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HIGGS BUNDLES AND REPRESENTATION SPACES ASSOCIATED TO MORPHISMS

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Abstract. Let G be a connected reductive affine algebraic group defined over the complex numbers, and $K \subset G$ be a maximal compact subgroup. Let X, Y be irreducible smooth complex projective varieties and $f\colon X \to Y$ an algebraic morphism, such that $\pi_1(Y)$ is virtually nilpotent and the homomorphism $f_*\colon \pi_1(X) \to \pi_1(Y)$ is surjective. Define

$$\mathcal{R}^f\Big(\pi_1(X), G\Big) = \{\rho \in \operatorname{Hom}\Big(\pi_1(X), G\Big) \mid A \circ \rho \ \text{ factors through } \ f_*\}\,,$$

$$\mathcal{R}^f\Big(\pi_1(X), K\Big) = \{\rho \in \operatorname{Hom}\Big(\pi_1(X), K\Big) \mid A \circ \rho \ \text{ factors through } \ f_*\}\,,$$

where $A\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$ is the adjoint action. We prove that the geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X, x_0),\, G)/\!\!/G$ admits a deformation retraction to $\mathcal{R}^f(\pi_1(X, x_0),\, K)/K$. We also show that the space of conjugacy classes of n almost commuting elements in G admits a deformation retraction to the space of conjugacy classes of n almost commuting elements in K.

1. Introduction

Let G be a connected reductive affine algebraic group defined over the complex numbers. Consider an algebraic morphism

$$f\colon X \to Y$$

where X and Y are irreducible smooth complex projective varieties, and let

$$f_*\colon \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big)$$

be the induced morphism of fundamental groups, where $x_0 \in X$ is a base point. In certain situations, the representations

$$\rho\colon \pi_1(X, x_0) \to G$$

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that factor through f_* have special geometric properties. See [9], where necessary and sufficient conditions for such a factorization are given in terms of the spectral curve of the G-Higgs bundle associated to ρ.

In this article, we are interested in the whole moduli space of representations that factor in a similar way, and in its topological properties. Under some assumptions on f and Y, we provide a natural deformation retraction between two such representation spaces, described as follows.

The Lie algebra of G will be denoted by $\mathfrak{g}$. Let $A\colon G \to \mathrm{GL}(\mathfrak{g})$ be the homomorphism given by the adjoint action of G on $\mathfrak{g}$. Fix a maximal compact subgroup $K \subset G$ and define:

$$\mathcal{R}^f\big(\pi_1(X,x_0),G\big) = \{\rho \in \mathrm{Hom}\big(\pi_1(X,x_0),G\big) \mid A\circ\rho \ \text{ factors through } \ f_*\}\,,$$

$$\mathcal{R}^f\big(\pi_1(X,x_0),K\big) = \{\rho \in \mathrm{Hom}\big(\pi_1(X,x_0),K\big) \mid A\circ\rho \ \text{ factors through } \ f_*\}\,.$$

We note that the group G (respectively, K) acts on $\mathcal{R}^f(\pi_1(X,x_0),G)$ (respectively, on $\mathcal{R}^f(\pi_1(X,x_0),K)$) via the conjugation action of G (respectively, K) on itself. The quotient $\mathcal{R}^f(\pi_1(X,x_0),K)/K$ is contained in the geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X,x_0),G)/\!\!/G$.

We prove the following in Theorem 2.6:

Suppose that the fundamental group of Y is virtually nilpotent, and the homomorphism f_* is surjective. Then $\mathcal{R}^f(\pi_1(X,x_0),G)/\!\!/G$ admits a deformation retraction to the subset $\mathcal{R}^f(\pi_1(X,x_0),K)/K$.

In Section 3, we consider spaces of almost commuting elements in K and in G. Define:

$$\mathrm{AC}^n(K) = \{(g_1,\ldots,g_n) \in K^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_K \quad \forall\, i\,,j\}\,,$$

where Z_K denotes the center of K. The moduli space of conjugacy classes:

$$\mathrm{AC}^n(K)\,/\,K\,,$$

where K acts by simultaneous conjugation, was studied in [6], [8], and plenty of information is known in the cases $n=2$ and $n=3$. For instance, the number of components of $\mathrm{AC}^3(K)\,/\,K$ has been related in [6] to the Chern-Simons invariants associated to flat connections on a 3-torus.

In a similar fashion, we define $\mathrm{AC}^n(G)/\!\!/G$, the moduli space of conjugacy classes of n almost commuting elements in G. For example, if G has trivial center, then $\mathrm{AC}^{2n}(G)/\!\!/G$ coincides with

$$\mathrm{Hom}(\pi_1(X,x_0),G)/\!\!/G\,,$$

where X is an abelian variety of complex dimension n. In Proposition 3.1, we show that $\mathrm{AC}^n(G)\,/\,G$ admits a deformation retraction to $\mathrm{AC}^n(K)\,/\,K$, and that the same holds for $\mathrm{A}C^n(G)$ and $\mathrm{A}C^n(K)$, extending one of the main results in [7] and [4].

2. Representation spaces associated to a morphism

Let X be an irreducible smooth complex projective variety. Fix a point $x_0 \in X$. Let

$$f\colon X \to Y$$

be an algebraic morphism, where Y is also an irreducible smooth complex projective variety, such that:

(1) the fundamental group $\pi_1(Y, f(x_0))$ is virtually nilpotent, and

(2) the homomorphism of fundamental groups induced by f

$$f_*\colon \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big) \tag{2.1}$$

is surjective.

Using the homomorphism f_* in (2.1), we will consider $\pi_1(Y, f(x_0))$ as a quotient of the group $\pi_1(X, x_0)$.

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. Let

$$A\colon G \to \mathrm{GL}(\mathfrak{g}) \tag{2.2}$$

be the homomorphism given by the adjoint action of G on $\mathfrak{g}$. The affine algebraic variety (not necessarily irreducible) of representations

$$\rho\colon \pi_1(X, x_0) \to G$$

will be denoted by $\mathrm{Hom}(\pi_1(X, x_0), G)$.

Definition 2.1. Let $\rho \in \mathrm{Hom}(\pi_1(X, x_0),\, G)$. We sat that $A \circ \rho$ factors through f_* in (2.1) (or that $A \circ \rho$ *factors geometrically* through $f : X \to Y$, see [9]) if there exists a homomorphism $\rho' \in \mathrm{Hom}(\pi_1(Y, f(x_0)), \mathrm{GL}(\mathfrak{g}))$ such that

$$\rho' \circ f_* = A \circ \rho\,. \tag{2.3}$$

Remark 2.2. (1) Clearly, if ρ itself factorizes as $\rho = \tilde{\rho} \circ f_*$ for some $\tilde{\rho} \in \mathrm{Hom}(\pi_1(X, x_0), G)$, then $A \circ \rho$ factorizes through f_* as in the definition; the converse is not always true.

(2) It is clear that $A \circ \rho \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}(\mathfrak{g}))$ factors through f_* as in (2.3), if and only if $A \circ \rho$ is trivial on the kernel of f_*. Moreover, when $A \circ \rho$ factors through f_*, a homomorphism $\rho' \in \mathrm{Hom}(\pi_1(Y, f(x_0)), \mathrm{GL}(\mathfrak{g}))$ satisfying equation (2.3) is unique, because f_* is surjective.

In the framework of non-abelian Hodge theory, there is a correspondence between semistable G-Higgs bundles over X and representations in $\mathrm{Hom}(\pi_1(X, x_0), G)$, [11], [5]. Denote by (E_ρ, θ_ρ) the semistable G-Higgs bundle on X associated to ρ under this correspondence. We note that (E_ρ, θ_ρ) is semistable with respect to every polarization on X.

Lemma 2.3. *Let $\rho \in \mathrm{Hom}(\pi_1(X, x_0), G)$ be such that $A \circ \rho$ factors through f_*. Then, the above principal G-bundle E_ρ on X is semistable.*

Proof. Let

$$\mathrm{ad}(E_\rho) := E_\rho \times^A \mathfrak{g} \to X$$

be the adjoint vector bundle of E_ρ. The Higgs field on $\mathrm{ad}(E_\rho)$ induced by θ_ρ will be denoted by $\mathrm{ad}(\theta_\rho)$.

Let $\rho' : \pi_1(Y, f(x_0)) \to \mathrm{GL}(\mathfrak{g})$ be the unique homomorphism satisfying equation (2.3); the uniqueness of ρ' is a consequence of the surjectivity of f_* as remarked above. Let (E', θ') be the semistable Higgs vector bundle on Y associated to this homomorphism ρ'. Since the fundamental group of Y is virtually nilpotent, we know that the vector bundle E' is semistable [3, Proposition 3.1]. Let $c_i(E')$, $i \geq 0$, be the sequence of Chern classes of the bundle E'. Then, $c_i(E') = 0$ for all $i > 0$ because the C^∞ complex vector bundle underlying E' admits a flat connection (it is isomorphic to the C^∞ complex vector bundle underlying the flat vector bundle associated to ρ'). Therefore, by [2, p. 39, Theorem 5.1], the vector bundle E' admits a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = E'$$

of holomorphic subbundles such that each successive quotient V_i/V_{i-1}, $1 \leq i \leq \ell$, admits a flat unitary connection. Consider the pulled back filtration

$$0 = f^*V_0 \subset f^*V_1 \subset \cdots \subset f^*V_{\ell-1} \subset f^*V_\ell = f^*E' \,. \tag{2.4}$$

A flat unitary connection on V_i/V_{i-1} pulls back to a flat unitary connection on

$$f^*V_i/(f^*V_{i-1}) = f^*(V_i/V_{i-1}) \,.$$

Since each successive quotient for the filtration of f^*E' in (2.4) admits a flat unitary connection, we conclude that the holomorphic vector bundle f^*E' is semistable.

From (2.3) it follows that

$$\big(\mathrm{ad}(E_\rho), \mathrm{ad}(\theta_\rho)\big) = (f^*E', f^*\theta') \,. \tag{2.5}$$

Since f^*E' is semistable, from (2.5) it follows that $\mathrm{ad}(E_\rho)$ is semistable. This implies that the principal G-bundle E_ρ is semistable [1, p. 214, Proposition 2.10]. $\square$

Lemma 2.3 has the following corollary:

Corollary 2.4. *For any Higgs field θ, the G-Higgs bundle (E_ρ, θ) is semistable.*

Let

$$\rho^\lambda : \pi_1(X, x_0) \to G \tag{2.6}$$

be a homomorphism corresponding to the Higgs G-bundle $(E_\rho, \lambda \cdot \theta_\rho)$, which is semistable by Corollary 2.4. We note that although ρ^λ is not uniquely determined by $(E_\rho, \lambda \cdot \theta_\rho)$, the point in the quotient space

$$\mathrm{Hom}(\pi_1(X, x_0), G)/G$$

given by ρ^λ does not depend on the choice of ρ^λ. In other words, any two different choices of ρ^λ differ by an inner automorphism of the group G.

Lemma 2.5. *For every $\lambda \in \mathbb{C}$, the homomorphism $A \circ \rho^\lambda$ factors through f_*, where ρ^λ is defined in* (2.6).

Proof. Let $(\mathrm{ad}(E_\rho)^\lambda, \mathrm{ad}(\theta_\rho)^\lambda)$ be the Higgs vector bundle associated to the homomorphism $A\circ\rho^\lambda$. We note that $(\mathrm{ad}(E_\rho)^\lambda, \mathrm{ad}(\theta_\rho)^\lambda)$ is isomorphic to $(f^*E', f^*(\lambda\cdot\theta'))$, because the Higgs bundle (E', θ') corresponds to ρ', and (2.3) holds. We saw in the proof of Lemma 2.3 that E' is semistable with $c_i(E') = 0$ for all $i > 0$. Since $(\mathrm{ad}(E_\rho)^\lambda, \mathrm{ad}(\theta_\rho)^\lambda)$ is isomorphic to the pullback of a semistable Higgs vector bundle on Y such that all the Chern classes of positive degrees of the underlying vector bundle on Y vanish, it can be deduced that $A\circ\rho^\lambda$ factors through the quotient $\pi_1(Y, f(x_0))$. In fact, if

$$\delta\colon \pi_1(Y, f(x_0)) \to \mathrm{GL}(\mathfrak{g})$$

is a homomorphism corresponding to the Higgs vector bundle $(E', \lambda\cdot\theta')$, then

- the homomorphism $A\circ\rho^\lambda$ factors through the quotient $\pi_1(Y, f(x_0))$, and
- the homomorphism $\pi_1(Y, f(x_0)) \to \mathrm{GL}(\mathfrak{g})$ resulting from $A\circ\rho^\lambda$ differs from δ by an inner automorphism of $\mathrm{GL}(\mathfrak{g})$.

This completes the proof. □

Fix a maximal compact subgroup

$$K \subset G\,.$$

Define

$$\mathcal{R}^f\big(\pi_1(X, x_0), G\big) = \{\rho \in \mathrm{Hom}(\pi_1(X, x_0),\, G) \mid A\circ\rho \ \text{ factors through } f_*\}\,,$$

$$\mathcal{R}^f(\pi_1(X, x_0),\, K) = \{\rho \in \mathrm{Hom}(\pi_1(X, x_0),\, K) \mid A\circ\rho \ \text{ factors through } \ f_*\}\,.$$

Since $\pi_1(X, x_0)$ is a finitely presented group, the affine algebraic structure of G produces an affine algebraic structure on $\mathcal{R}^f(\pi_1(X, x_0),\, G)$. The group G acts on $\mathcal{R}^f(\pi_1(X, x_0),\, G)$ via the conjugation action of G on itself. Let

$$\mathcal{R}^f(\pi_1(X, x_0),\, G)/\!\!/G$$

be the corresponding geometric invariant theoretic quotient. We note that this geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X, x_0),\, G)/\!\!/G$ is a complex affine algebraic variety. Let

$$\mathcal{R}^f(\pi_1(X, x_0),\, K)/K$$

be the quotient of $\mathcal{R}^f(\pi_1(X, x_0),\, K)$ for the adjoint action of K on itself.

The inclusion of K in G produces an inclusion of $\mathcal{R}^f(\pi_1(X, x_0), K)$ in $\mathcal{R}^f(\pi_1(X, x_0), G)$, which, in turn, gives an inclusion

$$\mathcal{R}^f(\pi_1(X, x_0),\, K)/K \hookrightarrow \mathcal{R}^f(\pi_1(X, x_0),\, G)/\!\!/G\,. \tag{2.7}$$

Instead of working with the Zariski topology on $\mathcal{R}^f(\pi_1(X, x_0),\, G)/\!\!/G$, we consider on it the Euclidean topology which is induced from an embedding of this space in a complex affine space. Indeed, such an embedding can always be obtained by considering a finite set of generators of the algebra of G-invariant regular functions on $\mathcal{R}^f(\pi_1(X, x_0),\, G)$. Moreover, this topology is independent of the choice of such embedding, and compatible with the inclusion (2.7).

Theorem 2.6. *The topological space $\mathcal{R}^f(\pi_1(X, x_0),\, G)/\!\!/G$ admits a deformation retraction to the above subset $\mathcal{R}^f(\pi_1(X, x_0),\, K)/K$.*

Proof. Two elements of $\mathrm{Hom}(\pi_1(X,x_0),\, G)$ are called equivalent if they differ by an inner automorphism of G. Points of $\mathcal{R}^f(\pi_1(X,x_0),\, G)/\!\!/G$ correspond to the equivalence classes of homomorphisms $\rho \in \mathrm{Hom}(\pi_1(X,x_0),\, G)$ such that the action of $\pi_1(X,x_0)$ on $\mathfrak{g}$ given by $A\circ\rho$ is completely reducible, meaning that $\mathfrak{g}$ is a direct sum of irreducible $\pi_1(X,x_0)$-modules. Let $(E_\rho\,,\theta_\rho)$ be the semistable G-Higgs bundle corresponding to the above homomorphism ρ, and let $(\mathrm{ad}(E_\rho)\,,\mathrm{ad}(\theta_\rho))$ be the semistable adjoint Higgs vector bundle associated to $(E_\rho\,,\theta_\rho)$. The above condition that the action of $\pi_1(X,x_0)$ on $\mathfrak{g}$ given by $A\circ\rho$ is completely reducible is equivalent to the condition that the semistable Higgs vector bundle $(\mathrm{ad}(E_\rho)\,,\mathrm{ad}(\theta_\rho))$ is polystable.

Let

$$\phi\colon \big(\mathcal{R}^f(\pi_1(X,x_0),\, G)/\!\!/G\big)\times[0\,,1]\to \mathcal{R}^f\big(\pi_1(X,x_0),\, G\big)/\!\!/G$$

be the map defined by $(\rho,\lambda)\longmapsto \rho^{1-\lambda}$ (defined in (2.6)), where $\rho\in\mathrm{Hom}(\pi_1(X,x_0),\, G)$ satisfies the condition that the action of $\pi_1(X,x_0)$ on $\mathfrak{g}$ given by $A\circ\rho$ is completely reducible. It is easy to see that ϕ is well-defined. We note that the point in the geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X,x_0),G)/\!\!/G$ given by ρ lies in the subset $\mathcal{R}^f(\pi_1(X,x_0),K)/K$ if and only if the Higgs field θ_ρ on the principal G–bundle E_ρ vanishes identically (as before, (E_ρ,θ_ρ) is the Higgs G-bundle corresponding to ρ).

The following are straightforward to check:

- $\phi(z\,,0)=z$ for all $z\in\mathcal{R}^f(\pi_1(X,x_0),G)/\!\!/G$,
- $\phi(z,1)\in\mathcal{R}^f(\pi_1(X,x_0),K)/K$ for all $z\in\mathcal{R}^f(\pi_1(X,x_0),G)/\!\!/G$, and
- $\phi(z,\lambda)=z$ for all $z\in\mathcal{R}^f(\pi_1(X,x_0),K)/K$ and $\lambda\in[0,1]$.

Therefore, the above map ϕ produces a deformation retraction of $\mathcal{R}^f(\pi_1(X,x_0),G)/\!\!/G$ to $\mathcal{R}^f(\pi_1(X,x_0),K)/K$. ☐

Remark 2.7. Lemma 2.3 and Theorem 2.6 are also valid for morphisms $f\colon X\to Y$ in the category of compact Kähler manifolds, under the same assumptions on Y and f_*. The proofs of these results are analogous, by replacing semistability with the notion of *pseudostability* (see [5], [3]).

3. Deformation retraction of the space of almost commuting elements

Again, let G be a connected complex reductive group, and K be a maximal compact subgroup. Let

$$Z_G\subset G$$

be the center of G and let

$$PG:=G/Z_G$$

be the quotient group. We note that the center of PG is trivial. Let

$$q\colon G\to PG \tag{3.1}$$

be the quotient map. The image

$$PK:=q(K)\subset PG$$

is a maximal compact subgroup of PG. We have $q^{-1}(PK) = K$.

Fix a positive integer n. Define

$$\mathrm{AC}^n(G) = \{(g_1, \ldots, g_n) \in G^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_G \ \ \forall\, i, j\}\,.$$

It is a subscheme of the affine variety G^n. The group G acts on $\mathrm{AC}^n(G)$ as simultaneous conjugation of the n factors. Let

$$\mathrm{ACE}^n(G) \ := \ \mathrm{AC}^n(G)/\!\!/G$$

be the geometric invariant theoretic quotient. Also, define

$$\mathrm{AC}^n(K) = \{(g_1, \ldots, g_n) \in K^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_G \quad \forall\, i, j\}\,.$$

So $\mathrm{AC}^n(K) = \mathrm{AC}^n(G) \bigcap K^n$. Let

$$\mathrm{ACE}^n(K) := \mathrm{AC}^n(K)/K$$

be the quotient for the simultaneous conjugation action of K on the n factors. Note that the inclusion of K in G produces an inclusion

$$\mathrm{ACE}^n(K) \hookrightarrow \mathrm{ACE}^n(G)\,.$$

Proposition 3.1. *Let G be semisimple. Then, the topological space* $\mathrm{ACE}^n(G)$ *admits a deformation retraction to the above subset* $\mathrm{ACE}^n(K)$*.*

Proof. When G is semisimple, Z_G is a finite subgroup of G, so that the map (3.1) is a Galois covering. Also, $Z_G \subset K$. Define $\mathrm{AC}^n(PG)$ and $\mathrm{ACE}^n(PG)$ by substituting PG in place of G in the above constructions. Note that $\mathrm{AC}^n(PG)$ parametrizes commuting n elements of PG because the center of PG is trivial. Similarly, define $\mathrm{AC}^n(PK)$ and $\mathrm{ACE}^n(PK)$ by substituting PK in place of K. So $\mathrm{AC}^n(PK)$ parametrizes commuting n elements of PK. The projection

$$\beta\colon\ \mathrm{ACE}^n(G) \to \mathrm{ACE}^n(PG) \tag{3.2}$$

constructed using the the projection q in (3.1) is a Galois covering with Galois group Z_G^n. However it should be mentioned that $\mathrm{ACE}^n(G)$ need not be connected. Let

$$\gamma\colon\ \mathrm{ACE}^n(K) \to \mathrm{ACE}^n(PK)$$

be the projection constructed similarly using q. Clearly, γ coincides with the restriction of β to $\mathrm{ACE}^n(K) \subset \mathrm{ACE}^n(G)$.

There is a deformation retraction of $\mathrm{ACE}^n(PG)$ to $\mathrm{ACE}^n(PK)$

$$\varphi\colon\ \mathrm{ACE}^n(PG) \times [0,1] \to \mathrm{ACE}^n(PG)$$

[7, Theorem 1.1] (see also [4]). In particular, $\varphi|_{\mathrm{ACE}^n(PG)\times\{0\}}$ is the identity map of $\mathrm{ACE}^n(PG)$.

Applying the homotopy lifting property to the covering β in (3.2), there is a unique map

$$\widetilde{\varphi}\colon\ \mathrm{ACE}^n(G) \times [0,1] \to \mathrm{ACE}^n(G)$$

such that

(1) $\beta \circ \widetilde{\varphi} = \varphi \circ (\beta \times \mathrm{Id}_{[0,1]})$, and

(2) $\widetilde{\varphi}|_{\mathrm{ACE}^n(G)\times\{0\}}$ is the identity map of $\mathrm{ACE}^n(G)$.

This map $\widetilde{\varphi}$ is a deformation retraction of $\mathrm{ACE}^n(G)$ to $\mathrm{ACE}^n(K)$, because φ is a deformation retraction. □

Proposition 3.1 remains valid in the more general situation when G is reductive.

Theorem 3.2. *Let G be a connected reductive affine algebraic group over $\mathbb{C}$. Then, $\mathrm{ACE}^n(G)$ admits a deformation retraction to the subset $\mathrm{ACE}^n(K)$.*

Proof. First, note that Proposition 3.1 is clearly valid if G is a product of copies of the multiplicative group $\mathbb{C}^*$. Hence it remains valid for any G which is a product of a semisimple group and copies of $\mathbb{C}^*$. For a general connected reductive group G, consider the natural homomorphism

$$\eta\colon G \to PG \times (G/[G,G])\,.$$

It is a surjective Galois covering map, the quotient $PG := G/Z_G$ is semisimple, while the quotient $G/[G,G]$ is a product of copies of $\mathbb{C}^*$. As mentioned above Proposition 3.1 is valid for $PG \times (G/[G,G])$. Using this and the above homomorphism η it follows that Proposition 3.1 is valid for G. □

3.1. **Deformation retraction of the space of n commuting elements.** Finally, we note that the analogous result is also verified for the space of n commuting elements, $\mathrm{AC}^n(G)$.

Theorem 3.3. *Let G be a connected reductive affine algebraic group over $\mathbb{C}$. Then, the space $\mathrm{AC}^n(G)$ admits a deformation retraction to the subset $\mathrm{AC}^n(K)$.*

Proof. Since PG and PK have trivial center, the spaces $\mathrm{AC}^n(PG)$ and $\mathrm{AC}^n(PK)$ consist of n commuting elements: If $(g_1, \ldots, g_n) \in \mathrm{AC}^n(PG)$, then

$$g_i g_j = g_j g_i\,, \quad \text{for all } i, j \in \{1, \ldots, n\}\,.$$

Therefore, it is known that $\mathrm{AC}^n(PG)$ admits a deformation retraction to $\mathrm{AC}^n(PK)$ [10, p. 2514, Theorem 1.1]. In view of this, imitating the proof of Proposition 3.1 it follows that $\mathrm{AC}^n(G)$ admits a deformation retraction to $\mathrm{AC}^n(K)$. □

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