introduction

When there are considered some physical or mechanical problems described by differential or difference equations, then there arises the problem of exponential and asymptotical stability [1].

Our program is as follows:

First, on the basis of some transformation, we reduce the differential system

$$\frac{dx}{dt} = A(t)x\,, \quad x(t_0) = x_0\,, \quad t \ge t_0 \ge 0\,, \tag{1}$$

where $A(t)$ is a $n \times n$ real continuous matrix function, $x = \mathrm{col}(x_1, \dots, x_n)$, and the difference system

$$x(n+1) = A(n)x(n)\,, \quad x(n_0) = x_0\,, \quad n \ge n_0 \ge 0\,, \tag{2}$$

where $A(n)$ is a $m \times m$ matrix function, $x = \mathrm{col}(x_1, \dots, x_m)$, to Volterra equations and we present then sufficient conditions for exponential stability of systems (1) and (2).

Next we study the stability of large scale functional discrete systems

$$x_i(n+1) = A_{ii}(n)x_i(n) + \sum_{j=1, i\neq j}^{r} A_{ij}(n)x_j(n) + \sum_{j=1}^{r} B_{ij}(n)x_j(n-l)\,, \tag{3}$$

where $n \ge 0$, $i = 1, \dots, r$, $l \ge 0$, $x_i = \mathrm{col}(x_1^{(i)}, \dots, x_{m_i}^{(i)})$, $\sum_{i=1}^{r} m_i = m$, $A_{ij}(n)$, $B_{ij}(n)$, $(i, j = 1, \dots, r)$ are $m_i \times m_j$ matrices.

As the definitions of exponential stability are not quite so standard, we state them below [3].

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Definition 1. The zero solution of system (1) is said to be exponentially stable if there exist constants $k > 0$, $C > 0$, $\delta > 0$, independent of $t_0 \geq 0$, such that any solution x of (1), for which $\|x(t_0)\| < \delta$, satisfies condition

$$\|x(t)\| \leq C\|x(t_0)\| \exp\big(-k(t-t_0)\big)\,, \quad t \geq t_0\,,$$

$\|x\|$ is the Euclidean norm of the vector $x \in R^n$.

Definition 2. The zero solution of system (2) is said to be exponentially stable if there exist constants $a > 1$, $\alpha > 0$, $C > 0$, $\delta > 0$, independent of n_0, such that any solution x of (2), for which $\|x(n_0)\| < \delta$, satisfies condition

$$\|x(n)\| \leq C\|x_0\| a^{-\alpha(n-n_0)}\,, \quad n \geq n_0 \geq 0\,.$$

Due to the linearity of Eq. (1) or (2), if one solution is exponentially stable then the same is true for all solutions and Eq. (1) or (2) is said to be exponentially stable.

It can be easily seen that the exponential stability implies the uniform asymptotic stability (see [3, 4, 5] for the definition). Moreover, for finite order linear difference equations also the converse is true, i.e., the exponential stability is equivalent to the uniform asymptotic stability [3, 4, 5].

2. Exponential stability

Let in Eq. (1) $A(t) = \big[a_{ij}(t)\big]$, $(i, j = 1, 2, 3)$, where $a_{ij}(t)$ are real and twice differentiable functions for $t \geq t_0$. Putting

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{2}}[y_1(t) - y_2(t)]\cos\theta(t)\,,\\ x_2(t) &= \frac{1}{\sqrt{2}}[y_1(t) - y_2(t)]\sin\theta(t)\,,\\ x_3(t) &= \frac{1}{\sqrt{2}}[y_1(t) + y_2(t)] \end{aligned} \tag{4}$$

in (1) we obtain

$$\begin{aligned} \frac{dy_1}{dt} &= D_{11}(t)y_1 + D_{12}(t)y_2\,,\\ \frac{dy_2}{dt} &= D_{21}(t)y_1 + D_{22}(t)y_2\,,\\ \frac{d\theta}{dt} &= \big[\sin\theta\cos\theta(a_{22}(t) - a_{11}(t)) + a_{21}(t)\cos^2\theta - a_{12}(t)\sin^2\theta\big]\\ &\quad + \frac{y_1 + y_2}{y_1 - y_2}\big(a_{23}(t)\cos\theta - a_{13}(t)\sin\theta\big)\,, \end{aligned} \tag{5}$$

where

$$\begin{aligned}
D_{11}(t) &= \frac{1}{2}\big[\big(A_{11}(t) + A_{22}(t)\big) + \big(A_{12}(t) + A_{21}(t)\big)\big]\,,\\
D_{12}(t) &= \frac{1}{2}\big[\big(A_{22}(t) - A_{11}(t)\big) + \big(A_{12}(t) - A_{21}(t)\big)\big]\,,\\
D_{21}(t) &= \frac{1}{2}\big[\big(A_{22}(t) - A_{11}(t)\big) - \big(A_{12}(t) - A_{21}(t)\big)\big]\,,\\
D_{22}(t) &= \frac{1}{2}\big[\big(A_{11}(t) + A_{22}(t)\big) - \big(A_{12}(t) + A_{21}(t)\big)\big]\,,\\
A_{11}(t) &= a_{11}(t)\cos^2\theta + a_{22}(t)\sin^2\theta + \big(a_{12}(t) + a_{21}(t)\big)\sin\theta\cos\theta\,,\\
A_{12}(t) &= a_{13}(t)\cos\theta + a_{23}(t)\sin\theta\,,\\
A_{21}(t) &= a_{31}(t)\cos\theta + a_{32}(t)\sin\theta\,,\\
A_{22}(t) &= a_{33}(t)\,,
\end{aligned}$$

and $\theta = \theta(t)$.

Remark 1. It is easy to see that the exponential stability of solution of the system (1) is implied by the exponential stability of the system composed of two first equations forming the system (5). Really, let

$$\|y(t)\| = \sqrt{(y_1(t))^2 + (y_2(t))^2} \le C\|y(t_0)\|\exp\big[-k(t-t_0)\big]\,,$$

then

$$\begin{aligned}
\|x(t)\| &= \sqrt{(x_1(t))^2 + (x_2(t))^2 + (x_3(t))^2}\\
&= \sqrt{(y_1(t))^2 + (y_2(t))^2} \le C\|x(t_0)\|\exp\big[-k(t-t_0)\big]\,.
\end{aligned}$$

In the sequel we shall use the following Theorems given in [7].

Proposition ([7])**.** *Suppose that in the interval* $\langle x_0, \overline{x})$, $\overline{x} \le \infty$, *there exist* $R^{'''}(x)$, $a_2^{''}(x)$, $a_1^{''}(x)$, $a_0^{'}(x)$ *and* $B^{'}(x)$, *and that* $R(x) \ne 0$, $a_2(x) \ne 0$. *Moreover, suppose that there exists a solution* $\overline{y}(x)$ *of the integral equation*

$$y(x) = f(x) + \int_{x_0}^{x} K(x,t)y(t)\,dt\,, \quad x_0 \le x \le \overline{x}\,, \tag{6}$$

where

$$\begin{aligned}
f(x) &= C + \frac{C_1}{g(x)} + \frac{1}{g(x)}\int_{x_0}^{x} g^{'}(t)B(t)\,dt\,,\\
g(x) &= \exp\Big(\int_{x_0}^{x}\frac{dt}{R(t)}\Big)\,,\\
K(x,t) &= \frac{g^{'}(t)}{g(x)}\phi(t) - \psi(t)\,,\\
\phi(x) &= R^{'}(x) + R(x)\Big\{\psi(x) - \frac{a_1(x)}{a_2(x)}\Big\} + 1\,,\\
\psi(x) &= R^{''}(x) - \Big[\frac{a_1(x)}{a_2(x)}R(x)\Big]^{'} + \frac{a_0(x)}{a_2(x)}R(x), \quad C,\ C_1 \text{ are constants.}
\end{aligned}$$

Then $\overline{y}(x)$ *satisfies the differential equation*

$$a_2(x)y^{''} + a_1(x)y^{'} + a_0(x)y = b(x)\,, \quad x \in \langle x_0\,, \overline{x}\rangle\,, \tag{7}$$

where

$$b(x) = \frac{a_2(x)}{R(x)} B^{'}(x)\,.$$

Theorem A ([7])**.** *Suppose that for* $x \geq x_0$

1) *there exist continuous* $R^{'''}(x)$, $a_2^{''}(x)$, $a_1^{''}(x)$, $a_0^{'}(x)$ *and* $b(x)$ $(x \geq x_0)$,
2) $R(x) \neq 0$, $a_2(x) \neq 0$, $(x \geq x_0)$,
3) $\int\limits_{x_0}^{\infty} \frac{dx}{|R(x)|} = \infty$,
4) $\lim\limits_{x\to\infty} \phi(x) = 0$,

5a) $\overline{\lim\limits_{x\to\infty}} \int\limits_{x_0}^{x} \left|\frac{R(t)b(t)}{a_2(t)}\right| dt = m < \infty$

or

5b) $\int\limits_{x_0}^{\infty} \frac{R(t)b(t)}{a_2(t)} dt = s$, $(|s| < \infty)$,

6) $\int\limits_{x_0}^{\infty} |\psi(x)|\, dx < \infty$,

where $\phi(x)$, $\psi(x)$ *are defined as in proposition.*

Then the differential equation (7) *for* $\overline{x} = \infty$ *has an integral* $\overline{y}(x)$ *bounded for* $x \to \infty$ *in case* 5a), *and convergent in case* 5b)*.*

We now consider conditions under which the system (5) is exponentially stable.

Theorem 1. *Suppose that*

1) $D_{ij}(t) \in C^3\langle 0, \infty)$, $(i, j = 1, 2)$, $D_{ij}(t) \neq 0$, $i \neq j$, $t \in \langle 0, \infty)$,
2) $R(t) \in C^3\langle 0, \infty)$, $\int\limits_{t_0}^{\infty} \frac{dt}{|R(t)|} = \infty$, $t_0 \geq 0$,
3) $\int\limits_{t_0}^{\infty} |C_2(s)|\, ds \leq K_2 < \infty$, $C_2(t) = D_{21}(t) \exp\big(\int\limits_{t_0}^{t} (D_{11}(s) - D_{22}(s))\, ds\big)$,
4) $K_3 \sum\limits_{i=1}^{2} \exp\big(\int\limits_{t_0}^{t} D_{ii}(s)\, ds\big) \leq r(t - t_0)$, $r(t) \in C\langle t_0, \infty)$, $\inf\limits_{t\geq 0} r(t) < 1$, K_3 − constant > 0,
5) $\lim\limits_{t\to\infty} \phi(t) = 0$, $\phi(t) = R^{'}(t) + R(t)\big[\psi(t) + C_3^{'}(t)C_3^{-1}(t)\big] + 1$, $\psi(t) = R^{''}(t) + \big(C_3^{'}(t)C_3^{-1}(t)R(t)\big)^{'} - D_{12}(t)D_{21}(t)R(t)$, $C_3(t) = D_{12}(t) \exp\big(\int\limits_{t_0}^{t} \big(D_{22}(s) - D_{11}(s)\big)\, ds\big)$,
6) $\int\limits_{t_0}^{\infty} |\psi(t)|\, dt < \infty$.

Then the zero solution of the system

$$\begin{aligned} \frac{dy_1}{dt} &= D_{11}(t)y_1 + D_{12}(t)y_2\,, \\ \frac{dy_2}{dt} &= D_{21}(t)y_1 + D_{22}(t)y_2\,, \end{aligned} \tag{8}$$

consisting of the first two equations forming the system (5), *is exponentially stable.*

Proof. The substitutions

$$\begin{aligned} y_1(t) &= u(t)\exp\Big(\int_{t_0}^{t} D_{11}(s)\,ds\Big)\,, \\ y_2(t) &= v(t)\exp\Big(\int_{t_0}^{t} D_{22}(s)\;ds\Big)\,, \end{aligned} \tag{9}$$

where $u(t),\ v(t) \in C^2\langle t_0, \infty)$, transform the system of differential equations (8) into the system of equations

$$\frac{du}{dt} = C_3(t)v\,, \quad \frac{dv}{dt} = C_2(t)u\,, \tag{10}$$

where

$$\begin{aligned} C_2(t) &= D_{21}(t)\exp\Big(\int_{t_0}^{t} \big(D_{11}(s) - D_{22}(s)\big)\;ds\Big)\,, \\ C_3(t) &= D_{12}(t)\exp\Big(\int_{t_0}^{t} \big(D_{22}(s) - D_{11}(s)\big)\;ds\Big)\,. \end{aligned}$$

Hence, we obtain

$$\frac{d^2u}{dt^2} - C_3^{'}(t)C_3^{-1}(t)\frac{du}{dt} - D_{12}(t)D_{21}(t)u = 0\,. \tag{11}$$

Applying the substitution

$$u(t) = \sqrt{\Big|\frac{C_3(t)}{C_3(t_0)}\Big|}\,w(t)$$

we obtain

$$\frac{d^2w}{dt^2} + (\mu_{C_3^2}(t) - D_{12}(t)D_{21}(t))w = 0\,, \tag{12}$$

where

$$\mu_{C_3^2}(t) \equiv \frac{1}{2}\frac{C_3^{''}(t)}{C_3(t)} - \frac{3}{4}\Big(\frac{C_3^{'}(t)}{C_3(t)}\Big)^2\,.$$

Let

$$\begin{aligned} a_2(t) &= 1\,,\\ a_1(t) &= -C_3'(t)C_3^{-1}(t)\,,\\ a_0(t) &= -D_{12}(t)D_{21}(t)\,,\\ \phi(t) &= R'(t) + R(t)\{\psi(t) - a_1(t)\} + 1\,,\\ \psi(t) &= R''(t) - (a_1(t)R(t))' + a_0(t)R(t)\,,\\ b(t) &= 0\,. \end{aligned}$$

Then under our hypothesis, in virtue of Theorem A, the differential equation (11) has an integral $u(t)$ bounded for $t \to \infty$, i.e., there exists a constant $K_1 > 0$ such that

$$|u(t)| \le K_1 \|y_1(t_0)\| \quad \text{for} \quad t \ge t_0\,.$$

Next, by integrating the second equation in (10), we obtain the estimation

$$|v(t)| \le (1 + K_1K_2)\|y(t_0)\|\,.$$

Now, from (9) we get

$$\|y(t)\| \le K_3 \|y(t_0)\| \sum_{i=1}^{2} \exp\Big(\int_{t_0}^{t} D_{ii}(s)\,ds\Big)\,,$$

where $\max\big(K_1^2, (1 + K_1K_2)^2\big) = K_3^2$. □

In the next part we will need the following lemma.

Lemma 1 (Massera–Schaffer [2])**.** *Assume that*

1) $\psi(t)$, $\rho(t)$ *are continuous and positive functions for* $t \ge 0$,
2) $\inf \rho(t) < 1$ *for* $t \ge 0$,
3) $\psi(t) \le \rho(t - t_0)\psi(t_0)$ *for every* $t \ge t_0 \ge 0$.

Then there exist constants α, $\beta > 0$ *such that*

$$\psi(t) \le \beta e^{-\alpha(t-t_0)}\psi(t_0)$$

for all $t \ge t_0 \ge 0$.

Now, using assumption 4) of Theorem and Lemma 1 (Massera–Schaffer [2]) we obtain

$$\|y(t)\| \le B\|y(t_0)\| e^{-k(t-t_0)}\,.$$

This completes the proof.

Remark 2. The condition 3) of theorem will be satisfied if

$$\begin{aligned} \int_{t_0}^{\infty} \Big[\sum_{i=1}^{3} |a_{ii}(t)| + |a_{12}(t) + a_{21}(t)| + |a_{13}(t) - a_{31}(t)| + |a_{23}(t) - a_{32}(t)|\Big] \\ \times \exp\Big(\int_{t_0}^{t} \Big[|a_{13}(s) + a_{31}(s)| + |a_{23}(s) + a_{32}(s)|\Big]\,ds\Big)\,dt \\ \le K_2 < \infty\,. \end{aligned}$$

Using the method given above we can study the following system

$$x(n+1) = A(n)x(n)\,, \tag{13}$$

where $n \in N(n_0) = \{n_0, n_0+1, \dots\}$, $n_0 \in N$ or $n_0 = 0$, $A(n) = [a_{ij}(n)]_{3\times 3}$.

We substitute

$$\begin{aligned} x_1(n) &= \frac{1}{\sqrt{2}}[y_1(n) - y_2(n)]\cos\theta(n)\,,\\ x_2(n) &= \frac{1}{\sqrt{2}}[y_1(n) - y_2(n)]\sin\theta(n)\,,\\ x_3(n) &= \frac{1}{\sqrt{2}}[y_1(n) + y_2(n)] \end{aligned} \tag{14}$$

where $\theta\colon N(n_0) \to R$, into (13) and we obtain relations

$$\begin{aligned} [y_1(n+1) - y_2(n+1)]\cos\theta(n+1) &= a_{11}(n)\big(y_1(n) - y_2(n)\big)\cos\theta(n)\\ &\quad + a_{12}(n)\big(y_1(n) - y_2(n)\big)\sin\theta(n) + a_{13}(n)\big(y_1(n) + y_2(n)\big)\,, \end{aligned}$$

$$\begin{aligned} [y_1(n+1) - y_2(n+1)]\sin\theta(n+1) &= a_{21}(n)\big(y_1(n) - y_2(n)\big)\cos\theta(n)\\ &\quad + a_{22}(n)\big(y_1(n) - y_2(n)\big)\sin\theta(n) + a_{23}(n)\big(y_1(n) + y_2(n)\big)\,, \end{aligned}$$

$$\begin{aligned} y_1(n+1) + y_2(n+1) &= a_{31}(n)\big(y_1(n) - y_2(n)\big)\cos\theta(n)\\ &\quad + a_{32}(n)\big(y_1(n) - y_2(n)\big)\sin\theta(n) + a_{33}(n)\big(y_1(n) + y_2(n)\big)\,. \end{aligned}$$

In the formal way we obtain from this

$$\begin{aligned} y_1(n+1) &= D_{11}(n)y_1(n) - D_{12}(n)y_2(n)\,,\\ y_2(n+1) &= D_{21}(n)y_1(n) + D_{22}(n)y_2(n)\,, \end{aligned} \tag{15}$$

where

$$\begin{aligned} D_{11}(n) = \frac{1}{2}\Big\{&\big(a_{11}(n)\cos\theta(n) + a_{12}(n)\sin\theta(n) + a_{13}(n)\big)\cos\theta(n+1)\\ &+ \big(a_{21}(n)\cos\theta(n) + a_{22}(n)\sin\theta(n) + a_{23}(n)\big)\sin\theta(n+1)\\ &+ \big(a_{31}(n)\cos\theta(n) + a_{32}(n)\sin\theta(n) - a_{33}(n)\big)\Big\}\,, \end{aligned}$$

$$\begin{aligned} D_{12}(n) = \frac{1}{2}\Big\{&\big(a_{11}(n)\cos\theta(n) + a_{12}(n)\sin\theta(n) - a_{13}(n)\big)\cos\theta(n+1)\\ &+ \big(a_{21}(n)\cos\theta(n) + a_{22}(n)\sin\theta(n) - a_{23}(n)\big)\sin\theta(n+1)\\ &+ \big(a_{31}(n)\cos\theta(n) + a_{32}(n)\sin\theta(n) - a_{33}(n)\big)\Big\}\,, \end{aligned}$$

$$D_{21}(n) = \frac{1}{2}\Big\{\big(- a_{11}(n)\cos\theta(n) - a_{12}(n)\sin\theta(n) - a_{13}(n)\big)\cos\theta(n+1)$$
$$+ \big(- a_{21}(n)\cos\theta(n) - a_{22}(n)\sin\theta(n) - a_{23}(n)\big)\sin\theta(n+1)$$
$$+ \big(a_{31}(n)\cos\theta(n) + a_{32}(n)\sin\theta(n) + a_{33}(n)\big)\Big\}\,,$$

$$D_{22}(n) = \frac{1}{2}\Big\{\big(a_{11}(n)\cos\theta(n) + a_{12}(n)\sin\theta(n) - a_{13}(n)\big)\cos\theta(n+1)$$
$$+ \big(a_{21}(n)\cos\theta(n) + a_{22}(n)\sin\theta(n) - a_{23}(n)\big)\sin\theta(n+1)$$
$$+ \big(- a_{31}(n)\cos\theta(n) - a_{32}(n)\sin\theta(n) + a_{33}(n)\big)\Big\}\,.$$

Substituting into (15)

$$\begin{aligned} y_1(n) &= u(n)w(n)\,, \\ y_2(n) &= v(n)z(n)\,, \quad n \ge n_0 \ge 0\,, \end{aligned} \tag{16}$$

where the functions $w(n)$ and $z(n)$ will be defined later, we obtain the system

$$\begin{aligned} u(n+1)w(n+1) &= D_{11}(n)u(n)w(n) - D_{12}(n)v(n)z(n)\,, \\ v(n+1)z(n+1) &= D_{21}(n)u(n)w(n) + D_{22}(n)v(n)z(n) \end{aligned} \tag{17}$$

or

$$\begin{aligned} w(n+1)\Delta u(n) &= (D_{11}(n)w(n) - w(n+1))u(n) - D_{12}(n)v(n)z(n)\,, \\ z(n+1)\Delta v(n) &= (D_{22}(n)z(n) - z(n+1))v(n) + D_{21}(n)u(n)w(n)\,. \end{aligned} \tag{18}$$

Now the functions $w(n)$ and $z(n)$ we define as follows:

$$w(n+1) = D_{11}(n)w(n)\,,$$

and

$$z(n+1) = D_{22}(n)z(n)\,.$$

Starting with the initial values $w(n_0) = 1$, $z(n_0) = 1$ it gives

$$\begin{aligned} w(n) &= \prod_{s=n_0}^{n-1} D_{11}(s)\,, \\ z(n) &= \prod_{s=n_0}^{n-1} D_{22}(s)\,. \end{aligned} \tag{19}$$

Hence system (18) may be written as

$$\Delta u(n) = \frac{-D_{12}(n)z(n)}{w(n+1)}v(n)\,,$$

$$\Delta v(n) = \frac{D_{21}(n)w(n)}{z(n+1)}u(n)\,.$$

Let

$$\widetilde{C_1}(n) = \frac{-D_{12}(n)\prod\limits_{s=n_0}^{n-1} D_{22}(s)}{\prod\limits_{s=n_0}^{n} D_{11}(s)} \neq 0, \quad n \geq n_0 \geq 0,$$

$$\widetilde{C_2}(n) = \frac{D_{21}(n)\prod\limits_{s=n_0}^{n-1} D_{11}(s)}{\prod\limits_{s=n_0}^{n} D_{22}(s)} \neq 0, \quad n \geq n_0 \geq 0.$$

Then the system (18) gives

$$\begin{aligned} \Delta u(n) &= \widetilde{C_1}(n)v(n)\,, \\ \Delta v(n) &= \widetilde{C_2}(n)u(n)\,, \end{aligned} \tag{20}$$

hence

$$\Delta^2 u(n) - \frac{\Delta\widetilde{C_1}(n)}{\widetilde{C_1}(n)}\Delta u(n) - \widetilde{C_2}(n)\widetilde{C_1}(n+1)u(n) = 0\,. \tag{21}$$

Proposition ([6]). *Suppose that there exist $R(n) > 1$, $a_0(n)$, $a_1(n)$, $a_2(n) \neq 0$, $B(n)$ for $n \geq n_0 \geq 0$. Moreover, suppose that there exists a solution $\overline{y}(n)$ of the equation*

$$y(n) = f(n) + \sum_{s=n_0}^{n-1} K(n,s)y(s)\,,$$

where

$$f(n) = C + \frac{C_1}{g(n)} + \frac{1}{g(n)}\sum_{s=n_0}^{n-1} B(s)\Delta g(s)\,,$$

$$g(n) = \prod_{s=n_0}^{n-1}\Big(1 + \frac{1}{R(n)-1}\Big), \qquad K(n,s) = \frac{\Delta g(s)}{g(n)}\phi(s) - \psi(s)\,,$$

$$\phi(n) = R(n)\Big(\psi(n) - \frac{a_1(n-1)}{a_2(n-1)}\Big) + 1 + \Delta R(n-1)\,,$$

$$\psi(n) = \Delta^2 R(n-1) + R(n+1)\frac{a_0(n)}{a_2(n)} - \Delta\Big(R(n)\frac{a_1(n-1)}{a_2(n-1)}\Big)\,,$$

C, C_1 are constants.

Then $\overline{y}(n)$ satisfies the difference equation

$$a_2(n)\Delta^2 y(n) + a_1(n)\Delta y(n) + a_0(n)y(n) = b(n)\,,$$

where $b(n) = a_2(n)\frac{\Delta B(n)}{R(n+1)}$.

Theorem 2 ([6, Theorem 4]). *Let R, $a_2 \neq 0$, a_1, a_0 be defined for $n \geq n_0 \geq 0$. Suppose that*

1) $\sum_{n=n_0}^{\infty} \frac{1}{R(n)} = \infty$,
2) $\lim_{n\to\infty} \phi(n) = 0$,
3) $\sum_{n=n_0}^{\infty} |\psi(n)| < \infty$,

where $R(n) > 1$, $\phi(n) = R(n)\big(\psi(n) - a_1(n-1)\big) + 1 + \Delta R(n-1)$,
$\psi(n) = \Delta^2 R(n-1) + R(n+1)a_0(n) - \Delta\big(R(n)a_1(n-1)\big)$.

Then the difference equation

$$a_2(n)\Delta^2 u(n) + a_1(n)\Delta u(n) + a_0(n)u(n) = 0 \tag{22}$$

has a solution $u(n)$ bounded for $n \to \infty$.

In the next theorem we show that the zero solution of system (15) is exponentially stable.

We set

$$a_2(n) = 1\,, \quad a_1(n) = \frac{-\Delta\widetilde{C_1}(n)}{\widetilde{C_1}(n)}\,, \quad a_0(n) = -\widetilde{C_2}(n)\widetilde{C_1}(n+1)\,.$$

Theorem 3. *Assume that*

1) *the assumptions of* [6, Theorem 4] *hold,*
2) $\sum_{n=0}^{\infty} |\widetilde{C_2}(n)| \leq B_1 < \infty$,
3) *there exist constants $\alpha > 0$, $a > 1$ and $A_3 > 0$ such that* $\sum_{i=1}^{2}\big(\prod_{s=n_0}^{n-1} |D_{ii}(s)|\big) \leq A_3 a^{-\alpha(n-n_0)}$, $n \geq n_0 \geq 0$.

Then the zero solution of (15) *is exponentially stable.*

Proof. From [6, Theorem 4] there exists a positive constant A_1 such that

$$|u(n)| \leq A_1\|y_0\| \quad \text{for} \quad n \geq n_0 \geq 0\,.$$

Similarly, from the second equation in (20), we have

$$v(n) = v(n_0) + \sum_{s=n_0}^{n-1} \widetilde{C_2}(s)u(s)\,.$$

Hence by assumption 2) and the estimation $|u(n)| \leq A_1\|y_0\|$, we have

$$|v(n)| \leq (1 + A_1B_1)\|y_0\|\,, \quad n \geq n_0 \geq 0\,.$$

Using the above inequalities, equations (16), (19) and the assumption 3) we have

$$\begin{aligned}\|y(n)\| &= \sqrt{y_1^2(n) + y_2^2(n)}\\ &\leq \|y_0\| \max(A_1, (1 + A_1B_1)) \sum_{i=1}^{2}\Big(\prod_{s=n_0}^{n-1} |D_{ii}(s)|\Big) \leq B\|y_0\|a^{-\alpha(n-n_0)}\,,\end{aligned}$$

where $B = A_3 \max\big(A_1, (1 + A_1B_1)\big)$.

This completes the proof of Theorem 3. □

From Remark 2 and the above results we conclude about exponential stability of solution of (13).

3. Asymptotic stability

Now we consider the system

$$x_i(n+1) = A_{ii}(n)x_i(n) + \sum_{j=1,\, i\neq j}^{r} A_{ij}(n)x_j(n) + \sum_{j=1}^{r} B_{ij}x_j(n-l)\,, \tag{23}$$

where $i = 1,\dots,r$, $n \in N_0 = \{0,1,2,\dots\}$, $x_i = \operatorname{col}\big(x_1^{(i)},\dots,x_{m_i}^{(i)}\big)$, $\sum_{i=1}^{r} m_i = m$, $x^T = (x_1^T,\dots,x_r^T)$, $A_{ij}(n)$, $B_{ij}(n)$, $(i,j = 1,\dots,r)$ are $m_i \times m_j$ real matrix functions on N_0, l — is a nonnegative integer. The initial condition is

$$x_i(n) = \phi_i(n)\,, \quad -l \le n \le 0\,, \quad i = 1,\dots,r\,, \tag{24}$$

where $\phi_i(n)$ are defined on $\langle -l, 0\rangle$.

Assumption A. *Suppose that* $\|A_{ij}(n)\| \le a_{ij}$, $(i \neq j,\, i,j = 1,2,\dots,r)$, $\|B_{ij}(n)\| \le b_{ij}$, $(i,j = 1,2,\dots,r)$, *where* a_{ij}, b_{ij} *are constants,* $\|\cdot\|$ *is a vector or matrix norm in the real Euclidean space. Define* $\|\phi_i\| = \sup_{-l\le n\le 0} \|\phi_i(n)\|$, $(i = 1,\dots,r)$.

We regard (23) as a perturbed system of the system

$$x_i(n+1) = A_{ii}(n)x_i(n)\,, \quad (i = 1,\dots,r)\,, \tag{25}$$

in order to obtain some new results on the asymptotic behaviour of solutions of (23) using a fundamental matrix $X_i(n)$ of (25).

The main tool in our analysis is the variation of constants formula, then the solution of (23) with the initial function ϕ_i on $[-l,0]$ is given by

$$\begin{aligned} x_i(n) &= Y_i(n,0)\phi_i(0) + \sum_{s=0}^{n-1} Y_i(n,s+1) \\ &\quad \times \Big\{ \sum_{j=1,\, i\neq j}^{r} A_{ij}(s)x_j(s) + \sum_{j=1}^{r} B_{ij}(s)x_j(s-l)\Big\}\,, \end{aligned} \tag{26}$$

where $n \in N_0$, $Y_i(n,s) = X_i(n)X_i^{-1}(s)$.

We give some definitions of stability of zero solution of (23). Since now we assume that $x(n) = 0$ is a solution of (23).

Definition 3. The zero solution of (23) is stable, if for every $\varepsilon > 0$ and any $n_0 \in N_0$ there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that $\|\phi\| < \delta$ and $n \in N_0$ imply $\|x(n)\| < \varepsilon$.

Definition 4. The zero solution of (23) is asymptotically stable if it is stable and for any $n \in N_0$ there exists $\delta_0(n_0) > 0$ such that $\|\phi\| < \delta_0$ implies $\|x(n)\| \to 0$ as $n \to \infty$.

Theorem 4. *If the system* (23) *satisfies the following conditions:*

1) $||Y_i(n,s)|| \le L_i p_i^{n-m}$, *where* $L_i \ge 1$, $0 < p_i < 1$,

2) *there exist constants* $\alpha_i > 0$, $k > 1$ *such that* $\sum\limits_{i=1}^{r} \alpha_i(\overline{a}_{ij} + k\overline{b}_{ij}) < 0$, *where* $\overline{a}_{ii} = a_{ii} = p_i - 1$, $\overline{b}_{ij} = L_i b_{ij}$,

then the zero solution of (23) *is asymptotically stable.*

Proof. From assumptions and (26), we have

$$\|x_i(n)\| \le \|\phi_i\| L_i p_i^n + L_i \sum_{s=0}^{n-1} p_i^{n-s-1} \times \Big[\sum_{j=1,\, i\neq j}^{r} a_{ij}\|x_j(s)\| + \sum_{j=1}^{r} b_{ij}\|x_j(s-l)\|\Big] .$$

Let

$$P_i(n) = \begin{cases} \|\phi_i\| \quad \text{for} \quad -l \le n \le 0\,, \\ \|\phi_i\| L_i p_i^n + L_i \sum_{s=0}^{n-1} p_i^{n-s-1} \Big[\sum_{j=1,\, i\neq j}^{r} a_{ij}\|x_j(s)\| + \sum_{j=1}^{r} b_{ij}\|x_j(s-l)\|\Big] , \end{cases}$$

for $n \in N_0$. Then $\|x_i(n)\| \le P_i(n),\ n \in \langle -l, \infty),\ i = 1, \ldots, r$.

Moreover,

$$\Delta P_i(n) \le \sum_{j=1}^{r} \overline{a}_{ij} P_j(n) + \sum_{j=1}^{r} \overline{b}_{ij} P_j(n-l)\,,$$

where $a_{ii} = \overline{a}_{ii} = p_i - 1$, $\overline{a}_{ij} = L_i a_{ij}$, $\overline{b}_{ij} = L_i b_{ij}$, $i \neq j$.

Let

$$V(n) = \sum_{i=1}^{r} \Bigg\{ \alpha_i \Big[P_i(n) + k \sum_{s=n-l}^{n-1} \Big(\sum_{j=1}^{r} \overline{b}_{ij} P_j(s) \Big) \Big] \Bigg\} ,$$

where α_i, k — some positive constants.

Then

$$\Delta V(n) = \sum_{i=1}^{r} \Bigg\{ \alpha_i \Big[P_i(n+1) + k \sum_{s=n+1-l}^{n} \Big(\sum_{j=1}^{r} \overline{b}_{ij} P_j(s) \Big) \Big] \Bigg\} - \sum_{i=1}^{r} \Bigg\{ \alpha_i \Big[P_i(n) + k \sum_{s=n-l}^{n-1} \Big(\sum_{j=1}^{r} \overline{b}_{ij} P_j(s) \Big) \Big] \Bigg\}$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \alpha_i \Delta P_i(n) + \sum_{i=1}^{r} \alpha_i k \sum_{j=1}^{r} \overline{b}_{ij}\big(P_j(n) - P_j(n-l)\big) \\
&\le \sum_{j=1}^{r} \Big(\sum_{i=1}^{r} (\alpha_i \overline{a}_{ij} + \alpha_i k \overline{b}_{ij}) \Big) P_j(n) \\
&\quad + \sum_{j=1}^{r} \Big(\sum_{i=1}^{r} (\alpha_i \overline{b}_{ij} - \alpha_i k \overline{b}_{ij}) \Big) P_j(n-l) \\
&\le -\beta \sum_{j=1}^{r} P_j(n)\,, \qquad (27)
\end{aligned}
$$

where $-\beta = \max\limits_{1\le j\le r} \sum\limits_{i=1}^{r} \alpha_i(\overline{a}_{ij} + k\overline{b}_{ij})$, $\beta > 0$.

From (27) it follows that

$$
V(n) + \beta \sum_{s=0}^{n-1} \Big(\sum_{j=1}^{r} P_j(s) \Big) \le V(0) \qquad (28)
$$

Notice that

$$
V(n) \ge \sum_{i=1}^{r} \alpha_i P_i(n) \ge \overline{\alpha} \sum_{i=1}^{r} P_i(n) \ge \overline{\alpha} \sum_{i=1}^{r} \|x_i(n)\|\,,
$$

where $\overline{\alpha} = \min\limits_{1\le i\le r} \alpha_i$, and

$$
\begin{aligned}
V(0) &= \sum_{i=1}^{r} \Big\{ \alpha_i \Big[P_i(0) + k \sum_{s=-l}^{-1} \Big(\sum_{j=1}^{r} \overline{b}_{ij} P_j(s) \Big) \Big] \Big\} \\
&\le \overline{\overline{\alpha}} \sum_{i=1}^{r} \|\phi_i\| + kbrl \sum_{i=1}^{r} \|\phi_i\| \\
&= (\overline{\overline{\alpha}} + kbrl) \sum_{i=1}^{r} \|\phi_i\|\,,
\end{aligned}
$$

where $\overline{\overline{\alpha}} = \max\limits_{1\le i\le r} (\alpha_i)$, $b = \max\limits_{1\le i,j\le r} (\overline{b}_{ij})$.

Since

$$
\overline{\alpha} \sum_{i=1}^{r} \|x_i(n)\| \le V(n) \le V(0) \le (\overline{\overline{\alpha}} + kbrl) \sum_{i=1}^{r} \|\phi_i\|\,,
$$

$$
\sum_{i=1}^{r} \|x_i(n)\| \le \frac{\overline{\overline{\alpha}} + kbrl}{\overline{\alpha}} \sum_{i=1}^{r} \|\phi_i\|\,,
$$

so the zero solution of (23) is stable.

From

$$\overline{\alpha}\sum_{i=1}^{r} P_i(n) \le V(0)$$

it follows that $\sum_{i=1}^{r} P_i(n)$ is bounded for all $n \in N_0$.

From (28) we see that $\sum_{n=0}^{\infty}\left(\sum_{j=1}^{r} P_j(n)\right)$ is convergent.

Then

$$\sum_{i=1}^{r} P_i(n) \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\sum_{i=1}^{r} \|x_i(n)\| \to 0 \quad \text{as} \quad n \to \infty\,.$$

Thus the zero solution of (23) is asymptotically stable and the proof is completed. □

Remark 3. Analogously we can study the following nonlinear large scale system

$$x_i(n+1) = A_{ii}(n)x_i(n) + f_i(n, x(n), x(n-l))\,, \qquad i = 1, \dots, r\,,$$

where $f_i(n,0,0) \equiv 0$ and

$$\big\|f_i\big(n, x(n), x(n-l)\big)\big\| \le \sum_{j=1}^{r} a_{ij}\|x_j(n)\| + \sum_{j=1}^{r} b_{ij}\|x_j(n-l)\|$$

for $a_{ij} \ge 0$, $b_{ij} \ge 0$, $(i, j = 1, \dots, r)$.

Remark 4. The method used above can be adapted to establish criteria for stability with respect to a part of variables involved in difference equations.

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