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ON THOSE ORDINARY DIFFERENTIAL EQUATIONS THAT ARE SOLVED EXACTLY BY THE IMPROVED EULER METHOD

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Abstract. As a numerical method for solving ordinary differential equations $y' = f(x,y)$, the improved Euler method is not assumed to give exact solutions. In this paper we classify all cases where this method gives the exact solution for all initial conditions. We reduce an infinite system of partial differential equations for $f(x,y)$ to a finite system that is sufficient and necessary for the improved Euler method to give the exact solution. The improved Euler method is the simplest explicit second order Runge-Kutta method.

1. Introduction

In general, numerical schemes do not give exact solutions. A scheme may, however, give the exact solution to certain problems for all step sizes and all initial conditions. The classical fourth-order method of Runge [16], applied to equations on the form $y' = f(x)$, is identical to Simpson's rule applied on the integral in the solution $y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$. Simpson's rule is exact when $f(x) = p(x)$ is a polynomial of degree three or less. Therefore, Runge's method solves $y' = p(x)$ exactly.

What are those first order ODEs $y' = f(x,y)$ that are solved exactly by the improved Euler method, the simplest explicit second order Runge-Kutta method? Although numerical methods for solving differential equations have been present since the 18th century [6], the answer is not known. We will answer that question in the this paper.

The method is briefly explained as follows: Exactness of the numerical method leads to an infinite number of non-linear partial differential equations with f as the dependent variable and x and y as the free variables. We reduce this infinite set of partial differential equations to a finite set.

Our method has never been applied to other numerical schemes. The formulas, however, have been applied in many studies; Euler introduced these formulas to find higher derivatives of $y(x)$, where $y(x)$ is a solution of $y' = f(x,y)$. Later, Cauchy used the same formulas in 1824 to prove convergence of the classical Euler method.

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The same formulas were used by Runge in his works on higher order numerical schemes [15, 16].

Problems similar to the one in this paper have been studied [1, 4, 13]: To find numerical schemes that solve exactly a specific equation. An important reason to find such schemes is to make better numerical schemes. For more references see [5, 10, 12].

Although error bounds for Runge-Kutta methods are known from the works of many authors [3, 7, 8, 9, 11, 14], no error bounds tell us when a scheme solves an equation exactly. Therefore, better error bounds than these could exist.

2. On the improved Euler method

Consider the linear ordinary differential equation

$$y' = f(x, y)\,, \tag{1}$$

where y is an unknown real function in the variable x. Let $y(x_0) = y_0$ be any initial condition. The improved Euler method gives approximate values $y_1, y_2, \dots$ for the solution $y(x)$ at the points $x_1, x_2, \dots$:

$$\begin{aligned} \alpha &= hf(x_n, y_n) \\ \beta &= hf(x_n + h, y_n + \alpha) \\ y_{n+1} &= y_n + \frac{1}{2}(\alpha + \beta) \\ x_{n+1} &= x_n + h \end{aligned}$$

The step size h is a positive number. It is known that if the exact solutions of the original ODE are linear or if $f(x, y) = ax + b$, $a,\ b \in \mathbb{R}$, then the improved Euler method gives the exact solutions. The result is:

Theorem. *Given a differential equation $y' = f(x, y)$. The improved Euler method gives the exact solution for all small step sizes and all initial conditions $y(x_0) = y_0$ if and only if one of the following conditions holds true:*

1) $f(x, y) = -\dfrac{y - a}{x - b} + c, \qquad a, b, c \in \mathbb{R}.$
2) $f(x, y) = ax + b, \qquad a, b \in \mathbb{R}.$
3) *The exact solutions are linear:* $y(x) = y_0 + f(x_0, y_0)(x - x_0)$.

The "if" part of this theorem is easy to establish by inspection. In the present article we will prove the "only if" part of the theorem.

3. The derived equations

Let $y(x)$ be a solution of $y' = f(x, y)$. Define the operators $D_f = \frac{\partial}{\partial x} + f(x, y(x))\frac{\partial}{\partial y}$ and $D_f^n = D_f D_f \dots D_f$ (n times). Higher order derivatives of $y(x)$ are obtained by $y^{(n+1)}(x) = [D_f^n f](x, y(x))$.
Define the operator $D_f^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^i \frac{\partial^n}{\partial x^{n-i} \partial y^i}$.

The first step of the improved Euler equation is a function of the step size.

$$y_1(h) = y_0 + \frac{1}{2}\Big[h\, f(x_0, y_0) + h\, f\big(x_0 + h, y_0 + h\, f(x_0, y_0)\big)\Big]\,.$$

We need its derivative.

Lemma 1. *The n-th derivative of $y_1(h)$ is*

$$y_1^{(n)}(h) = \frac{1}{2}\Big[n\, D_f^{(n-1)} f\big(x_0 + h, y_0 + h\, f(x_0, y_0)\big) + h\, D_f^{(n)} f\big(x_0 + h, y_0 + h\, f(x_0, y_0)\big)\Big]\,.$$

Proof. The lemma follows from induction. □

Proposition 1. *If $y' = f(x, y)$ is solved exactly by the improved Euler method for all initial conditions and all small step sizes, the following differential equations are satisfied.*

$$(n+1)D_f^{(n)} f = 2D_f^n f\,, \quad n \geq 2\,. \tag{2}$$

Proof. Given any initial condition $y(x_0) = y_0$. If two functions are equal in a neighborhood about x_0, they have the same value and derivatives in x_0. The derivatives of $y(x)$ in x_0 are

$$y^{(n)}(x_0) = \big\{D_f^{n-1} f\big\}(x_0, y_0)\,. \tag{3}$$

The derivatives of $y_1(h)$ in $h = 0$ are

$$y_1^{(n)}(0) = \frac{n}{2}\big\{D_f^{(n-1)} f\big\}(x_0, y_0)\,. \tag{4}$$

The derivatives of $y_1(h)$ was given in Lemma 1. A necessary condition for $y(x_0+h) = y_1(h)$ is therefore that (3) and (4) are equal. □

Proposition 2. *Let D_f and $D_f^{(n)}$ be as above and let g be any smooth function in x and y. The following identity holds:*

$$D_f(D_f^{(n)} g) = D_f^{(n+1)} g + n(D_f f)\big(D_f^{(n-1)} \partial g/\partial y\big) \tag{5}$$

for all $n \geq 1$.

Proof. We have

$$\begin{aligned} D_f(D_f^{(n)} g) &= \frac{\partial}{\partial x}\Big(\sum_{i=0}^{n} \binom{n}{i} f^i \frac{\partial^n g}{\partial x^{n-i}\partial y^i}\Big) + f \frac{\partial}{\partial y}\Big(\sum_{i=0}^{n} \binom{n}{i} f^i \frac{\partial^n g}{\partial x^{n-i}\partial y^i}\Big) \\ &= \sum_{i=1}^{n} i \binom{n}{i} f^{i-1} \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y}\right) \frac{\partial^n g}{\partial x^{n-i}\partial y^i} \\ &\quad + \sum_{i=0}^{n} \binom{n}{i} f^i \frac{\partial^{n+1} g}{\partial x^{n+1-i}\partial y^i} + \sum_{i=0}^{n} \binom{n}{i} f^{i+1} \frac{\partial^{n+1} g}{\partial x^{n-i}\partial y^{i+1}} \\ &= (n\, D_f f)\left(D_f^{(n-1)} \partial g/\partial y\right) + D_f^{(n+1)} g\,. \end{aligned}$$

We have used that $i\binom{n}{i} = \binom{n-1}{i-1}$ and that $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$. □

By using the n-th equation in (2) and replacing g with f in (5) we get

$$\frac{2}{n+1} D_f^{n+1} f = D_f^{(n+1)} f + n(D_f f)\big(D_f^{(n-1)} \partial f/\partial y\big) . \tag{6}$$

By using (6) in the $(n+1)$th equation in (2) we get the following lemma.

Lemma 2. *If $y' = f(x, y)$ is solved exactly by the improved Euler method for all initial conditions and all small step sizes, the following differential equations are satisfied.*

$$D_f^{(n)} f = n(n-1)(D_f f)\big(D_f^{(n-2)} \partial f/\partial y\big) , \quad n \geq 2 . \tag{7}$$

4. General solution of the derived equations

Take $n = 2$ in equation (7):

$$f_{xx} + 2f f_{xy} + f^2 f_{yy} = 2f_x f_y + 2f f_y^2 . \tag{8}$$

Apply D_f to both sides:

$$f_{xxx} + 3f f_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} = 2f_y(f^2 f_{yy} + f_{xx} + f_x f_y + 2f f_{xy} + f f_y^2) . \tag{9}$$

By using equation (8) we eliminate f_{xx}:

$$f_{xxx} + 3f f_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} = 6f_y^2(f_x + f f_y) . \tag{10}$$

For $n = 3$, equation (7) yields:

$$f_{xxx} + 3f f_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} = 6(f_x + f f_y)(f_{xy} + f f_{yy}) . \tag{11}$$

Notice that the left hand sides of equation (11) and equation (10) are equal. The same must be true for the right hand sides: $6f_y^2(f_x + f f_y) = 6(f_x + f f_y)(f_{xy} + f f_{yy})$. Thus we see that f must satisfy either

$$\text{a) } f_{xy} + f f_{yy} = f_y^2 \qquad \text{or} \qquad \text{b) } f_x + f f_y = 0 . \tag{12}$$

Proposition 3. *If f is constant along all solutions of $y' = f(x, y)$, then $f_x + f f_y = 0$.*

Proof. Since f is constant along a solution y of (1) then $0 = f(x, y)' = f_x + y' f_y = f_x + f f_y$. □

Proposition 4. *If $D_f f = 0$, then the exact solution of $y' = f(x, y)$, $y(x_0) = y_0$ is linear.*

Proof. $y'' = \frac{d}{dx} f(x, y) = f_x(x, y) + f_y(x, y)y' = D_f f(x, y) = 0$. So $y' = k = f(x_0, y_0)$ and $y = y_0 + k(x - x_0)$. □

We apply the operator D_f on both sides of (7). When $n = 2$, this gives

$$D_f D_f^{(2)} f = 2(D_f^2 f)\partial f/\partial y + 2(D_f f)D_f \partial f/\partial y . \tag{13}$$

For $n \geq 3$,

$$\begin{aligned} D_f^{(n+1)} f + n(D_f f)D_f^{(n-1)} \partial f/\partial y &= n(n-1)(D_f^2 f)\big(D_f^{(n-2)} \partial f/\partial y\big) \\ &+ n(n-1)(D_f f)D_f^{(n-1)} \partial f/\partial y + n(n-1)(n-2)(D_f f)^2\big(D_f^{(n-3)} \partial^2 f/\partial y^2\big) . \end{aligned} \tag{14}$$

We have used proposition 2 on the left hand side. By substituting n with $n+1$ in the equation (7) we get $D_f^{(n+1)} f = n(n+1)(D_f f)\left(D_f^{(n-1)} \partial f/\partial y\right)$, $n \geq 1$. We use this to eliminate $D_f^{(n+1)} f$ from (14):

$$\begin{aligned} 3(D_f f)\big(D_f^{(n-1)} \partial f/\partial y\big) = (n-1)(D_f^2 f)\big(D_f^{(n-2)} \partial f/\partial y\big) \\ + (n-1)(n-2)(D_f f)^2\big(D_f^{(n-3)} \partial^2 f/\partial y^2\big) \, . \end{aligned} \tag{15}$$

In the rest of the article we will assume that $D_f f \neq 0$. By adding equation (2) for $n = 2$ to 3 times equation (5) for $n = 1$ and $g = f$, one obtains $D_f^2 f = 3(D_f f) f_y$. Substituting $3(D_f f) f_y$ for $D_f^2 f$ and dividing by $D_f f$ in (15), we obtain the following equation:

$$\begin{aligned} 3\big(D_f^{(n-1)} \partial f/\partial y\big) = 3(n-1) f_y \big(D_f^{(n-2)} \partial f/\partial y\big) \\ + (n-1)(n-2)(D_f f)\big(D_f^{(n-3)} \partial^2 f/\partial y^2\big) \, . \end{aligned} \tag{16}$$

For $n = 3$ we obtain:

$$3 D_f^{(2)} f_y = 6 f_y D_f f_y + 2(D_f f) f_{yy} \, . \tag{17}$$

We apply D_f on both sides of (12a). By (5) and the multiplication law, this simplifies to:

$$D_f^{(2)} f_y + (D_f f) f_{yy} = 2 f_y D_f(f_y) \, . \tag{18}$$

By subtracting equation (18) three times from equation (17) one obtains $(D_f f) f_{yy} = 0$. Since $D_f f \neq 0$, this implies that $f_{yy} = 0$. Therefore (12a) splits into

$$f_{yy} = 0 \quad \text{and} \quad f_{xy} = f_y^2 \, . \tag{19}$$

From the first of these two, the solution must be on the form

$$f(x, y) = F(x) y + G(x) \, . \tag{20}$$

The second of the two equations in (19) is then the Riccati equation $F'(x) = F^2$, which has solutions $F(x) = 0$ and $F = -1/(x-b)$, where b is a constant. From (8) and (19) one has

$$f_{xx} = 2 f_x f_y \, . \tag{21}$$

For equation (20) this gives $\frac{G''(x)}{G'(x)} = -\frac{2}{x-b}$ for $F \not\equiv 0$. This equation has general solution $G(x) = \frac{a}{x-b} + c$. For $F \equiv 0$, we have $G''(x) = 0$ which has the solution $ax + b$. Thus the system of equations in (21) and (19) has the general solution

$$f(x, y) = -\frac{y-a}{x-b} + c \qquad \text{or} \qquad f(x, y) = ax + b \, . \tag{22}$$

This concludes the proof of our theorem.

5. Discussion

This article introduces the method of prolonging the infinite set of partial differential equations that describes the exactness of a numerical method for solving first order ODEs. The improved Euler method was chosen as an example of this method. Our theorem is an example of those theorems that can be proved by using this prolongation method. I conjecture that for any Runge-Kutta method [2, 15] a finite number of prolongations will reduce the infinite number of equations for exactness to a finite number.

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