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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR SOME DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

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Abstract. In this article we are interested in the existence and uniqueness of solutions for the Dirichlet problem associated with the degenerate nonlinear elliptic equations

$$\Delta(v(x)\,|\Delta u|^{p-2}\Delta u) - \sum_{j=1}^{n} D_j\Big[\omega(x)\mathcal{A}_j(x,u,\nabla u)\Big] = f_0(x) - \sum_{j=1}^{n} D_j f_j(x)\,,\quad \text{in}\quad \Omega$$

in the setting of the weighted Sobolev spaces.

1. Introduction

In this article we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $X = W^{2,p}(\Omega,v)\cap W_0^{1,p}(\Omega,\omega)$ (see Definition 2.7) for the Dirichlet problem

$$\text{(P)}\qquad \begin{cases} Lu(x) = f_0(x) - \sum_{j=1}^n D_j f_j(x)\,, & \text{in}\quad \Omega\\ u(x) = 0\,, \quad \text{on} & \partial\Omega \end{cases}$$

where L is the partial differential operator

$$Lu(x) = \Delta(v(x)\,|\Delta u|^{p-2}\Delta u) - \sum_{j=1}^{n} D_j\big[\omega(x)\mathcal{A}_j(x,u(x),\nabla u(x))\big]$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω and v are two weight functions, Δ is the Laplacian operator, $1 < p < \infty$ and the functions $\mathcal{A}_j\colon \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1,\dots,n)$ satisfies the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$
$(\eta,\xi)\mapsto\mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

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(H2) there exists a constant $\theta_1 > 0$ such that

$$\big[\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi')\big]\cdot(\xi-\xi') \geq \theta_1\,|\xi-\xi'|^p,$$

whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi),\dots,\mathcal{A}_n(x,\eta,\xi))$ (where a dot denote here the Euclidian scalar product in $\mathbb{R}^n$).

(H3) $\mathcal{A}(x,\eta,\xi)\cdot\xi \geq \lambda_1|\xi|^p$, where λ_1 is a positive constant.

(H4) $|\mathcal{A}(x,\eta,\xi)| \leq K_1(x) + h_1(x)\,|\eta|^{p/p'} + h_2(x)\,|\xi|^{p/p'}$, where K_1, h_1 and h_2 are non-negative functions, with h_1 and $h_2\in L^\infty(\Omega)$, and $K_1\in L^{p'}(\Omega,\omega)$ (with $1/p + 1/p' = 1$).

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x\in\mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [2, 1] and [4]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [11]). These classes have found many useful applications in harmonic analysis (see [13]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [10]). There are, in fact, many interesting examples of weights (see [9] for p-admissible weights).

In the non-degenerate case (i.e. with $v(x)\equiv 1$), for all $f\in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x)\,, & \text{in} \quad \Omega \\ u(x) = 0\,, & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ (see [8]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x)\,, & \text{in} \quad \Omega \\ u(x) = 0\,, & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [3]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator have been studied by many authors (see [12] and the references therein), and the degenerated p-Laplacian has been studied in [4].

The following theorem will be proved in Section 3.

Theorem 1.1. *Assume* (H1)–(H4). *If* $v,\,\omega\in A_p$ *(with* $1<p<\infty$*),* $f_j/\omega\in L^{p'}(\Omega,\omega)$ $(j=0,1,\dots,n)$ *then the problem* (P) *has a unique solution* $u\in X = W^{2,p}(\Omega,v)$

$\cap\, W_0^{1,p}(\Omega,\omega)$. *Moreover, we have*

$$\|u\|_X \leq \frac{1}{\gamma^{p'/p}}\Big(C_\Omega \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\Big)^{p'/p},$$

where $\gamma = \min\{\lambda_1, 1\}$.

2. Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\Big(\frac{1}{|B|}\int_B \omega(x)dx\Big)\Big(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\Big)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [7], [9] or [13] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x;r)) \leq C\,\mu(B(x;2\,r))$, for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [9]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [13]).

If $\omega \in A_p$, then $\left(\dfrac{|E|}{|B|}\right)^p \leq C\dfrac{\mu(E)}{\mu(B)}$ whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [9]). Therefore, if $\mu(E) = 0$ then $|E| = 0$.

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \Big(\int_\Omega |f(x)|^p \omega(x)\,dx\Big)^{1/p} < \infty\,.$$

If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [14]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$ and $\omega \in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \leq |\alpha| \leq k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \Big(\int_\Omega |u(x)|^p\,\omega(x)\,dx + \sum_{1\leq|\alpha|\leq k}\int_\Omega |D^\alpha u(x)|^p\,\omega(x)\,dx\Big)^{1/p}. \tag{2.1}$$

We also define $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega,\omega)}$.

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Theorem 2.1.4 in [14]). The spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces and the spaces $W^{k,2}(\Omega,\omega)$ and $W_0^{k,2}(\Omega,\omega)$ are Hilbert spaces.

It is evident that the weights ω which satisfy $0 < c_1 \leq \omega(x) \leq c_2$ for $x \in \Omega$ (c_1 and c_2 positive constants), give nothing new (the space $\mathrm{W}_0^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}_0^{k,p}(\Omega)$). Consequently, we shall interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

In this paper we use the following results.

Theorem 2.3. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that*

(i) *$u_{m_k}(x) \to u(x)$, $m_k \to \infty$, μ - a.e. on Ω;*

(ii) *$|u_{m_k}(x)| \leq \Phi(x)$, μ - a.e. on Ω;*
(where $\mu(E) = \int_E \omega(x)\,dx$).

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [6]. □

Theorem 2.4 (The weighted Sobolev inequality)**.** *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ ($1 < p < \infty$). There exist constants C_Ω and δ positive such that for all $u \in C_0^\infty(\Omega)$ and all k satisfying $1 \leq k \leq n/(n-1) + \delta$,*

$$\|u\|_{L^{kp}(\Omega,\omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega,\omega)}\,.$$

Proof. See Theorem 1.3 in [5]. □

Lemma 2.5. *Let $1 < p < \infty$.*

(a) *There exists a constant α_p such that*

$$\big|\,|x|^{p-2}x - |y|^{p-2}y\big| \leq \alpha_p\,|x-y|(|x|+|y|)^{p-2}\,,$$

for all x, $y \in \mathbb{R}^n$;

(b) *There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$*

$$\beta_p\,(|x|+|y|)^{p-2}|x-y|^2 \leq (|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y) \leq \gamma_p\,(|x|+|y|)^{p-2}|x-y|^2\,.$$

Proof. See [3], Proposition 17.2 and Proposition 17.3. □

Definition 2.6. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $v, \omega \in A_p$, $1 < p < \infty$. We denote by $X = W^{2,p}(\Omega,v) \cap W_0^{1,p}(\Omega,\omega)$ with the norm

$$\|u\|_X = \Big(\int_\Omega |\nabla u|^p\,\omega\,dx + \int_\Omega |\Delta u|^p\,v\,dx\Big)^{1/p}\,.$$

Definition 2.7. We say that an element $u \in X = W^{2,p}(\Omega, v) \cap W_0^{1,p}(\Omega, \omega)$ is a (weak) solution of problem (P) if for all $\varphi \in X$ we have

$$\int_\Omega |\Delta u|^{p-2}\,\Delta u\,\Delta\varphi\,v\,dx + \sum_{j=1}^n \int_\Omega \omega\,\mathcal{A}_j\big(x, u(x), \nabla u(x)\big) D_j\varphi\,dx = \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^n \int_\Omega f_j\,D_j\varphi\,dx\,.$$

3. Proof of Theorem 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Let $A\colon X \to X^*$ be an operator on the real Banach space X. A is said to be hemicontinuous iff the real function $t \mapsto \langle A(u_1 + t\,u_2), u_3\rangle$ is continuous on $[0,1]$ for all $u_1,\ u_2,\ u_3 \in X$ (see Definition 26.1 in [15]) (where $\langle f, x\rangle$ denotes the value of the linear functional f at the point x).

Theorem 3.1. *Let $A\colon X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*

(a) *For each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$;*

(b) *If the operator A is strictly monotone, then equation $A\,u = T$ is uniquely solvable in X.*

Proof. See Theorem 26.A in [15]. □

We define B, B_1, $B_2\colon X \times X \to \mathbb{R}$ and $T\colon X \to \mathbb{R}$ by

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi)$$

$$B_1(u,\varphi) = \sum_{j=1}^n \int_\Omega \omega\,\mathcal{A}_j(x,u,\nabla u) D_j\varphi\,dx = \int_\Omega \omega\,\mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,dx$$

$$B_2(u,\varphi) = \int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,v\,dx$$

$$T(\varphi) = \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^n \int_\Omega f_j\,D_j\varphi\,dx\,.$$

Then $u \in X$ is a (weak) solution to problem (P) if

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi)\,, \quad \text{for all} \quad \varphi \in X\,.$$

Step 1. For $j = 1, \dots, n$ we define the operator $F_j\colon X \to L^{p'}(\Omega, \omega)$ by

$$(F_j u)(x) = \mathcal{A}_j\big(x, u(x), \nabla u(x)\big)\,.$$

We have that the operator F_j is bounded and continuous. In fact:

(i) Using (H4) we obtain

$$
\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u(x)|^{p'}\omega\,dx = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\omega\,dx \\
&\le \int_\Omega \left(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\right)^{p'}\omega\,dx \\
&\le C_p \int_\Omega \left[(K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega\right]dx \\
&= C_p\Big[\int_\Omega K_1^{p'}\,\omega\,dx + \int_\Omega h_1^{p'}|u|^p\,\omega\,dx + \int_\Omega h_2^{p'}|\nabla u|^p\omega\,dx\Big], \qquad (3.1)
\end{aligned}
$$

where the constant C_p depends only on p.

We have, by Theorem 2.4,

$$
\begin{aligned}
\int_\Omega h_1^{p'}|u|^p\,\omega\,dx &\le \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\,\omega\,dx \\
&\le C_\Omega^p\,\|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\,\omega\,dx \\
&\le C_\Omega^p\,\|h_1\|^{p'}_{L^\infty(\Omega)}\,\|u\|^p_X\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega h_2^{p'}|\nabla u|^p\omega\,dx &\le \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\,\omega\,dx \\
&\le \|h_2\|^{p'}_{L^\infty(\Omega)}\|u\|^p_X\,.
\end{aligned}
$$

Therefore, in (3.1) we obtain

$$
\|F_j u\|_{L^{p'}(\Omega,\omega)} \le C_p\left(\|K\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\,\|u\|_X^{p/p'}\right).
$$

(ii) Let $u_m \to u$ in X as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,\omega)$.

If $u_m \to u$ in X, then $u_m \to u$ in $L^p(\Omega,\omega)$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega)$. Using Theorem 2.3, there exist a subsequence $\{u_{m_k}\}$ and functions Φ_1 and Φ_2 in $L^p(\Omega,\omega)$ such that

$$
\begin{aligned}
&u_{m_k}(x) \to u(x), && \mu_1-\ \text{a.e. in}\ \ \Omega\,,\\
&|u_{m_k}(x)| \le \Phi_1(x)\,, && \mu_1-\ \text{a.e. in}\ \ \Omega,\\
&|\nabla u_{m_k}(x)| \to |\nabla u(x)|\,, && \mu_1-\ \text{a.e. in}\ \ \Omega\,,\\
&|\nabla u_{m_k}(x)| \le \Phi_2(x)\,, && \mu_1-\ \text{a.e. in}\ \ \Omega\,,
\end{aligned}
$$

where $\mu_1(E) = \int_E \omega(x)\,dx$. Hence, using (H4), we obtain

$$
\begin{aligned}
\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'}\omega\,dx \\
&= \int_\Omega |\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k}) - \mathcal{A}_j(x,u,\nabla u)|^{p'}\,\omega\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C_p \int_\Omega \big(|\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,u,\nabla u)|^{p'}\big)\omega\,dx \\
&\le C_p \Big[\int_\Omega \big(K_1 + h_1|u_{m_k}|^{p/p'} + h_2|\nabla u_{m_k}|^{p/p'}\big)^{p'}\,\omega\,dx \\
&\quad + \int_\Omega \big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\big)^{p'}\,\omega\,dx\Big] \\
&\le 2\,C_p \int_\Omega \big(K_1 + h_1\Phi_1^{p/p'} + h_2\Phi_2^{p/p'}\big)^{p'}\,\omega\,dx \\
&\le 2\,C_p \Big[\int_\Omega K_1^{p'}\,\omega\,dx + \int_\Omega h_1^{p'}\,\Phi_1^p\,\omega\,dx + \int_\Omega h_2^{p'}\,\Phi_2^p\,\omega\,dx\Big] \\
&\le 2\,C_p\Big[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_1^p\omega\,dx \\
&\quad + \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_2^p\,\omega\,dx\Big] \\
&\le 2\,C_p\Big[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\;\|\Phi_1\|^p_{L^p(\Omega,\omega)} \\
&\quad + \|h_2\|^{p'}_{L^\infty(\Omega)}\|\Phi_2\|^p_{L^p(\Omega,\omega)}\Big]\,.
\end{aligned}
$$

By condition (H1), we have

$$F_j u_m(x) = \mathcal{A}_j(x,u_m(x),\nabla u_m(x)) \to \mathcal{A}_j(x,u(x),\nabla u(x)) = F_j u(x)\,,$$

as $m\to+\infty$. Therefore, by Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega)} \to 0\,,$$

that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega,\omega)$. By Convergence principle in Banach spaces (see Proposition 10.13 in [16]), we have

$$F_j u_m \to F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega)\,. \tag{3.2}$$

Step 2. We define the operator $G\colon X \to L^{p'}(\Omega,v)$ by $(Gu)(x) = |\Delta u(x)|^{p-2}\,\Delta u(x)$.

We also have that the operator G is continuous and bounded. In fact,

(i) We have

$$
\begin{aligned}
\|Gu\|^{p'}_{L^{p'}(\Omega,v)} &= \int_\Omega \big|\,|\Delta u|^{p-2}\,\Delta u\big|^{p'}\,v\,dx \\
&= \int_\Omega |\Delta u|^{(p-2)\,p'}|\Delta u|^{p'}\,v\,dx \\
&= \int_\Omega |\Delta u|^p v\,dx \le \|u\|^p_X\,.
\end{aligned}
$$

Hence, $\|Gu\|_{L^{p'}(\Omega,v)} \le \|u\|_X^{p/p'}$.

(ii) If $u_m \to u$ in X then $\Delta u_m \to \Delta u$ in $L^p(\Omega, v)$. By Theorem 2.3, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_3 \in L^p(\Omega, v)$ such that

$$\begin{aligned} &\Delta u_{m_k}(x) \to \Delta u(x)\,, && \mu_2 - \text{a.e. in} \quad \Omega \\ &|\Delta u_{m_k}(x)| \le \Phi_3(x), && \mu_2 - \text{a.e. in} \quad \Omega\,. \end{aligned}$$

where $\mu_2(E) = \int_E v(x)\,dx$. Hence, using Lemma 2.5 (a), we obtain, if $p \neq 2$

$$\begin{aligned} \|Gu_{m_k} - Gu\|^{p'}_{L^{p'}(\Omega,v)} &= \int_\Omega |\, Gu_{m_k} - Gu|^{p'}\, v\,dx \\ &= \int_\Omega \big|\, |\Delta u_{m_k}|^{p-2}\, \Delta u_{m_k} - |\,\Delta u|^{p-2}\,\Delta u\big|^{p'}\, v\,dx \\ &\le \int_\Omega \big[\alpha_p\, |\Delta u_{m_k} - \Delta u|\,(\,|\Delta u_{m_k}| + |\Delta u|)^{(p-2)}\big]^{p'}\, v\,dx \\ &\le \alpha_p^{p'} \int_\Omega |\,\Delta u_{m_k} - \Delta u|^{p'}\,(2\,\Phi_3)^{(p-2)\,p'}\, v\,dx \\ &\le \alpha_p^{p'}\, 2^{(p-2)p'} \Big(\int_\Omega |\Delta u_{m_k} - \Delta u|^p\, v\,dx\Big)^{p'/p} \\ &\quad \times \Big(\int_\Omega \Phi_3^{(p-2)\,p\,p'/(p-p')}\, v\,dx\Big)^{(p-p')/p} \\ &\le \alpha_p^{p'}\, 2^{(p-2)\,p'}\, \|u_{m_k} - u\|^{p'}_X\, \|\Phi\|^{p-p'}_{L^p(\Omega,v)}, \end{aligned}$$

since $(p-2)\,p\,p'/(p-p') = p$ if $p \neq 2$. If $p = 2$, we have

$$\|Gu_{m_k} - Gu\|^2_{L^2(\Omega,v)} = \int_\Omega |\Delta u_{m_k} - \Delta u|^2\, v\,dx \le \|u_{m_k} - u\|^2_X\,.$$

Therefore (for $1 < p < \infty$), by Dominated Convergence Theorem, we obtain

$$\|Gu_{m_k} - Gu\|_{L^{p'}(\Omega,v)} \to 0\,,$$

that is, $Gu_{m_k} \to Gu$ in $L^{p'}(\Omega, v)$. By Convergence principle in Banach spaces (see Proposition 10.13 in [16]), we have

$$Gu_m \to Gu \quad \text{in} \quad L^{p'}(\Omega, v)\,. \tag{3.3}$$

Step 3. We have, by Theorem 2.4,

$$
\begin{aligned}
|T(\varphi)| &\le \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx \\
&= \int_\Omega \frac{|f_0|}{\omega}|\varphi|\omega\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega}|D_j\varphi|\,\omega\,dx \\
&\le \|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)} \\
&\le \Big(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\Big)\|\varphi\|_X\,.
\end{aligned}
$$

Moreover, using (H4) and the Hölder inequality, we also have

$$
\begin{aligned}
|B(u,\varphi)| &\le |B_1(u,\varphi)| + |B_2(u,\varphi)| \\
&\le \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,dx + \int_\Omega |\,\Delta u|^{p-2}\,|\,\Delta u|\,|\,\Delta\varphi|\,v\,dx\,. \qquad (3.4)
\end{aligned}
$$

In (3.4) we have

$$
\begin{aligned}
\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega\,dx &\le \int_\Omega \big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\big)|\nabla\varphi|\,\omega\,dx \\
&\le \|K_1\|_{L^{p'}(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{L^p(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} \\
&\quad + \|h_2\|_{L^\infty(\Omega)}\|\nabla u\|^{p/p'}_{L^p(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} \\
&\le \big(\|K_1\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}\big)\|\varphi\|_X\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega |\,\Delta u|^{p-2}\,|\,\Delta u|\,|\,\Delta\,\varphi|\,v\,dx &= \int_\Omega |\,\Delta u|^{p-1}\,|\,\Delta\,\varphi|\,v\,dx \\
&\le \Big(\int_\Omega |\,\Delta u|^p\,v\,dx\Big)^{1/p'}\Big(\int_\Omega |\,\Delta\varphi|^p\,v\,dx\Big)^{1/p} \\
&\le \|u\|_X^{p/p'}\,\|\varphi\|_X\,.
\end{aligned}
$$

Hence, in (3.4) we obtain, for all $u,\varphi\in X$

$$
\begin{aligned}
&|B(u,\varphi)| \\
&\le \big[\|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{p/p'}\big]\,\|\varphi\|_X\,.
\end{aligned}
$$

Since $B(u,\cdot)$ is linear, for each $u\in X$, there exists a linear and continuous operator $A\colon X\to X^*$ such that $\langle Au,\varphi\rangle = B(u,\varphi)$, for all $u,\ \varphi\in X$ (where $\langle f,x\rangle$ denotes the value of the linear functional f at the point x) and

$$
\|Au\|_* \le \|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{p/p'}\,.
$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T\,, \quad u \in X\,.$$

Step 4. Using condition (H2) and Lemma 2.5 (b), we have

$$\begin{aligned}
\langle Au_1 - Au_2, u_1 - u_2\rangle &= B(u_1, u_1 - u_2) - B(u_2, u_1 - u_2) \\
&= \int_\Omega \omega\, \mathcal{A}(x, u_1, \nabla u_1)\cdot \nabla(u_1 - u_2)\, dx + \int_\Omega |\,\Delta u_1|^{p-2}\,\Delta u_1\,\Delta(u_1 - u_2)\, v\, dx \\
&\quad - \int_\Omega \omega\, \mathcal{A}(x, u_2, \nabla u_2)\cdot \nabla(u_1 - u_2)\, dx - \int_\Omega |\,\Delta u_2|^{p-2}\,\Delta u_2\,\Delta(u_1 - u_2)\, v\, dx \\
&= \int_\Omega \omega\, \big(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2)\big)\cdot \nabla(u_1 - u_2)\, dx \\
&\quad + \int_\Omega (|\,\Delta u_1|^{p-2}\,\Delta u_1 - |\,\Delta u_2|^{p-2}\,\Delta u_2)\,\Delta(u_1 - u_2)\, v\, dx \\
&\geq \theta_1 \int_\Omega \omega\, |\nabla(u_1 - u_2)|^p\, dx + \beta_p \int_\Omega (|\,\Delta u_1| + |\,\Delta u_2|)^{p-2}\, |\Delta u_1 - \Delta u_2|^2\, v\, dx \\
&\geq \theta_1 \int_\Omega \omega\, |\nabla(u_1 - u_2)|^p\, dx + \beta_p \int_\Omega (|\,\Delta u_1 - \Delta u_2|)^{p-2}\, |\Delta u_1 - \Delta u_2|^2\, v\, dx \\
&= \theta_1 \int_\Omega \omega\, |\nabla(u_1 - u_2)|^p\, dx + \beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^p\, v\, dx \\
&\geq \theta\, \|u_1 - u_2\|_X^p
\end{aligned}$$

where $\theta = \min\{\theta_1, \beta_p\}$.

Therefore, the operator A is strictly monotone. Moreover, using (H3), we obtain

$$\begin{aligned}
\langle Au, u\rangle &= B(u, u) = B_1(u, u) + B_2(u, u) \\
&= \int_\Omega \omega\, \mathcal{A}(x, u, \nabla u)\cdot \nabla u\, dx + \int_\Omega |\,\Delta u|^{p-2}\,\Delta u\,\Delta u\, v\, dx \\
&\geq \int_\Omega \lambda_1 |\nabla u|^p\, \omega\, dx + \int_\Omega |\,\Delta u|^p\, v\, dx \geq \gamma\, \|u\|_X^p
\end{aligned}$$

where $\gamma = \min\{\lambda_1, 1\}$. Hence, since $p > 1$, we have

$$\frac{\langle Au, u\rangle}{\|u\|_X} \to +\infty, \quad \text{as} \quad \|u\|_X \to +\infty\,,$$

that is, A is coercive.

Step 5. We need to show that the operator A is continuous.

Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}|B_1(u_m,\varphi)-B_1(u,\varphi)| &\le \sum_{j=1}^{n}\int_\Omega |\mathcal{A}_j(x,u_m,\nabla u_m)-\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,dx\\ &= \sum_{j=1}^{n}\int_\Omega |F_ju_m-F_ju||D_j\varphi|\,\omega\,dx\\ &\le \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)}\\ &\le \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_X\,,\end{aligned}$$

and

$$\begin{aligned}|B_2(u_m,\varphi)-B_2(u,\varphi)| &= \Big|\int_\Omega |\Delta u_m|^{p-2}\Delta u_m\,\Delta\varphi\,v\,dx-\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,v\,dx\Big|\\ &\le \int_\Omega \Big||\Delta u_m|^{p-2}\,\Delta u_m-|\Delta u|^{p-2}\Delta u\Big|\,|\Delta\varphi|\,v\,dx\\ &= \int_\Omega |Gu_m-Gu|\,|\Delta\varphi|\,v\,dx\\ &\le \|Gu_m-Gu\|_{L^{p'}(\Omega,v)}\,\|\varphi\|_X,\end{aligned}$$

for all $\varphi\in X$. Hence,

$$\begin{aligned}|B(u_m,\varphi)-B(u,\varphi)| &\le \big|B_1(u_m,\varphi)-B_1(u,\varphi)|+|B_2(u_m,\varphi)-B_2(u,\varphi)\big|\\ &\le \Big[\sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}+\|Gu_m-Gu\|_{L^{p'}(\Omega,v)}\Big]\|\varphi\|_X\,.\end{aligned}$$

Then we obtain

$$\|Au_m-Au\|_*\le \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}+\|Gu_m-Gu\|_{L^{p'}(\Omega,v)}\,.$$

Therefore, using (3.2) and (3.3) we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous (and this implies that A is hemicontinuous, see Proposition 27.12 in [15]).

Therefore, by Theorem 3.1, the operator equation $Au=T$ has a unique solution $u\in X$ and it is the unique solution for problem (P).

Step 6. In particular, by setting $\varphi=u$ in Definition 2.7, we have

$$B(u,u)=B_1(u,u)+B_2(u,u)=T(u)\,. \tag{3.5}$$

Hence, using (H3) and $\gamma = \min\{\lambda_1, 1\}$, we obtain

$$\begin{aligned}B_1(u,u) + B_2(u,u) &= \int_\Omega \omega\, \mathcal{A}(x,u,\nabla u).\nabla u\, dx + \int_\Omega |\,\Delta u|^{p-2}\, \Delta u\, \Delta u\, v\, dx \\ &\geq \int_\Omega \lambda_1\, |\nabla u|^p + \int_\Omega |\,\Delta u|^p\, v\, dx \geq\ \gamma \|u\|_X^p\end{aligned}$$

and

$$\begin{aligned}T(u) &= \int_\Omega f_0\, u\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j u\, dx \\ &\leq \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} \|u\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega|_{L^{p'}(\Omega)} \|D_j u\|_{L^p(\Omega,\omega)} \\ &\leq \Big(C_\Omega\, \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega)}\Big)\, \|u\|_X\,.\end{aligned}$$

Therefore, in (3.5), we obtain

$$\gamma\, \|u\|_X^p \leq \Big(C_\Omega\, \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\Big)\, \|u\|_X\,,$$

and we obtain

$$\|u\|_X \leq \frac{1}{\gamma^{p'/p}}\, \Big(C_\Omega\, \|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\Big)^{p'/p}\,.$$

Example. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and consider the weights functions $\omega(x,y) = (x^2+y^2)^{-1/2}$ and $v(x,y) = (x^2+y^2)^{-2/3}$ ($\omega, v \in A_3$, $p=3$), and the function

$$\mathcal{A}\colon \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$\mathcal{A}\big((x,y),\eta,\xi\big) = h_2(x,y)\, |\xi|\, \xi\,,$$

where $h(x,y) = 2\, \mathrm{e}^{(x^2+y^2)}$. Let us consider the partial differential operator

$$Lu(x,y) = \Delta\big((x^2+y^2)^{-2/3} |\Delta u|\, \Delta u\big) - \operatorname{div}\big((x^2+y^2)^{-1/2}\, \mathcal{A}\big((x,y),u,\nabla u\big)\big)\,.$$

Therefore, by Theorem 1.1, the problem

$$\text{(P)} \qquad \begin{cases} Lu(x) = \dfrac{\cos(xy)}{(x^2+y^2)} - \dfrac{\partial}{\partial x}\Big(\dfrac{\sin(xy)}{(x^2+y^2)}\Big) - \dfrac{\partial}{\partial y}\Big(\dfrac{\sin(xy)}{(x^2+y^2)}\Big)\,, & \text{in } \Omega \\ u(x) = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in X = W^{2,3}(\Omega, v) \cap W_0^{1,3}(\Omega,\omega)$.

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