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EFFECTIVE CHAIN COMPLEXES FOR TWISTED PRODUCTS

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Abstract. In the paper weak sufficient conditions for the reduction of the chain complex of a twisted cartesian product $F \times_\tau B$ to a chain complex of free finitely generated abelian groups are found.

1. Introduction

When making algorithmic calculations with simplicial sets in algebraic topology on the level of chain complexes, it is often useful to replace the chain complex $C_*(X)$ associated to a simplicial set X with another chain complex EC_* where all the groups EC_n are finitely generated free abelian. Such chain complexes are called *effective*. This replacement is usually obtained using a reduction or a strong equivallence.

It is therefore natural to ask if standard topological constructions with simplicial sets are reflected by our replacements. For example by the theorem of Eilenberg and Zilber we know that given simplicial sets X, Y and their effective chain complexes $EC_*(X)$, $EC_*(Y)$, the simplicial set $X \times Y$ has an effective chain complex $EC_*(X) \otimes EC_*(Y)$.

Let $F \to E \to B$ be a Kan fibration of simplicial sets. By [6] we may think of the total space E as $E = F \times_\tau B$, i.e. a twisted cartesian product. We want to find an effective chain complex of the total space E from the knowledge of effective chain complexes of F and B and the twisting operator τ.

In [9] (Theorem 132) the solution of this problem was given in the case when the space B is 1–reduced, which means that the 1–skeleton of B is a point. However, this condition seems to be too restrictive and not necessary. For example if we aim to generalize the results in the paper [2] and construct an equivariant version of the Postnikov tower one cannot assume the base spaces are even 0–reduced (see [1]). In Theorem 10 and Corollary 12 we give weaker conditions under which an effective chain complex for the twisted cartesian product can be found. Our approach is based on the results by Shih as presented in [10] and on the approach from the paper [5].

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2. Basic notions

Let (C_*, ∂), (D_*, ∂) be chain complexes. The triple of maps $\rho = (f, g, h)$ where $f\colon C_* \to D_*$, $g\colon D_* \to C_*$ are chain homomorphisms and $h\colon C_* \to C_{*+1}$ is a chain homotopy such that

$$\begin{aligned} gf - \mathrm{id}_{C_*} &= \partial h + h\partial\,, \qquad & hh &= 0\,, \\ gh &= 0\,, & fh &= 0 \end{aligned}$$

is called a reduction. The chain complex D_* is said to be a reduct of C_*. We will denote this by $C_* \Rightarrow D_*$.

This definition of reduction coincides with the one given in [9, Definition 42], or [5, 2.1]. It is easy to observe that a composition of reductions is a reduction. We say, there is a *strong equivalence* between chain complexes C_* and C'_* if there exists a chain complex D_* together with two reductions $\rho_1 = (f_1, g_1, h_1)\colon D_* \Rightarrow C_*$ and $\rho_2 = (f_2, g_2, h_2)\colon D_* \Rightarrow C'_*$. We denote this by $C_* \Leftarrow D_* \Rightarrow C'_*$ or $C_* \Leftrightarrow C'_*$. The following lemma shows that strong equivalences are in some sense composable.

Lemma 1 ([9, Proposition 125])**.** *Let $A_* \Leftrightarrow B_*$ and $B_* \Leftrightarrow C_*$ be strong equivalnces of chain complexes. Then there is a strong equivalence $A_* \Leftrightarrow C_*$.*

We omit the proof, it can be found in [9]. We will make use of the following "tensor product" of reductions.

Lemma 2. *Let $\rho_C = (f_C, g_C, h_C)\colon C_* \Rightarrow C'_*$ and $\rho_D = (f_D, g_D, h_D)\colon D_* \Rightarrow D'_*$ be reductions. Then there is a reduction*

$$\rho_{C\otimes D} = (f_{C\otimes D}, g_{C\otimes D}, h_{C\otimes D})\colon C_* \otimes D_* \Rightarrow C'_* \otimes D'_*\,.$$

Proof. The new reduction is defined by $f_{C\otimes D} = f_C \otimes f_D$, $g_{C\otimes D} = g_C \otimes g_D$, $h_{C\otimes D} = h_C \otimes \mathrm{id}_D + g_C f_C \otimes h_D$, or $h_{C\otimes D} = h_C \otimes g_D f_D + \mathrm{id}_C \otimes h_D$. ☐

Further we will deal only with chain complexes which are formed by free abelian groups. For any simplicial set X there is a canonically associated chain complex $C_*(X)$ where the group $C_n(X)$ is freely generated by nondegenerate n–simplices of X and the boundary homomorphism ∂_n is induced by face maps in X_n as follows

$$\partial_n = \sum_{i=0}^{n} (-1)^n d_i\,.$$

Let (C_*, ∂) be a chain complex. A collection of maps $\delta_n\colon C_n \to C_{n-1}$ is called a *perturbation* if $(\partial_n + \delta_n)^2 = 0$ for all $n \in \mathbb{N}$. We will now introduce the Basic Perturbation Lemma. It is a powerful tool that enables us to construct new reductions.

Proposition 3 (Basic Perturbation Lemma, [10])**.** *Let $\rho = (f, g, h)\colon (C_*, \partial) \Rightarrow (D_*, \partial')$ be a reduction and let δ be a perturbation of the differential ∂. If for every $c \in C_n$ there exists an $\alpha \in \mathbb{N}$ such that $(h\delta)^\alpha(c) = 0$, then there is a reduction*

$$\rho' = (f', g', h')\colon (C_*, \partial + \delta) \Rightarrow (D_*, \partial' + \delta')\,,$$

where δ' is a perturbation of the differential ∂'.

Proof. The maps involved in the reduction ρ' are given explicitly as follows:

$$f' = f \circ (1 + (\delta h) + (\delta h)^2 + (\delta h)^3 + \dots)\,,$$
$$g' = (1 + (h\delta) + (h\delta)^2 + (h\delta)^3 + \dots) \circ g\,,$$
$$h' = (1 + (h\delta) + (h\delta)^2 + (h\delta)^3 + \dots) \circ h = h \circ (1 + (\delta h) + (\delta h)^2 + (\delta h)^3 + \dots)\,,$$
$$\delta' = f \circ \delta \circ (1 + (h\delta) + (h\delta)^2 + (h\delta)^3 + \dots) \circ g\,.$$

The proof can be found in [9, Theorem 50]. □

On the other hand, if we add a perturbation to the differential of the other chain complex, we easily get the following result:

Lemma 4 (Easy Perturbation Lemma)**.** *Let $\rho = (f, g, h)\colon (C_*, \partial) \Rightarrow (D_*, \partial')$ be a reduction and let δ' be a perturbation of the differential ∂'. Then there is a reducion $\rho = (f, g, h)\colon (C_*, \partial + \delta) \Rightarrow (D_*, \partial' + \delta')$, where $\delta = g\delta' f$.*

The difficulty with the BPL consists in the fact that it is sometimes difficult to verify the nilpotency assumption. Instead of looking for a description of $(h\delta)$ we can find a filtration and check how how the perturbation δ changes the filtration.

Definition 5. Let B and F be simplicial sets and let $E = F \times B$. Let $(y, b) \in E$. We may assume $b = s_* b' \in B$, where s_* is a composition of degeneracy operators and b' is nondegenerate. The filtration degree of (y, b) is the dimension of b'. The filtration degree of an nonzero element $y \otimes b \in C_*(F) \otimes C_*(B)$ is the dimension of b.

3. Twisting cochains

The twisted cartesian product (TCP) is defined as follows:

Definition 6. Let B, F be simplicial sets and G a simplicial group with a right action $\cdot\colon F \times G \to F$. A function $\tau_n\colon B_n \to G_{n-1}$, $n \geq 1$, is said to be a *twisting operator*, if it satisfies the following properties:

(1) $d_0\tau(b) = \tau(d_1 b) \cdot \tau(d_0 b)^{-1}$,
(2) $d_i\tau(b) = \tau(d_{i+1} b)$, $i > 0$,
(3) $s_i\tau(b) = \tau(s_{i+1} b)$, $i \geq 0$,
(4) $\tau(s_0 b) = e_m$, if $b \in B_{m+1}$ where e_m is the unit element of G_m.

The twisted cartesian product with the base B, the fiber F and the group G is a simplicial set denoted E or $F \times_\tau B$ where $E_n = F_n \times B_n$ has the following face and degeneracy operators:

(1) $d_0(y, b) = (d_0(y) \cdot \tau(b), d_0(b))$,
(2) $d_i(y, b) = (d_i(y), d_i(b))$, $i > 0$,
(3) $s_i(y, b) = (s_i(y), s_i(b))$, $i \geq 0$.

The face and degeneracy operations on E naturally define a differential ∂_τ on the chain complex $C_*(E)$. Note that $\partial_\tau(y_0, b_0) = 0$ for $(y_0, b_0) \in F_0 \times_\tau B_0$ since $d_0(y_0, b_0)$ is not defined.

We now introduce the following notation: If X is a simplicial set and $x \in X_n$ we put $\tilde{d}^{n-i}x = d_{i+1} \cdots d_n x$ and $\tilde{d}^0 x = x$. Given $(x, y) \in (X \times Y)_n$ we define the Alexander-Whitney operator:

$$\mathrm{AW}(x, y) = \sum_{i=0}^{n} \tilde{d}^{n-i}x \otimes {d_0}^i y \,.$$

For a non-twisted product $F \times B$, there exists a reduction

$$(\mathrm{AW}, \mathrm{EML}, \mathrm{SH})\colon \big(C(F \times B), \partial\big) \Rightarrow \big(C_*(F) \otimes C_*(B), \partial^F_\otimes\big)\,,$$

known as the Eilenberg–Zilber reduction. For the full description of the reduction see [3].

The only difference between the chain complexes $(C_*(F \times_\tau B), \partial_\tau)$ and $(C_*(F \times B), \partial)$ is in their differentials and it is easy to see that

$$\partial_\tau = \partial + \big(d_0(y) \cdot \tau(b), d_0(b)\big) - \big(d_0(y), d_0(b)\big)\,.$$

So the differential ∂_τ of $C_*(E)$ is just the ∂ with the added perturbation

$$\delta_\tau = \big(d_0(y) \cdot \tau(b), d_0(b)\big) - \big(d_0(y), d_0(b)\big)\,.$$

Proposition 7 ([9], Theorem 131)**.** *Let $F \times_\tau B$ be a twisted product of simplicial sets. Then the Basic Perturbation Lemma can be applied to the reduction data $(\mathrm{AW}, \mathrm{EML}, \mathrm{SH})\colon C_*(F \times B, \partial) \Rightarrow (C_*(F) \otimes C_*(B), \partial^F_\otimes)$ to obtain the reduction*

$$(f, g, h)\colon \big(C_*(F \times_\tau B), \partial_\tau\big) \Rightarrow \big(C_*(F) \mathbin{\hat{\otimes}} C_*(B), \partial^F_\tau\big)\,,$$

where $C_*(F) \mathbin{\hat{\otimes}} C_*(B)$ is just $C_*(F) \otimes C_*(B)$ with a new differential ∂^F_τ.

According to [10], the perturbation $\partial^F_\tau - \partial^F_\otimes$ can be seen as a cap product with so called twisting cochain, which is induced by τ. We will now give definitions of those notions.

Let $t\colon C_*(B) \to C_{*-1}(G)$ be a sequence of abelian group homomorphisms $t_n\colon C(B)_n \to C(G)_{n-1}$. We define a few operators that will be used within the construction:

$$D = \mathrm{AW} \circ C_*(\Delta)\colon C_*(B) \to C_*(B) \otimes C_*(B),$$

where $C_*(\Delta)$ is induced by the diagonal map $\Delta\colon B \to B \times B$ and

$$\sigma = C(\cdot) \circ \mathrm{EML}\colon C_*(F) \otimes C_*(G) \to C_*(F)\,.$$

Finally, we define the cap product $(t\cap)\colon C_*(F) \otimes C_*(B) \to C_*(F) \otimes C_*(B)$ as a composition

$$(\sigma \otimes 1)(1 \otimes t \otimes 1)(1 \otimes D)\,.$$

Observe, that the cap product is a homomorphism of graded abelian groups and not of chain complexes. We say that t is a *twisting cochain* if

$$\big(\partial^F_\otimes + (t\cap)\big)^2 = \partial^F_\otimes(t\cap) + (t\cap)\partial^F_\otimes + (t\cap)(t\cap) = 0\,.$$

We saw that the twisting operator τ induces via the BPL a new differential ∂^F_τ on the chain complex $C_*(F) \otimes C_*(B)$. Then the same twisting operator τ (this time seen as a part of the twisted cartesian product $G \times_\tau B$) also induces a differential ∂^G_τ on the chain complex $C_*(G) \otimes C_*(B)$.

According to [10], the twisting operator induces a *twisting cochain* $t : C_*(B) \to C_{*-1}(G)$ as follows:

$$t_n \colon C_n(B) \xrightarrow{e_0 \otimes 1} C_0(G) \otimes C_n(B) \xrightarrow{\lambda_0(\partial_\tau^G - \partial_\otimes^G)} C_{n-1}(G) \otimes C_0(B) \xrightarrow{p} C_{n-1}(G)\,,$$

where e_0 is the unit element of G_0, λ_0 is a projection on the summand $C_{n-1}(G) \otimes C_0(B)$ of the sum

$$\big(C_*(G) \otimes C_*(B)\big)_{n-1} = \sum_{i=0}^{n-1} C_{n-1-i}(G) \otimes C_i(B)$$

and $p(x \otimes b) = (\varepsilon b)x$ where the map $\varepsilon \colon C_0(B) \to \mathbb{Z}$ is the augmentation.

The following proposition was formulated and proved by Shih in [10] and describes the relation between t and ∂_τ^F.

Proposition 8 ([10], Theorem 2)**.** *Let $F \times_\tau B$ be a TCP and let t be the twisting cochain induced by the differential ∂_τ^G of the chain complex $C_*(G) \hat{\otimes} C_*(B)$. Then $\partial_\tau^F - \partial_\otimes^F = t\cap$.*

Let $E = F \times_\tau B$ be a twisted product of simplicial sets, t be a twisting cochain induced by the differential ∂_τ^G on the chain complex $C_*(G) \hat{\otimes} C_*(B)$ and $b \in B_n$, $y \in F_k$. Then using the definition of AW and $t\cap$ together with the fact that $t(\tilde{d}^n b) = 0$ we obtain the following formula:

$$(1)\ \ t \cap (y \otimes b) = (-1)^k \sigma\big(y \otimes t(\tilde{d}^{n-1} b)\big) \otimes d_0 b + \sum_{i=2}^{n} (-1)^k \sigma\big(y \otimes t(\tilde{d}^{n-i} b)\big) \otimes {d_0}^i b\,.$$

Using this formula we can summarize some properties of $t\cap$.

Corollary 9 ([5], Lemma 3.4)**.** *Let $E = F \times_\tau B$ be a twisted product of simplicial sets and let t be a twisting cochain induced by the differential ∂_τ^G on the chain complex $C_*(G) \hat{\otimes} C_*(B)$. Then the following holds:*

1. *The perturbation $(t\cap) \colon C_*(F) \otimes C_*(B) \to C_*(F) \otimes C_*(B)$ lowers the filtration degree by at least one.*
2. *If for all $b \in B_1$, $t(b) = 0$, then the perturbation $(t\cap)$ lowers the filtration degree by at least two.*

Proof. The first part is clear by the formula (1). If $t(\tilde{d}^{n-1} b) = 0$ for all $b \in B_n$, then

$$t \cap (y \otimes b) = \sum_{i=2}^{n} (-1)^k \sigma\big(y \otimes t(\tilde{d}^{n-i} b)\big) \otimes {d_0}^i b.$$

which proves the second part. □

4. Effective chain complex for twisted product

We would like to find an answer to the following problem: Let B and F be simplicial sets, G a simplicial group, $E = F \times_\tau B$ a TCP, and $\rho_B \colon C_*(B) \Rightarrow EC_*(B)$, $\rho_F \colon C_*(F) \Rightarrow EC_*(F)$ be reductions to effective chain complexes. Is there a reduction of the chain complex $C_*(E)$ to an effective chain complex which can be obtained from ρ_B, ρ_F and τ by the application of the Basic Perturbation Lemma?

Our aim is to find an answer using the composition of given reductions. Having reductions ρ_B, ρ_F we can by the Lemma 2 construct the reduction

$$\rho_{F\otimes B}\colon C_*(F)\otimes C_*(B) \Rightarrow EC_*(F)\otimes EC_*(B)\,.$$

We know that the chain homotopy $h_{F\otimes B}$ from the reduction $\rho_{F\otimes B}$ raises the filtration degree by at most 1. This follows from the fact that h_B raises the filtration degree by at most 1 and the proof of Lemma 2. We can use the BPL to construct a reduction $\rho_E = (f,g,h)\colon C_*(E) \Rightarrow C_*(F)\,\hat{\otimes}\,C_*(B)$. From Corollary 9, the perturbation operator $\partial^F_\tau - \partial^F_\otimes = t\cap$ lowers the filtration degree by at least one. If the composition $h_{F\otimes B}\circ(\partial^F_\tau - \partial^F_\otimes)$ decreased the filtration, it would be nilpotent and hence we could use the BPL on the reduction data $\rho_{F\otimes B}$ and the perturbation $\partial^F_\tau - \partial^F_\otimes$ to get a reduction

$$\rho_t\colon C_*(F)\,\hat{\otimes}\,C_*(B) \Rightarrow EC_*(F)\,\hat{\otimes}\,EC_*(B)$$

to an effective chain complex $EC_*(F)\,\hat{\otimes}\,EC_*(B)$ which is $EC_*(F)\otimes EC_*(B)$ with a new differential obtained from the BPL. However, in full generality $h_{F\otimes B}\circ(\partial^F_\tau - \partial^F_\otimes) = h_{F\otimes B}\circ(t\cap)$ preserves the filtration degree.

From (1) we see that in the composition $h_{F\otimes B}\circ(t\cap)(y\otimes b)$, where $b\in B_n$, there is only one element with the filtration degree n, namely

$$g_F f_F \sigma\big(y\otimes t(\tilde{d}^{n-1}b)\big)\otimes h_B d_0 b \tag{2}$$

and the degree n element in $(h_{F\otimes B}\circ(t\cap))^i(y\otimes b)$ is $y_i\otimes b_i$ where

$$\begin{aligned} b_0 &= b\,, & b_{i+1} &= h_B d_0 b_i = (h_B d_0)^i b\,,\\ y_0 &= y\,, & y_{i+1} &= g_F f_F \sigma\big(y_i\otimes t(\tilde{d}^{n-1}b_i)\big)\,. \end{aligned}$$

Now we can establish conditions for $(h_{F\otimes B}\circ(t\cap))^i$ to decrease the filtration and prove the following theorem.

Theorem 10. *Let B and F be simplicial sets, G a simplicial group with an action on F, $E = F\times_\tau B$ a TCP, and $\rho_B\colon C_*(B)\Rightarrow EC_*(B)$, $\rho_F\colon C_*(F)\Rightarrow EC_*(F)$ be reductions to effective chain complexes.*

If for all $n\in\mathbb{N}$, $b\in B_n$, $y\otimes b\in C_*(F)\otimes C_*(B)$, there exists $i\in\mathbb{N}$ such that $(h_B d_0)^i b = 0$ (thus $h_B\ d_0$ is nilpotent) or $y_i = 0$, then there is a reduction from the chain complex $C_*(E)$ to an effective chain complex $EC_*(F)\,\hat{\otimes}\,EC_*(B)$ which can be obtained from ρ_B, ρ_F and τ by the application of the Basic Perturbation Lemma.

Corollary 11. *If G is 0-reduced or ρ_B is trivial (i.e. $f_B = g_B = \mathrm{id}$, $h_B = 0$), $C_*(E)$ can be reduced to an effective chain complex using the BPL.*

Proof. If the reduction ρ_B is trivial, then the chain homotopy h_B is trivial, so $h_B = 0$ and hence $b_1 = h_B d_0 = 0$. To prove the case when G is 0-reduced we compute $t(b)$ where $b\in B_1$. According to the definition we get

$$t(b) = t_1(b) = p\lambda_0(\partial^G_\tau - \partial^G_\otimes)(e_0\otimes b)\,.$$

From the Basic Perturbation Lemma we get

$$\begin{aligned}(\partial_\tau^G - \partial_\otimes^G)(e_0 \otimes b) &= \mathrm{AW}\big(1 + \delta_\tau \mathrm{SH} + (\delta_\tau \mathrm{SH})^2 + (\delta_\tau \mathrm{SH})^3 + \dots\big)\delta_\tau \mathrm{EML}(e_0 \otimes b)\\ &= \mathrm{AW}\big(1 + \delta_\tau \mathrm{SH} + (\delta_\tau \mathrm{SH})^2 + (\delta_\tau \mathrm{SH})^3 + \dots\big)\delta_\tau\big(s_0(e_0), b\big)\\ &= \mathrm{AW}\big(1 + \delta_\tau \mathrm{SH} + (\delta_\tau \mathrm{SH})^2 + (\delta_\tau \mathrm{SH})^3 + \dots\big)\\ &\quad \big(d_0 s_0(e_0)\cdot \tau(b), d_0(b)\big) - \big(d_0 s_0(e_0), d_0(b)\big)\\ &= \mathrm{AW}\big(1 + \delta_\tau \mathrm{SH} + (\delta_\tau \mathrm{SH})^2 + (\delta_\tau \mathrm{SH})^3 + \dots\big)\\ &\quad \big(\tau(b), d_0(b)\big) - \big(e_0, d_0(b)\big)\,.\end{aligned}$$

As the operator $\mathrm{SH} = 0$ on $(F \times B)_0$ the only nonzero term of $(\partial_\tau^G - \partial_\otimes^G)(e_0 \otimes b)$ is

$$\mathrm{AW}\big(\tau(b), d_0(b)\big) - \big(e_0, d_0(b)\big) = \big(\tau(b) \otimes d_0(b)\big) - \big(e_0 \otimes d_0(b)\big)\,,$$

so we have

$$t(b) = t_1(b) = p\lambda_0\big(\tau(b) \otimes d_0(b)\big) - \big(e_0 \otimes d_0(b)\big) = \tau(b) - e_0\,.$$

If the group G is 0-reduced, $\tau(b) = e_0$ as e_0 is the only element in G_0 and we have $t(b) = 0$ for $b \in B_1$. That is why $y_1 = g_F f_F \sigma(y \otimes t(\tilde{d}^{n-1}b)) = 0$ and we can apply the previous theorem. □

Now we turn to strong equivalences.

Corollary 12. *Let B and F be simplicial sets, G a simplicial group, $E = F \times_\tau B$ a TCP, and $C_*(B) \Leftrightarrow EC_*(B)$, $C_*(F) \Leftrightarrow EC_*(F)$ strong equivalences with effective chain complexes. If G is 0-reduced or ρ_B is trivial (i.e. $EC_*(B) = C_*(B)$ and all reductions are trivial) then $C_*(F \times_\tau B)$ is strongly equivalent to an effective chain complex $EC_*(F) \hat{\otimes} EC_*(B)$ which can be obtained from the strong equivalences for $C_*(B)$ and $C_*(F)$ representing $C_*(E)$ and an effective chain complex using the Basic and Easy Perturbation Lemmas.*

Proof. By Proposition 7 we have a reduction $C_*(F \times_\tau B) \Rightarrow C_*(F) \hat{\otimes} C_*(B)$. Since strong equivalences are composable, it remains to show that there is a strong equivalence $C_*(F) \hat{\otimes} C_*(B) \Leftrightarrow EC_*(F) \hat{\otimes} EC_*(B)$.

Having strong equivalences $C_*(B) \Leftarrow D_*(B) \Rightarrow EC_*(B)$ and $C_*(F) \Leftarrow D_*(F) \Rightarrow EC_*(F)$ then by Lemma 2 there is a strong equivalence

$$C_*(F) \otimes C_*(B) \Leftarrow D_*(F) \otimes D_*(B) \Rightarrow EC_*(F) \otimes EC_*(B)$$

consisting of two reductions:

$$\rho_1 = (f_1, g_1, h_1)\colon C_*(F) \otimes C_*(B) \Leftarrow D_*(F) \otimes D_*(B)\,,$$

$$\rho_2 = (f_2, g_2, h_2) : D_*(F) \otimes D_*(B) \Rightarrow EC_*(F) \otimes EC_*(B)\,.$$

Given the perturbation $(t\cap)$ on the chain complex $C_*(F) \otimes C_*(B)$, we can use the Easy Perturbation Lemma on the reduction $\rho_1 = (f_1, g_1, h_1)\colon C_*(F) \otimes C_*(B) \Leftarrow D_*(F) \otimes D_*(B)$ to get a new reduction

$$\rho_1 = (f_1, g_1, h_1)\colon C_*(F) \hat{\otimes} C_*(B) \Leftarrow D_*(F) \hat{\otimes} D_*(B)\,,$$

where we introduce a perturbation $g_1(t\cap)f_1$ to the differential of the chain complex $D_*(F) \otimes D_*(B)$ and the reduction data remains unchanged. If the nilpotency condition of the composition $(g_1(t\cap)f_1) \circ h_2$ was satisfied, we could apply the Basic Perturbation Lemma on the reduction data $\rho_2 = (f_2, g_2, h_2)\colon D_*(F) \otimes D_*(B) \Rightarrow EC_*(F) \otimes EC_*(B)$ to obtain a reduction

$$\rho_2'\colon D_*(F) \hat{\otimes} D_*(B) \Rightarrow EC_*(F) \hat{\otimes} EC_*(B)\,.$$

If G is 0-reduced, then the filtration degree of the perturbation $g_1(t\cap)f_1$ is -2 by Corollaries 9 and 11 and as the the filtration degree of h_2 is $+1$, the nilpotency condition is satisfied. For ρ_B trivial, h_2 is 0 and the nilpotency condition is trivially satisfied.

The reductions ρ_1, ρ_2' therefore establish a strong equivalence

$$C_*(F) \hat{\otimes} C_*(B) \Leftrightarrow EC_*(F) \hat{\otimes} EC_*(B)$$

and, as the strong equivalences are composable, we get $C_*(F \times_\tau B) \Leftrightarrow EC_*(F) \hat{\otimes} EC_*(B)$. □

5. Vector fields

We will now deal with the case in which we have more information about the reduction $\rho_B\colon C_*(B) \Rightarrow EC_*(B)$. In particular, ρ_B is obtained via a *discrete vector field*. A discrete vector field V on a simplicial set X is a set of ordered pairs (σ, τ), where σ, τ are nondegenerate simplices of X, $\sigma = d_i\tau$ for exactly one index i and for every two distinct pairs (σ, τ), (σ', τ') we have $\sigma' \neq \sigma$, $\tau' \neq \tau$, $\sigma' \neq \tau$ and $\tau' \neq \sigma$. By writing $V(\sigma) = \tau$, we mean $(\sigma, \tau) \in V$. Given a dicrete vector field V, the nondegenerate simplices of X are divided into three subsets $\mathcal{S}$, $\mathcal{T}$, $\mathcal{C}$ as follows:

- $\mathcal{S}$ is the set of *source* simplices i.e. the simplices σ such that $(\sigma, \tau) \in V$,
- $\mathcal{T}$ is the set of *target* simplices i.e. the simplices τ such that $(\sigma, \tau) \in V$,
- $\mathcal{C}$ is the set of *critical* simplices i.e the remaining ones, not occuring in any edge of V.

A discrete vector field V on a simplicial set X induces a reduction $\rho_X = (h_X, f_X, g_X)\colon C_*(X) \Rightarrow D_*(X)$ (see [7],[4]). We will concentrate on the properties of the induced chain homotopy h_X. It turns out that $h_X(\sigma) \in \mathbb{Z}\mathcal{T}$ for any σ and more importantly $h_X(\sigma) = 0$ whenever $\sigma \in \mathcal{C} \cup \mathcal{T}$.

Definition 13. Let X be a simplicial set. For any nondegenerate simplex $\sigma \in X_n$ we will consider the following condition:

$$(*) \qquad d_0\sigma \in \mathcal{S} \quad \text{implies} \quad \sigma \in \mathcal{S}\,.$$

We say that a discrete vector field V on a simplicial set satisfies $(*)$ if all nondegenerate simplices of X satisfy $(*)$.

Corollary 14. *Let B and F be simplicial sets, G a simplicial group, $E = F \times_\tau B$ a TCP and $\rho_B\colon C_*(B) \Rightarrow EC_*(B)$, $\rho_F\colon C_*(F) \Rightarrow EC_*(F)$ be reductions to effective chain complexes. If the reduction ρ_B is induced by a vector field satisfying $(*)$, then there exists a reduction from the chain complex $C_*(E)$ to an effective chain complex which can be obtained from ρ_B, ρ_F and τ.*

Proof. We show that $(h_B d_0)^2 = 0$. Under our conditions for any $b \in B_n$ we have $b_1 = h_B(d_0 b) \in \mathbb{Z}\mathcal{T}$. As h_B satisfies $(*)$, we see that $d_0 b_1 \in \mathbb{Z}(\mathcal{C} \cup \mathcal{T})$ and consequently, $b_2 = h_B d_0 b_1 = 0$ and we can apply Theorem 10. □

Example 15. An example of a vector field satisfying $(*)$ is so called Eilenberg–MacLane vector field. Let us have $X = K(\mathbb{Z}, 1)$. In the standart model which is infinite (see [6]), the simplex $\sigma \in X_n$ can be represented as an n-tuple $[a_1|\ldots|a_n]$, where $a_1, \ldots, a_n \in \mathbb{Z}$ (see [4], page 5). The face operators are $d_0\sigma = [a_2|\ldots|a_n]$, $d_n\sigma = [a_1|\ldots|a_{n-1}]$, $d_i\sigma = [a_1|\ldots|a_{i-1}|a_i + a_{i+1}|a_{i+2}|\ldots|a_n]$, where $1 < i < n$.

For any $\sigma = [a_1|\ldots|a_n] \in X_n$, where $a_n \neq 1$, we define the Eilenberg-MacLane vector field V_{EML} in the following way:

$$V_{\mathrm{EML}}(\sigma) = \begin{cases} [a_1|\ldots|a_{n-1}|a_n - 1|1] & \text{for} \quad a_n > 1\,, \\ [a_1|\ldots|a_{n-1}|1] & \text{for} \quad a_n < 0\,. \end{cases}$$

Now we can classify the simplices:

- $\sigma \in \mathcal{S}$ has the form $[a_1|\ldots|a_n]$, where $a_n \neq 1$ and $n > 0$.
- $\sigma \in \mathcal{T}$ has the form $[a_1|\ldots|a_{n-1}|1]$, where $n > 1$.
- $\sigma \in \mathcal{C}$ is $[]$ and $[1]$.

It is easy to check that the vector field V_{EML} satisfies $(*)$. Note that Corollary 14 implies that for any $E = F \times_\tau K(\mathbb{Z}, 1)$ there is a reduction $C_*(E) \Rightarrow EC_*(E)$ to an effective chain complex if there is a reduction $C_*(F) \Rightarrow EC_*(F)$ to an effective chain complex.

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