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COMPACT SPACE-LIKE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN LOCALLY SYMMETRIC LORENTZ SPACES

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Abstract. A new class of $(n+1)$-dimensional Lorentz spaces of index 1 is introduced which satisfies some geometric conditions and can be regarded as a generalization of Lorentz space form. Then, the compact space-like hypersurface with constant scalar curvature of this spaces is investigated and a gap theorem for the hypersurface is obtained.

1. Introduction

Let $\mathbb{N}_p^{n+p}$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of index p. It is called a semi-definite space of index p. When we refer to index p, we mean that there are only p negative eigenvalues of semi-Riemannian metric of $\mathbb{N}_p^{n+p}$ and the other eigenvalues are positive. In particular, $\mathbb{N}_1^{n+1}$ is called a Lorentz space when $p=1$. When the Lorentz space $\mathbb{N}_1^{n+1}$ is of constant curvature c, we call it Lorentz space form, denote it by $\mathbb{N}_1^{n+1}(c)$, with de Sitter space $\mathbb{S}_1^{n+1}(1)$ and anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ as its special cases. A hypersurface M of a Lorentz space is said to be space-like if the induced metric from that of the ambient space is positive definite.

The authors in [3] introduced a class of Lorentz spaces $\overline{M}$ of index 1. Let $\overline{\nabla}$, $\overline{K}$ and $\overline{R}$ denote the semi-Riemannian connection, sectional curvature and curvature tensor on $\overline{M}$, respectively. For constant c_1, c_2 and c_3, they considered Lorentz spaces which satisfy the following conditions:

(1) for any space-like vector u and any time-like vector v, $\overline{K}(u,v) = -\frac{c_1}{n}$,

(2) for any space-like vector u and v, $\overline{K}(u,v) \geq c_2$,

(3)

$$|\overline{\nabla}\,\overline{R}| \leq \frac{c_3}{n}\,.$$

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When $\overline{M}$ satisfies conditions (1) and (2), they say that $\overline{M}$ satisfies condition $(*)$. When $\overline{M}$ satisfies conditions (1) – (3), they say that $\overline{M}$ satisfies condition $(**)$.

Also they give some examples as following.

Example 1.1. The semi-Riemannian product manifold $H_1^k(-\frac{c_1}{n}) \times M^{n+1-k}(c_2)$, $c_1 > 0$. Its sectional curvature is given by

$$\overline{K}(u_1, u_b) = \overline{K}(u_a, u_b) = -\frac{c_1}{n}, \quad \overline{K}(u_a, u_r) = 0, \quad \overline{K}(u_r, u_s) = c_2,$$

where $a, b = 2, \ldots, k$; $r, s = k+1, \ldots, n+1$, u_1 and u_a, u_r denote time-like and space-like vectors respectively.

Example 1.2. The semi-Riemannian product manifold $R_1^k \times S^{n+1-k}(1)$. Its sectional curvature is given by

$$\overline{K}(u_1, u_a) = \overline{K}(u_a, u_b) = 0, \quad \overline{K}(u_1, u_r) = 0, \quad \overline{K}(u_r, u_s) = 1,$$

where $a, b = 2, \ldots, k$; $r, s = k+1, \ldots, n+1$. In particular, $R_1^1 \times S^n(1)$ is called Einstein Static Universe. Notice that it is not a Lorentz space form.

The authors in [2, 8] investigated complete space-like hypersurfaces M in a Lorentz space satisfying condition $(**)$. They estimate the square norm of the second fundamental form of M under some conditions. Baek-Cheng-Suh in [3] studied complete space-like hypersurfaces with constant mean curvature satisfying the condition $(*)$. Later, Xu and Chen in [9] generalized the related results in [3] by investigating complete space-like submanifolds with constant mean curvature in locally symmetric semi-Riemannian spaces. Recently, Liu and Wei in [4] obtained a gap theorem for complete space-like hypersurface with constant scalar curvature in locally symmetric Lorentz spaces.

Now we consider Lorentz spaces which satisfy another condition:

(4) for any space-like vectors u and v, $\overline{K}(u, v) \leq c_2$.

When $\overline{M}$ satisfies conditions (1) and (4), we shall say that $\overline{M}$ satisfies conditions $(\overline{*})$. When $\overline{M}$ satisfies conditions (1), (3) and (4), we shall say that $\overline{M}$ satisfies condition $(\overline{**})$. In this paper, we mainly discuss the compact space-like hypersurfaces with constant scalar curvature in a locally symmetric Lorentz spaces satisfying the condition $(\overline{*})$. It is worthy to point out that both Example 1.1 and 1.2 satisfy the condition $(\overline{*})$.

Remark 1.3. It is easy to see that a Lorentz space form $N_1^{n+1}(s)$ satisfies both conditions $(**)$ and $(\overline{**})$, where $-\frac{c_1}{n} = c_2 = s$.

Remark 1.4. If a Lorentz space $\overline{M}$ is locally symmetric, then the condition (3) holds naturally, because $\overline{\nabla}\,\overline{R} = 0$ in this situation.

Remark 1.5. As discussed in section 4, our theorem extend the results in [6] under some geometric conditions.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional Lorentz space of index 1. Throughout the paper, manifolds are assumed to be connected and geometric objects are assumed to be of class C^∞. For any point $p \in \overline{M}$, we choose a local field of semi-orthonormal frames $\{e_A\} = \{e_1, e_2, \ldots, e_{n+1}\}$ on a neighborhood of p, where $e_1, \ldots, e_n$ are space-like and e_{n+1} is time-like. We use the following convention on the range of indices throughout the paper

$$A, B, \ldots = 1, \ldots, n+1\,; \quad i, j, \ldots = 1, 2, \ldots, n\,.$$

Let $\{\omega_A\} = \{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ denote the dual frame fields of $\{e_A\}$ on $\overline{M}$. The metric tensor $\overline{g}$ of $\overline{M}$ satisfies $\overline{g}(e_A, e_B) = \epsilon_A \delta_{AB}$, where $\epsilon_1 = \ldots = \epsilon_n = 1$ and $\epsilon_{n+1} = -1$. The canonical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ satisfy the following structure equations

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B\,, \qquad \omega_{AB} + \omega_{BA} = 0\,, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \epsilon_C \epsilon_D \overline{R}_{ABCD} \omega_C \wedge \omega_D\,. \tag{2.2}$$

The components $\overline{R}_{CD}$ of the Ricci tensor and the scalar curvature $\overline{R}$ are given respectively by

$$\overline{R}_{CD} = \sum_B \epsilon_B \overline{R}_{BCDB}\,, \tag{2.3}$$

and

$$\overline{R} = \sum_A \epsilon_A \overline{R}_{AA}\,. \tag{2.4}$$

The components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\overline{R}$ are defined by

$$\begin{aligned} &\sum_E \epsilon_E \overline{R}_{ABCD;E} \\ &= d\overline{R}_{ABCD} - \sum_E \epsilon_E (\overline{R}_{EBCD}\omega_{EA} + \overline{R}_{AECD}\omega_{EB} + \overline{R}_{ABED}\omega_{EC} + \overline{R}_{ABCE}\omega_{ED})\,. \end{aligned} \tag{2.5}$$

Restricting the forms $\{\omega_A\}$ to a space-like hypersurface M in $\overline{M}$, we have

$$\omega_{n+1} = 0\,, \tag{2.6}$$

and the induced metric g of M is given by $g = \sum_i \omega_i \otimes \omega_i$. It is well known that by Cartan's Lemma we get

$$\omega_{(n+1)i} = \sum_j h_{ij} \omega_j\,, \qquad h_{ij} = h_{ji}\,, \tag{2.7}$$

where h_{ij} are the coefficients of the second fundamental form of M. Then we denote by $H = \frac{1}{n}\sum_i h_{ii}$ and $S = \sum_{ij} h_{ij}^2$ the mean curvature and squared norm of the second fundamental form of M, respectively.

The structure equations of M are given by

$$d\omega_i = -\sum_i \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0\,, \tag{2.8}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,. \tag{2.9}$$

The Gauss equation is given by

$$R_{ijkl} = \overline{R}_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk})\,. \tag{2.10}$$

The Ricci tensor and normalized scalar curvature of M are given respectively by

$$R_{ij} = \sum_k \overline{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}\,, \tag{2.11}$$

and

$$n(n-1)R = \sum_{j,k} \overline{R}_{kjjk} - n^2H^2 + S\,. \tag{2.12}$$

Let $\overline{M}$ be a locally symmetric Lorentz space satisfying the condition $(\overline{*})$. We know that the scalar curvature $\overline{R}$ of $\overline{M}$ is a constant. By using the structure equations of $\overline{M}$, we have

$$\overline{R} = \sum_A \epsilon_A \overline{R}_{AA} = -2\sum_i \overline{R}_{(n+1)ii(n+1)} + \sum_{i,j} \overline{R}_{ijji} = -2c_1 + \sum_{i,j} \overline{R}_{ijji}\,, \tag{2.13}$$

which means that $\sum_{i,j} \overline{R}_{ijji}$ is a constant. We assume from now that the scalar curvature R of M is constant. Together with the above equation and (2.12), we define a constant P by

$$n(n-1)P = n^2H^2 - S = \sum_{ij} \overline{R}_{ijji} - n(n-1)R\,. \tag{2.14}$$

By taking exterior differentiation of (2.7) and defining h_{ijk} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k (h_{kj}\omega_{ki} + h_{ik}\omega_{kj})\,, \tag{2.15}$$

we have the following Codazzi equation

$$h_{ijk} - h_{ikj} = \overline{R}_{(n+1)ijk}. \tag{2.16}$$

Similarly, we define h_{ijkl} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l (h_{ljk}\omega_{li} + h_{ilk}\omega_{lj} + h_{ijl}\omega_{lk})\,. \tag{2.17}$$

By taking exterior differentiation of (2.15), we have Ricci formula for the second fundamental form of M

$$h_{ijkl} - h_{ijlk} = -\sum_r (h_{ir}R_{rjkl} + h_{jr}R_{rikl})\,. \tag{2.18}$$

Restricting (2.5) on M, $\overline{R}_{(n+1)ijk;l}$ is given by

$$\begin{aligned}\overline{R}_{(n+1)ijk;l} = &\overline{R}_{(n+1)ijkl} + \overline{R}_{(n+1)i(n+1)k}h_{jl}\\ &+ \overline{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \overline{R}_{mijk}h_{ml}\,,\end{aligned} \tag{2.19}$$

where $\overline{R}_{(n+1)ijkl}$ denote the covariant derivative of $\overline{R}_{(n+1)ijk}$ as a tensor on M by

$$\begin{aligned}\sum_l \overline{R}_{(n+1)ijkl}\omega_l = d\overline{R}_{(n+1)ijk} &- \sum_l \overline{R}_{(n+1)ljk}\omega_{li}\\ &- \sum_l \overline{R}_{(n+1)ilk}\omega_{lj} - \sum_l \overline{R}_{(n+1)ijl}\omega_{lk}\,.\end{aligned} \tag{2.20}$$

Remark 2.1. If $\overline{M}$ is a Lorentz space form of index 1, by a straightforward calculation we check that the sum of the last three terms of right-hand side of (2.19) goes to zero. Then we have $\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl}$, which is the same as in the case that the ambient space is a space form.

It is well known that the Laplacian Δh_{ij} is defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}\,. \tag{2.21}$$

By using Codazzi equation and Ricci formula, we get

$$\begin{aligned}\Delta h_{ij} &= \sum_k h_{ikjk} + \sum_k \overline{R}_{(n+1)ijkk} = \sum_k h_{kijk} + \sum_k \overline{R}_{(n+1)ijkk}\\ &= \sum_k \Big(h_{kikj} - \sum_l (h_{kl}R_{lijk} + h_{il}R_{lkjk}) + \overline{R}_{(n+1)ijkk}\Big)\,.\end{aligned} \tag{2.22}$$

From the Codazzi equation $h_{ikjk} = h_{kkij} + \overline{R}_{(n+1)kikj}$, we have

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_k \left(\overline{R}_{(n+1)ijkk} + \overline{R}_{(n+1)kikj}\right) - \sum_{k,l} \left(h_{kl}R_{lijk} + h_{il}R_{lkjk}\right).$$

Together with Gauss equation and above equation and (2.19), we have

$$\begin{aligned}\Delta h_{ij} =& \sum_k h_{kkij} + \sum_k \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right)\\ &- \sum_{k,l}\left(2h_{kl}\overline{R}_{lijk} + h_{jl}\overline{R}_{lkik} + h_{il}\overline{R}_{lkjk}\right) + Sh_{ij}\\ &- \sum_k \left(h_{kk}\overline{R}_{(n+1)ij(n+1)} + h_{ij}\overline{R}_{(n+1)k(n+1)k}\right) - nH\sum_l h_{il}h_{jl}\,.\end{aligned} \tag{2.23}$$

Thus

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij} \\ =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij} + \sum_{i,j,k}\big(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\big)h_{ij} \\ &- \Big(nH\sum_{i,j} h_{ij}\overline{R}_{(n+1)ij(n+1)} + S\sum_k \overline{R}_{(n+1)k(n+1)k}\Big) + S^2 \\ &- \sum_{i,j,k,l} 2\big(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk}\big) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij}\,. \end{aligned} \tag{2.24}$$

3. Estimates of Laplacian and Key lemmas

Let $\overline{M}$ be a locally symmetric Lorentz space, i.e., $\overline{R}_{ABCD;E} = 0$. We also may choose a canonical bases $\{e_1, e_2, \dots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$, thus

$$\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j} = 0\,. \tag{3.1}$$

Noticing that $\overline{M}$ satisfies condition ($\overline{*}$), we have

$$\begin{aligned} &- \Big(nH\sum_{i,j} h_{ij}\overline{R}_{(n+1)ij(n+1)} + S\sum_k \overline{R}_{(n+1)k(n+1)k}\Big) \\ =& - \Big(nH\sum_i \lambda_i\overline{R}_{(n+1)ii(n+1)} + S\sum_i \overline{R}_{(n+1)i(n+1)i}\Big) \\ =& c_1(S - nH^2)\,. \end{aligned} \tag{3.2}$$

Also we have

$$\begin{aligned} &- \sum_{i,j,k,l} 2(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk}) \\ =& - 2\sum_{j,k}(\lambda_j\lambda_k - \lambda_k^2)\overline{R}_{kjjk} \le c_2\sum_{j,k}(\lambda_j - \lambda_k)^2 = 2c_2(nS - n^2H^2)\,. \end{aligned} \tag{3.3}$$

Substituting (3.1), (3.2) and (3.3) in to (2.24), it yields that

$$\frac{1}{2}\Delta S \le \sum_{i,k} h_{iik}^2 + \sum_i \lambda_i(nH)_{ii} + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH\sum_i \lambda_i^3)\,. \tag{3.4}$$

Lemma 3.1 ([7]). *Let $\{\mu_1, \mu_2, \dots, \mu_n\}$ be real numbers satisfying $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = A$, where A is a constant no less than zero. Then we have*

$$\Big|\sum_i \mu_i^3\Big| \le \frac{n-2}{\sqrt{n(n-1)}}A^{\frac{3}{2}}\,,$$

and the equality holds if and only if at least $n-1$ of the μ_i are equal, i.e.,

$$\mu_1 = \mu_2 = \ldots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}}A, \quad \mu_n = \sqrt{\frac{n-1}{n}}A\,.$$

Lemma 3.2. *Let M be a space-like hypersurface with constant normalized scalar curvature R in locally symmetric $(n+1)$-dimensional Lorentz space satisfying the condition $(\divideontimes)$. If $h_{ijk} \geq 0$, then*

$$\sum_{i,j,k} h_{ijk}^2 \leq n^2 |\nabla H|^2 .$$

Proof. Notice that the following equation holds:

$$\begin{aligned} n^2|\nabla H|^2 &= \sum_k \Big(\sum_{i,j} h_{ijk}\Big)^2 = \sum_{i,j,k,l,m} h_{ijk}h_{lmk} \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i\neq l,j,k,m} h_{ijk}h_{lmk} + \sum_{i,j\neq m,k} h_{ijk}h_{imk} . \end{aligned}$$

Then the proof follows from the above equation. □

Next we will use the well known self-adjoint operator □ introduced in [1] to the function nH and using (2.14), we have

$$\begin{aligned} \square(nH) &:= \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}\Delta(n(n-1)P) + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii} . \end{aligned} \tag{3.5}$$

By (2.14), we know that P is a constant, so we have $\frac{1}{2}\Delta(n(n-1)P) = 0$. Then substituting (3.4) to (3.5), we obtain

$$\square(nH) \leq \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH\sum_i \lambda_i^3) . \tag{3.6}$$

Lemma 3.3. *Let M be a compact space-like hypersurface of dimension n with constant scalar curvature in a locally symmetric Lorentz space which satisfies condition $(\divideontimes)$ and $h_{ijk} \geq 0$. Then we have the following inequality*

$$\square(nH) \leq \frac{n-1}{n}(S - nP)\phi_P(S) ,$$

where $\phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S + \frac{n-2}{n}\sqrt{(n(n-1)P + S)(s - nP)}$ and $c = 2c_2 + \frac{c_1}{n}$.

Proof. We denote

$$\mu_i = \lambda_i - H, \quad B = \sum_i \mu_i^2 .$$

It is obvious to see that

$$\sum_i \mu_i = 0 , \qquad B = S - nH^2 , \qquad \sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3HB + nH^3 .$$

By using Lemma 3.1, we have

$$\begin{aligned} -nH\sum_i \lambda_i^3 =& -n^2H^4 - 3nH^2B - nH\sum_i \mu_i^3 \\ \le& 2n^2H^4 - 3nSH^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^{\frac{3}{2}}\,. \end{aligned} \tag{3.7}$$

Substituting (3.7) to (3.6) and together with the Lemma 3.2, we get

$$\Box(nH) \le B\Big(nc - nH^2 + B + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^{\frac{1}{2}}\Big)\,. \tag{3.8}$$

It follows from (2.14) that

$$B = S - nH^2 = \frac{n-1}{n}(S - nP)\,. \tag{3.9}$$

Putting the above equation into (3.8), we get

$$\Box(nH) \le \frac{n-1}{n}(S-nP)\phi_H(S)\,, \tag{3.10}$$

where

$$\phi_H(S) = nc - 2nH^2 + S + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|\sqrt{S - nH^2}\,. \tag{3.11}$$

Putting (3.9) into (3.11), we have

$$\phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S + \frac{n-2}{n}\sqrt{(n(n-1)P + S)(S - nP)}\,.$$

Finally, (3.10) becomes

$$\Box(nH) \le \frac{n-1}{n}(S-nP)\phi_P(S)\,, \tag{3.12}$$

then we complete the proof. □

4. Main theorems and proofs

Theorem 4.1. *Let M be a compact space-like hypersurface of dimension n (where $n > 2$) with constant scalar curvature in a locally symmetric Lorentz space of dimension $n+1$ which satisfies condition $(\overline{*})$ and $h_{ijk} \ge 0$. If $0 \le c \le P$ or $c \le P < \frac{2}{n}c$ or $P > \frac{1}{n-1}c, c < 0$, then the norm square of the second fundamental form S satisfies*

$$S \ge \frac{n}{(n-2)(nP-2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big)\,,$$

where P is given by (2.14) and $c = 2c_2 + \frac{c_1}{n}$.

Proof. Since $\Box$ is a self-adjoint operator and M is compact, then we have

$$\int_M \Box(nH) * 1 = 0\,. \tag{4.1}$$

We notice that $S - nP \geq 0$ holds naturally by (3.9) because $S \geq nH^2$. By taking integration on both sides of (3.12), we get $\phi_P(S) \geq 0$. By directly calculation we see that $\phi_P(S) \geq 0$ is equivalent to

$$S \geq \frac{n}{n-2}\big(2(n-1)P - nc\big)$$

or

$$\frac{n}{(n-2)(nP-2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big) \leq S < \frac{n}{n-2}\big(2(n-1)P - nc\big)\,.$$

By solving the above inequalities, we complete the proof. □

Theorem 4.2. *Let M be a compact space-like hypersurface of dimension n (where $n > 2$) with constant scalar curvature in a locally symmetric Lorentz space of dimension $n+1$ which satisfies condition $(\ast)$ and $h_{ijk} \geq 0$ and $0 \leq c \leq P$ or $c \leq P < \frac{2}{n}c$ or $P > \frac{1}{n-1}c, c < 0$. If the norm square of the second fundamental form S satisfies*

$$nP \leq S \leq \frac{n}{(n-2)(nP-2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big), \tag{4.2}$$

then

(i) *$S = nP$ and M is totally umbilical, or*

(ii) *$S = \frac{n}{(n-2)(nP-2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big)$ and M has two distinct principal curvatures.*

Proof. Together with (4.2) and the definition of P, we see that the right-hand side term of (3.12) is non-positive. As in proof of Theorem 4.1, we take integration on both sides of (3.12) and notice (4.1), we have $(S - nP)\phi_P(S) = 0$. In particular, we notice that $\phi_P(S) = 0$ if and only if the equality holds in Lemma 3.1, thus we prove the theorem. □

Remark 4.3. Let $\overline{M}$ in Theorem 4.2 be a Lorentz space form with constant sectional curvature s. In particular, we assume that $s = 1$ such that $\overline{M}$ is nothing but a de Sitter space. As seen in Remark 1.3, we have $-\frac{c_1}{n} = c_2 = 1$. Thus c defined in Lemma 3.3 is 1. Then our theorem is just like Liu's corollary in [6].

Finally, we discuss the compact space-like surface in a locally symmetric Lorentz spaces of dimension 3, i.e., the version of $n = 2$ of the Theorem 4.1. We using the convention of the ranges of the indexes as following

$$i, j, k = 1, 2\,, \qquad A, B, C = 1, 2, 3\,.$$

Theorem 4.4. *Let M be a compact space-like surface with constant scalar curvature in a locally symmetric 3-Minkowski space which satisfies condition $(\ast)$ and $h_{ijk} \geq 0$. Then*

$$P \leq c\,,$$

where P is given by (2.14) and $c = 2c_2 + \frac{c_1}{n}$ and h_{ijk} is defined by (2.15).

Proof. We notice that when $n = 2$, (3.12) becomes $\Box(2H) \leq (S - 2P)(c - P)$. Taking integration on both sides of the inequality, then we complete the proof. □

Corollary 4.5. *Let M be a compact space-like surface with constant scalar curvature in a locally symmetric* 3*-Minkowski space which satisfies condition* ($\overline{*}$) *and* $h_{ijk} \geq 0$*. If $P \geq c$, then*

(i) $S = 2P$ *and M is totally umbilical, or*

(ii) $P = c$*.*

The proof is the same as the proof of Theorem 4.2.

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References

[1] Chen, S. Y., Yau, S. T., *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), 195–204.

[2] Choi, S. M., Kwon, J. H., Suh, Y. J., *Complete spacelike hypersurfaces in a Lorentz manifold*, Math. J. Toyama Univ. **22** (1999), 53–76.

[3] Jin, O. B., Cheng, Q. M., Young, J. S., *Complete spacelike hypersurfaces in locally symmetric Lorentz space*, J. Geom. Phys. **49** (2004), 231–247.

[4] Liu, J. C., Wei, L., *A gep theorem for complete spacelike hypersurface with constant scalar curvature in locally symmetric Lorentz space*, Turkish J. Math. **34** (2010), 105–114.

[5] Liu, X., *Space-like hypersurfaces of constant scalar in the de Sitter space*, Atti Sem. Mat. Fis. Univ. Modena **48** (2000), 99–106.

[6] Liu, X. M., *Complete spacelike hypersurfaces with constant scalar curvature*, Manuscripta Math. **105** (2001), 367–377.

[7] Okumura, M., *Hypersurfaces and a piching problem on the second fundamental thesor*, J. Math. Soc. Japan **19** (1967), 205–214.

[8] Suh, Y. J., Choi, Y. S., Yang, H. Y., *On spacelike hypersurfaces with constant mean curvature in Lorentz manifold*, Houston J. Math. **28** (2002), 47–70.

[9] Xu, S. L., Chen, D. M., *Complete space-like submanifolds in locally symmetric semi-definite spaces*, Anal. Theory Appl. **20** (2004), 383–390.

[10] Yau, S. T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[11] Zheng, Y. F., *Spacelike hypersurfaces with constant scalar curvature in the de Sitter space*, Differential Geom. Appl. **6** (1996), 51–54.

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