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CONDITIONS FOR INTEGRABILITY OF A 3-FORM

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Abstract. We find necessary and sufficient conditions for the integrability of one type of multisymplectic 3-forms on a 6-dimensional manifold.

Let V be a 6-dimensional real vector space. The general linear group $GL(V)$ operates naturally on the space of 3-forms $\Lambda^3 V^*$ by

$$\varphi\alpha(v,v',v'') = \alpha(\varphi^{-1}v, \varphi^{-1}v', \varphi^{-1}v'')\,, \quad \alpha \in \Lambda^3 V^*,\ \varphi \in GL(V)\,.$$

This action has six orbits, see e.g. [1]. They can be described by their representatives. Let us choose a basis $v_1, \dots, v_6$ of V, and let $\alpha_1, \dots, \alpha_6$ be the corresponding dual basis. Let us recall that a 3-form $\alpha \in \Lambda^3 V^*$ is called *regular* or *multisymplectic* if the linear mapping

$$\iota\colon V \to \Lambda^2 V^*, \quad \iota(v) = \iota_v \alpha$$

is injective. All the other forms are then called *singular*. Obviously, all forms belonging to an orbit are either regular or singular. We then speak about *regular orbits* and *singular orbits*. We denote R_+, R_- and R_0 the regular orbits and by ρ_+, ρ_-, ρ_0 their representatives. Similarly we denote S_1, S_2 and S_3 the singular orbits and by σ_1, σ_2, σ_3 their representatives.

(R_+) $\qquad \rho_+ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6\,,$

(R_-) $\qquad \rho_- = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6\,,$

(R_0) $\qquad \rho_0 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6 + \alpha_3 \wedge \alpha_6 \wedge \alpha_4\,,$

(S_1) $\qquad \sigma_1 = 0\,,$

(S_2) $\qquad \sigma_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3\,,$

(S_3) $\qquad \sigma_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5)\,.$

We recall that a 2-form β on a vector space is called *decomposable* if there exist 1-forms γ and γ' such that $\beta = \gamma \wedge \gamma'$. It is well known that a 2-form β is decomposable if and only if $\beta \wedge \beta = 0$.

With every 3-form $\alpha \in \Lambda^3 V^*$ we can associate a subset $\Delta(\alpha) \subset V$ defined by

$$\Delta(\alpha) = \{v \in V; \iota_v\alpha \wedge \iota_v\alpha = 0\}\,.$$

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In other words $\Delta(\alpha)$ consists of all $v \in V$ such hat the 2-form $\iota_v \alpha$ is decomposable.

1. Algebraic properties

We take now an element $\alpha \in R_0$. We find easily that

$$\Delta(\rho_0) = [v_1, v_2, v_3]\,.$$

This shows that the subset $\Delta(\alpha)$ is a 3-dimensional subspace of V. For simplicity we denote $V_0 = \Delta(\alpha)$. There is also another possible description of $\Delta(\alpha)$.

1. **Lemma.** $\Delta(\alpha) = \big\{v \in V; (\iota_v \alpha) \wedge \alpha = 0\big\}$.

Proof. Obviously it suffices to prove this equality for $\alpha = \rho_0$. We take $v = a_1 v_1 + \dots + a_6 v_6$ and we find

$$\begin{aligned}(\iota_v \rho_0) \wedge \rho_0 = &-2a_6 \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 + 2a_4 \alpha_1 \wedge \alpha_3 \wedge \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 \\ &- 2a_5 \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6\,.\end{aligned}$$

This proves the lemma. □

For ρ_0, and consequently for every $\alpha \in R_0$ we have the following lemma.

2. **Lemma.** *If $\alpha \in R_0$ and v, $v' \in \Delta(\alpha)$, then $\alpha(v, v', \cdot) = 0$.*

Inspired by ρ_0 we introduce the following definition.

3. **Definition.** A basis $w_1, \dots, w_6$ of V is called *canonical basis for α* if the following conditions are satisfied

$$\begin{aligned}
&\alpha(w_1, w_2, w_3) = 0\,, \quad \alpha(w_i, w_j, w_k) = 0 \quad \text{for} \quad 1 \le i < j \le 3,\ k = 4, 5, 6\,,\\
&\alpha(w_1, w_4, w_5) = 1\,, \quad \alpha(w_1, w_5, w_6) = 0\,, \quad \alpha(w_1, w_6, w_4) = 0\,,\\
&\alpha(w_2, w_4, w_5) = 0\,, \quad \alpha(w_2, w_5, w_6) = 1\,, \quad \alpha(w_2, w_6, w_4) = 0\,,\\
&\alpha(w_3, w_4, w_5) = 0\,, \quad \alpha(w_3, w_5, w_6) = 0\,, \quad \alpha(w_3, w_6, w_4) = 1\,,\\
&\alpha(w_4, w_5, w_6) = 0\,.
\end{aligned}$$

A dual basis $\beta_1, \dots, \beta_6$ to a canonical basis will be called *canonical dual basis* for α.

It is easy to see that $\beta_1, \dots, \beta_6$ is a canonical dual basis for α if and only if there is

$$\alpha = \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_5 \wedge \beta_6 + \beta_3 \wedge \beta_6 \wedge \beta_4\,.$$

Because the forms α and ρ_0 are equivalent (= belong to the same orbit), it is obvious that

4. **Lemma.** *Every 3-form $\alpha \in R_0$ has a canonical basis.*

Nevertheless for the later considerations within the framework of differential geometry we shall present a constructive proof.

Proof. We choose first a complement V_c of V_0 in V. In this complement we take three linearly independent vectors z_4, z_5, z_6. We denote $a = \alpha(z_4, z_5, z_6)$. Because the form α is regular, there is $v_0 \in V_0$ such that $\alpha(v_0, z_5, z_6) = b \neq 0$. Taking $w_4 = z_4 - (a/b)v_0$, $w_5 = z_5$, and $w_6 = z_6$ we get

$$\begin{aligned}\alpha(w_4, w_5, w_6) &= \alpha(z_4 - (a/b)v_0, z_5, z_6) \\ &= \alpha(z_4, z_5, z_6) - (a/b)\alpha(v_0, z_5, z_6) = a - (a/b)b = 0\,.\end{aligned}$$

Now we have on V_0 three linear forms, namely the forms $\alpha(\cdot, w_4, w_5)$, $\alpha(\cdot, w_5, w_6)$, and $\alpha(\cdot, w_6, w_4)$. The regularity of α implies again that these three forms are linearly independent. Consequently, there are uniquely determined w_1, w_2, $w_3 \in V_0$ such that

$$\begin{aligned}
&\alpha(w_1, w_4, w_5) = 1\,, \quad \alpha(w_1, w_5, w_6) = 0\,, \quad \alpha(w_1, w_6, w_4) = 0\,,\\
&\alpha(w_2, w_4, w_5) = 0\,, \quad \alpha(w_2, w_5, w_6) = 1\,, \quad \alpha(w_2, w_6, w_4) = 0\,,\\
&\alpha(w_3, w_4, w_5) = 0\,, \quad \alpha(w_3, w_5, w_6) = 0\,, \quad \alpha(w_3, w_6, w_4) = 1\,.
\end{aligned}$$

The equations $\alpha(w_1, w_2, w_3) = 0$ and $\alpha(w_i, w_j, w_k) = 0$ for $1 \le i < j \le 3$, $k = 4, 5, 6$ are satisfied automatically by virtue of Lemma 2. $\square$

Let us consider two canonical dual bases $\beta_1, \dots, \beta_6$ and $\beta_1', \dots, \beta_6'$. We can write

$$\begin{aligned}
\beta_1' &= c_{11}\beta_1 + c_{12}\beta_2 + c_{13}\beta_3 + c_{14}\beta_4 + c_{15}\beta_5 + c_{16}\beta_6\\
\beta_2' &= c_{21}\beta_1 + c_{22}\beta_2 + c_{23}\beta_3 + c_{24}\beta_4 + c_{25}\beta_5 + c_{26}\beta_6\\
\beta_3' &= c_{31}\beta_1 + c_{32}\beta_2 + c_{33}\beta_3 + c_{34}\beta_4 + c_{35}\beta_5 + c_{36}\beta_6\\
\beta_4' &= c_{44}\beta_4 + c_{45}\beta_5 + c_{46}\beta_6\\
\beta_5' &= c_{54}\beta_4 + c_{55}\beta_5 + c_{56}\beta_6\\
\beta_6' &= c_{64}\beta_4 + c_{65}\beta_5 + c_{66}\beta_6
\end{aligned}$$

We start with the equation

$$\beta_1' \wedge \beta_4' \wedge \beta_5' + \beta_2' \wedge \beta_5' \wedge \beta_6' + \beta_3' \wedge \beta_6' \wedge \beta_4' = \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_5 \wedge \beta_6 + \beta_3 \wedge \beta_6 \wedge \beta_4.$$

Comparing the coefficients at $\beta_1 \wedge \beta_4 \wedge \beta_5$, $\beta_1 \wedge \beta_5 \wedge \beta_6$, and $\beta_1 \wedge \beta_6 \wedge \beta_4$, we obtain

$$\begin{vmatrix} c_{21} & c_{44} & c_{45} \\ c_{31} & c_{54} & c_{55} \\ c_{11} & c_{64} & c_{65} \end{vmatrix} = 1, \qquad \begin{vmatrix} c_{21} & c_{45} & c_{46} \\ c_{31} & c_{55} & c_{56} \\ c_{11} & c_{65} & c_{66} \end{vmatrix} = 0, \qquad \begin{vmatrix} c_{21} & c_{46} & c_{44} \\ c_{31} & c_{56} & c_{54} \\ c_{11} & c_{66} & c_{64} \end{vmatrix} = 0\,.$$

Let us introduce the vectors

$$z = (c_{21}, c_{31}, c_{11}), z_4 = (c_{44}, c_{54}, c_{64}), z_5 = (c_{45}, c_{55}, c_{65}), z_6 = (c_{46}, c_{56}, c_{66})\,.$$

It is obvious that the vectors z_4, z_5, z_6 are linearly independent. The last two determinant identities show that z is a linear combination of z_5 and z_6 as well as a linear combination of z_6 and z_4. This implies that z is a multiple of z_6, i.e. $z = \tau z_6$. From the first determinant identity we get then

$$\tau \begin{vmatrix} c_{46} & c_{44} & c_{45} \\ c_{56} & c_{54} & c_{55} \\ c_{66} & c_{64} & c_{65} \end{vmatrix} = 1\,.$$

We denote

$$\delta = \begin{vmatrix} c_{44} & c_{45} & c_{46} \\ c_{54} & c_{55} & c_{56} \\ c_{64} & c_{65} & c_{66} \end{vmatrix} .$$

From the identity $z = \tau z_6$ we get

$$c_{11} = c_{66} \cdot \delta^{-1}, \quad c_{21} = c_{46} \cdot \delta^{-1}, \quad c_{31} = c_{56} \cdot \delta^{-1} .$$

Comparing coefficients at the monomials $\beta_2 \wedge \beta_4 \wedge \beta_5$, $\beta_2 \wedge \beta_5 \wedge \beta_6$, and $\beta_2 \wedge \beta_6 \wedge \beta_4$ we obtain along the same lines as above

$$c_{12} = c_{64} \cdot \delta^{-1}, \quad c_{22} = c_{44} \cdot \delta^{-1}, \quad c_{32} = c_{54} \cdot \delta^{-1} .$$

Further, comparing coefficients at the monomials $\beta_3 \wedge \beta_4 \wedge \beta_5$, $\beta_3 \wedge \beta_5 \wedge \beta_6$, and $\beta_3 \wedge \beta_6 \wedge \beta_4$ we have

$$c_{13} = c_{65} \cdot \delta^{-1}, \quad c_{23} = c_{45} \cdot \delta^{-1}, \quad c_{33} = c_{55} \cdot \delta^{-1} .$$

It remains to compare coefficients at $\beta_4 \wedge \beta_5 \wedge \beta_6$. Here we obtain the identity

$$(*) \qquad \begin{vmatrix} c_{14} & c_{15} & c_{16} \\ c_{44} & c_{45} & c_{46} \\ c_{54} & c_{55} & c_{56} \end{vmatrix} + \begin{vmatrix} c_{24} & c_{25} & c_{26} \\ c_{54} & c_{55} & c_{56} \\ c_{64} & c_{65} & c_{66} \end{vmatrix} + \begin{vmatrix} c_{34} & c_{35} & c_{36} \\ c_{64} & c_{65} & c_{66} \\ c_{44} & c_{45} & c_{46} \end{vmatrix} = 0.$$

We have thus proved the following

5. **Lemma.** *If $\beta'_1, \dots, \beta'_6$ and $\beta_1, \dots, \beta_6$ are canonical dual bases, then their transition matrix has the form*

$$\begin{pmatrix} c_{66} \cdot \delta^{-1} & c_{64} \cdot \delta^{-1} & c_{65} \cdot \delta^{-1} & c_{14} & c_{15} & c_{16} \\ c_{46} \cdot \delta^{-1} & c_{44} \cdot \delta^{-1} & c_{45} \cdot \delta^{-1} & c_{24} & c_{25} & c_{26} \\ c_{56} \cdot \delta^{-1} & c_{54} \cdot \delta^{-1} & c_{55} \cdot \delta^{-1} & c_{34} & c_{35} & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & c_{46} \\ 0 & 0 & 0 & c_{54} & c_{55} & c_{56} \\ 0 & 0 & 0 & c_{64} & c_{65} & c_{66} \end{pmatrix}$$

satisfying $(*)$. If $\beta_1, \dots, \beta_6$ is a canonical dual basis and $\beta'_1, \dots, \beta'_6$ is a basis of V^* such that the transition matrix between both bases has the above form and satisfies $(*)$, then $\beta'_1, \dots, \beta'_6$ is also a canonical dual basis.

2. Geometric properties

Now we start to consider a 6-dimensional differentiable manifold M. From now on all structures will be differentiable, i.e. of class C^∞. A 3-form ω on M will be called a *form of class R_0* if for every $x \in M$ there is an isomorphism $h_x \colon T_x M \to V$ such that $h_x^* \rho_0 = \omega_x$. (Quite analogical definitions can be introduced for other types of forms.) We consider now on M a 3-form of type R_0. We get easily on M a 3-dimensional distribution D defined by $D_x = \Delta(\omega_x)$. But here we need the following lemma.

6. **Lemma.** *The distribution D is differentiable.*

Proof. Around any point $x \in M$ we can find a local basis $X_1, \dots, X_6$ of TM. We take a vector field $X = f_1 X_1 + \dots + f_6 X_6$, where $f_1, \dots, f_6$ are (locally defined) differentiable functions. To find differentiable vector fields Y_1, Y_2, Y_3 which span the distribution D it is necessary to solve the equation $(\iota_X \omega) \wedge \omega = 0$. This leads to a system of six linear homogeneous equations the coefficients of which are differentiable functions. The rest of the proof is then completely standard. $\square$

7. **Definition.** A local basis $X_1, \dots, X_6$ of TM around a point $x \in M$ is called *local canonical basis for* ω if the following conditions are satisfied

$$\begin{aligned}
&\alpha(X_1, X_2, X_3) = 0\,, \quad \alpha(X_i, X_j, X_k) = 0 \quad \text{for} \quad 1 \le i < j \le 3,\ k = 4, 5, 6\,,\\
&\alpha(X_1, X_4, X_5) = 1\,, \quad \alpha(X_1, X_5, X_6) = 0\,, \quad \alpha(X_1, X_6, X_4) = 0\,,\\
&\alpha(X_2, X_4, X_5) = 0\,, \quad \alpha(X_2, X_5, X_6) = 1\,, \quad \alpha(X_2, X_6, X_4) = 0\,,\\
&\alpha(X_3, X_4, X_5) = 0\,, \quad \alpha(X_3, X_5, X_6) = 0\,, \quad \alpha(X_3, X_6, X_4) = 1\,,\\
&\alpha(X_4, X_5, X_6) = 0\,.
\end{aligned}$$

8. **Proposition.** *Around every point $x \in M$ there exists a canonical basis for the 3-form ω.*

Proof. We choose first a complement D_c of D in TM. This complement is also a differentiable distribution. In this complement we take locally three linearly independent vector fields Y_4, Y_5, Y_6. We denote $f = \omega(Y_4, Y_5, Y_6)$. Because the form ω_x is regular, there is $v_0 \in D_x$ such that $\omega_x(v_0, Y_{5,x}, Y_{6,x}) = b \neq 0$. Then we take a vector field Y_0 around x lying in D such that $X_{0,x} = v_0$. Obviously, then $\omega(Y_0, Y_5, Y_6) = g$ is non-zero in a neighborhood of x. Taking $X_4 = Y_4 - (f/g)Y_0$, $X_5 = Y_5$, and $X_6 = Y_6$ we get

$$\begin{aligned}
\omega(X_4, X_5, X_6) &= \alpha(Y_4 - (f/g)Y_0, Y_5, Y_6)\\
&= \omega(Y_4, Y_5, Y_6) - (f/g)\omega(Y_0, Y_5, Y_6) = f - (f/g)g = 0\,.
\end{aligned}$$

Now we have in a neighborhood of $x \in M$ three 1-forms, namely the forms $\omega(\cdot, X_4, X_5)$, $\omega(\cdot, X_5, X_6)$, and $\omega(\cdot, X_6, X_4)$. The regularity of ω_x implies again that these three forms are linearly independent. Consequently, there are uniquely determined vector fields X_1, X_2, X_3 in D such that

$$\begin{aligned}
&\omega(X_1, X_4, X_5) = 1\,, \quad \omega(X_1, X_5, X_6) = 0\,, \quad \omega(X_1, X_6, X_4) = 0\,,\\
&\omega(X_2, X_4, X_5) = 0\,, \quad \omega(X_2, X_5, X_6) = 1\,, \quad \omega(X_2, X_6, X_4) = 0\,,\\
&\omega(X_3, X_4, X_5) = 0\,, \quad \omega(X_3, X_5, X_6) = 0\,, \quad \omega(X_3, X_6, X_4) = 1\,.
\end{aligned}$$

The equations $\omega(X_1, X_2, X_3) = 0$ and $\omega(X_i, X_j, X_k) = 0$ for $1 \le i < j \le 3$, $k = 4, 5, 6$ are again satisfied automatically by virtue of Lemma 2. This finihes the proof. $\square$

Now it suffices to take dual 1-forms $\omega_1, \dots, \omega_6$ to the vector fields $X_1, \dots, X_6$ and we get the following proposition.

9. **Proposition.** *For a 3-form ω of type R_0 on M locally there exist 1-forms $\omega_1, \dots, \omega_6$ such that*

$$\omega = \omega_1 \wedge \omega_4 \wedge \omega_5 + \omega_2 \wedge \omega_5 \wedge \omega_6 + \omega_3 \wedge \omega_6 \wedge \omega_4 \,.$$

10. **Example.** On $\mathbb{R}^6$ let us consider the 3-form

$$\omega = dx_1 \wedge (dx_4 + x_1 dx_3) \wedge dx_5 + dx_2 \wedge dx_5 \wedge dx_6 + dx_3 \wedge dx_6 \wedge (dx_4 + x_1 dx_3)\,.$$

We have

$$d\omega = dx_1 \wedge dx_1 \wedge dx_3 \wedge dx_5 + dx_3 \wedge dx_6 \wedge dx_1 \wedge dx_3 = 0\,.$$

On the other hand the distribution $D = \Delta(\omega)$ is spanned by the vector fields

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4}$$

and is not integrable. This shows that the closeness of the 3-form ω does not imply the integrability of the associated distribution $\Delta(\omega)$.

We shall need a version of the Poincaré lemma. On $\mathbb{R}^6$ we take coordinates $(x_1, \dots, x_6)$ and consider an integrable 3-dimensional distribution D defined by the equations $dx_4 = dx_5 = dx_6 = 0$.

11. **Lemma.** *Let θ be a 2-form on $\mathbb{R}^6$ such that $d\theta = 0$ and $\theta | D = 0$. Then there exists a 1-form η on $\mathbb{R}^6$ such that $\theta = d\eta$ and $\eta \mid D = 0$.*

Proof. We denote Ω^k the vector space of k-forms on $\mathbb{R}^6$ and $Z(\Omega^k)$ the subspace consisting of closed forms. It is well known that there exists a linear mapping $E : Z(\Omega^2) \to \Omega^1$ such that for every $\xi \in Z(\Omega^2)$ there is $\xi = dE(\xi)$. The problem is that $E(\theta)$ need not satisfy $E(\theta)|D = 0$. But we have

$$dE(\theta)|D = \theta|D = 0\,.$$

On any leaf $L(c_4, c_5, c_6)$ of the distribution D (i.e. $x_4 = c_4$, $x_5 = c_5$, $x_6 = c_6$) we can again apply the Poincaré lemma and we find that there exists on $L(c_4, c_5, c_6)$ a function $f_{(c_4,c_5,c_6)}$ such that $E(\theta)|L(c_4, c_5, c_6) = df_{(c_4,c_5,c_6)}$. Of course, this does not solve our problem. But we can use an obvious parametric version of the Poincaré lemma. We can consider $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3). On $\mathbb{R}^3$ we take a family of 1-forms ζ_{c_4,c_5,c_6} depending on three parameters c_4, c_5, c_6. Namely, the 1-form ζ_{c_4,c_5,c_6} with parameters c_4, c_5, c_6 is the form $E(\theta) \mid L(c_4, c_5, c_6)$ transferred to $\mathbb{R}^3$ under the natural identification $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, c_4, c_5, c_6)$. Now the Poincaré lemma with three parameters gives us a three parametric system of functions f_{c_4,c_5,c_6} on $\mathbb{R}^3$ such that $\zeta_{c_4,c_5,c_6} = df_{c_4,c_5,c_6}$. In other words this means that the function $f(x_1, x_2, x_3, x_4, x_5, x_6) = f_{x_4,x_5,x_6}(x_1, x_2, x_3)$ satisfies

$$E(\theta)|D = df|D\,.$$

Taking now $\eta = E(\theta) - df$ we can see that $d\eta = \theta$ and $\eta \mid D = 0$. $\square$

Let us recall now the following definition.

12. **Definition.** A 3-form ω of type R_0 on a manifold M is called *integrable* if locally there exist coordinates $x_1, \dots, x_6$ such that

$$\omega = dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_5 \wedge dx_6 + dx_3 \wedge dx_6 \wedge dx_4\,.$$

It is obvious that if the 3-form ω is integrable then ω is closed and the associated distribution $\Delta(\omega)$ is integrable. Now we are going to prove that these two conditions are also sufficient for the integrability.

13. **Theorem.** *A 3-form ω of type R_0 on a manifold M is integrable if and only if the following two conditions are satisfied*

(1) $d\omega = 0$,

(2) *the distribution $D = \Delta(\omega)$ is integrable.*

Proof. We must show that the conditions are sufficient. Acording to Proposition 9 around every point $x \in M$ we can find 1-forms $\omega_1'', \dots, \omega_6''$ such that

$$\omega = \omega_1'' \wedge \omega_4'' \wedge \omega_5'' + \omega_2'' \wedge \omega_5'' \wedge \omega_6'' + \omega_3'' \wedge \omega_6'' \wedge \omega_4'' .$$

Because $\Delta(\omega)$ is integrable, we can find three functions f_4', f_5', f_6' such that their differentials df_4', df_5', df_6' are linearly independent and $df_4' \mid D = df_5' \mid D = df_6' \mid D = 0$. Then using Lemma 5 we can find 1-forms ω_1', ω_2', ω_3' such that

$$\omega = \omega_1' \wedge df_4' \wedge df_5' + \omega_2' \wedge df_5' \wedge df_6' + \omega_3' \wedge df_6' \wedge df_4' .$$

We denote $X_1', \dots, X_6'$ the canonical basis associated to the canonical dual basis $\omega_1', \omega_2', \omega_3', df_4', df_5', df_6'$. Obviously, we have

(d) $$0 = d\omega = d\omega_1' \wedge df_4' \wedge df_5' + d\omega_2' \wedge df_5' \wedge df_6' + d\omega_3' \wedge df_6' \wedge df_4' .$$

Applying $\iota_{X_2'}\iota_{X_1'}$ on both sides we get

$$0 = (\iota_{X_2'}\iota_{X_1'} d\omega_1') \cdot df_4' \wedge df_5' + (\iota_{X_2'}\iota_{X_1'} d\omega_2') \cdot df_5' \wedge df_6' + (\iota_{X_2'}\iota_{X_1'} d\omega_3') \cdot df_6' \wedge df_4' .$$

This shows that $d\omega_1'(X_1', X_2') = d\omega_2'(X_1', X_2') = d\omega_1'(X_1', X_2') = 0$. Similarly we find that $d\omega_1'(X_2, X_3) = d\omega_2'(X_2, X_3) = d\omega_3'(X_2, X_3) = 0$ and $d\omega_1'(X_3, X_1) = d\omega_2'(X_3, X_1) = d\omega_3'(X_3, X_1) = 0$. We have thus proved that

$$d\omega_1' \mid D = d\omega_2' \mid D = d\omega_3' \mid D = 0 .$$

Consequently $d\omega_1'$ must have the following form

$$\begin{aligned} d\omega_1' = {} & g_{114}\omega_1' \wedge df_4' + g_{115}\omega_1' \wedge df_5' + g_{116}\omega_1' \wedge df_6' \\ & + g_{124}\omega_2' \wedge df_4' + g_{125}\omega_2' \wedge df_5' + g_{126}\omega_2' \wedge df_6' \\ & + g_{134}\omega_3' \wedge df_4' + g_{135}\omega_3' \wedge df_5' + g_{136}\omega_3' \wedge df_6' \\ & + g_{145} df_4' \wedge df_5' + g_{156} df_5' \wedge df_6' + g_{164} df_6' \wedge df_4' . \end{aligned}$$

Similar formulas we can write for $d\omega_2'$ and $d\omega_3'$. Now taking into account the equation (d) we find the following identities.

$$\begin{aligned} g_{116} + g_{214} + g_{315} &= 0 , \\ g_{126} + g_{224} + g_{325} &= 0 , \\ g_{136} + g_{234} + g_{335} &= 0 . \end{aligned}$$

Let us consider now the 2-form $d\omega_1'$. This form is closed because $dd\omega_1' = 0$ and $d\omega_1'|D = 0$. According to Lemma 11 there exists a 1-form θ_1 such that $\theta_1|D = 0$ and $d\theta_1 = d\omega_1'$. Again similar considerations are possible with the 2-forms $d\omega_2'$ and

$d\omega_3'$. In this way we obtain three 1-forms θ_1, θ_2, and θ_3, which can be expressed in the form

$$\begin{aligned}
\theta_1 &= h_{14}df_4' + h_{15}df_5' + h_{16}df_6'\,,\\
\theta_2 &= h_{24}df_4' + h_{25}df_5' + h_{26}df_6'\,,\\
\theta_3 &= h_{34}df_4' + h_{35}df_5' + h_{36}df_6'\,.
\end{aligned}$$

The 1-forms $\omega_1' - \theta_1$, $\omega_2' - \theta_2$, and $\omega_3' - \theta_3$ are closed and consequently we can find functions f_1', f_2', f_3' such that $\omega_1' - \theta_1 = df_1'$, $\omega_2' - \theta_2 = df_2'$, and $\omega_3' - \theta_3 = df_3'$. Now it is obvious that the functions $f_1' \dots, f_6'$ represent a local coordinate system. The local dual basis df_1', df_2', df_3', df_4', df_5', df_6' is a relatively good basis, but unfortunately it need not be a canonical basis. The transition matrix from the canonical basis ω_1', ω_2', ω_3', df_4', df_5', df_6' to the last basis is

$$\begin{pmatrix}
1 & 0 & 0 & -h_{14} & -h_{15} & -h_{16}\\
0 & 1 & 0 & -h_{24} & -h_{25} & -h_{26}\\
0 & 0 & 1 & -h_{34} & -h_{35} & -h_{36}\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and it may happen that $h_{16} + h_{24} + h_{35} \neq 0$.

Considering the equations $d\omega_1' = d\theta_1$, $d\omega_2' = d\theta_2$, and $d\omega_3' = d\theta_3$ we get the identities

$$\begin{aligned}
X_1'h_{16} &= g_{116}\,, \quad & X_2'h_{16} &= g_{126}\,, \quad & X_3'h_{16} &= g_{136}\,,\\
X_1'h_{24} &= g_{214}\,, \quad & X_2'h_{24} &= g_{224}\,, \quad & X_3'h_{24} &= g_{234}\,,\\
X_1'h_{35} &= g_{315}\,, \quad & X_2'h_{35} &= g_{325}\,, \quad & X_3'h_{35} &= g_{335}\,.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
X_1(h_{16} + h_{24} + h_{35}) &= g_{116} + g_{214} + g_{315} = 0\,,\\
X_2(h_{16} + h_{24} + h_{35}) &= g_{126} + g_{224} + g_{325} = 0\,,\\
X_3(h_{16} + h_{24} + h_{35}) &= g_{136} + g_{234} + g_{335} = 0\,.
\end{aligned}$$

We can see that the function $h = h_{16} + h_{24} + h_{35}$ is constant on the leaves of the foliation associated with the distribution D. In our coordinate system $f_1', \dots, f_6'$ this means that h is a function of variables f_4', f_5', f_6' only. We can choose a function l of variables f_4', f_5', f_6' only such that $\partial l/\partial f_6' = h$. Now we take a dual basis in the form

$$df_1' + dl, df_2', df_3', df_4', df_5', df_6'\,.$$

The transition matrix of this basis with respect to the basis ω'_1, ω'_2, ω'_3, df'_4, df'_5, df'_6 is

$$\begin{pmatrix} 1 & 0 & 0 & -h_{14}+\partial l/\partial f'_4 & -h_{15}+\partial l/\partial f'_5 & -h_{16}+h \\ 0 & 1 & 0 & -h_{24} & -h_{25} & -h_{26} \\ 0 & 0 & 1 & -h_{34} & -h_{35} & -h_{36} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and obviously satisfies the condition (*). This implies that the dual basis df'_1+dl, df'_2, df'_3, df'_4, df'_5, df'_6 is canonical. Now it suffices to set $f_1=f'_1+l$, $f_2=f'_2$, $f_3=f'_3$, $f_4=f'_4$, $f_5=f'_5$, $f_6=f'_6$ and we have

$$\omega = df_1\wedge df_4\wedge df_5 + df_2\wedge df_5\wedge df_6 + df_3\wedge df_6\wedge df_4\,.$$

□

Let us assume now that there exists on M a symmetric connection ∇ such that $\nabla\omega=0$. Then using [2, Cor. 8.6], we find that $d\omega=\mathrm{Alt}(\nabla\omega)=0$. Next for arbitrary vector fields X, X_1, X_2, Y we can calculate

$$\begin{aligned}\big(\nabla_Y(\iota_X\omega)(X_1,X_2)\big) &= Y\big((\iota_X\omega)(X_1,X_2)\big)-(\iota_X\omega)(\nabla_Y X_1,X_2)-(\iota_X\omega)(X_1,\nabla_Y X_2)\\ &= Y(\omega(X,X_1,X_2))-\omega(X,\nabla_Y X_1,X_2)-\omega(X,X_1,\nabla_Y X_2)\\ &= (\nabla_Y\omega)(X,X_1,X_2)+\omega(\nabla_Y X,X_1,X_2)+\omega(X,\nabla_Y X_1,X_2)\\ &\quad +\omega(X,X_1,\nabla_Y X_2)-\omega(X,\nabla_Y X_1,X_2)-\omega(X,X_1,\nabla_Y X_2)\\ &= \omega(\nabla_Y X,X_1,X_2)=(\iota_{\nabla_Y X}\omega)(X_1,X_2)\,.\end{aligned}$$

Now let us assume that a vector field X lies in the distribution D. We have then $(\iota_X\omega)\wedge\omega=0$ and consequently

$$0=\nabla_Y((\iota_X\omega)\wedge\omega)=(\nabla_Y(\iota_X\omega))\wedge\omega=(\iota_{\nabla_Y X}\omega)\wedge\omega\,,$$

which show that ∇ preserves the distribution D. Because the connection ∇ is symmetric, this implies that the distribution D is integrable. Together this means that the 3-form ω is integrable. We will see that the converse is also true.

14. **Theorem.** *A 3-form ω of type R_0 on a paracompact manifold M is integrable if and only if there exists on M a symmetric connection ∇ such that $\nabla\omega=0$.*

Proof. We must prove that if ω is integrable then there exists a symmetric connection ∇ such that $\nabla\omega=0$. We can cover M by a locally finite open covering of M consisting of charts $\{U^\lambda\}_{\lambda\in I}$ with coordinates $x_1^\lambda,\dots,x_6^\lambda$ such that on U^λ we have

$$\omega = dx_1^\lambda\wedge dx_4^\lambda\wedge dx_5^\lambda + dx_2^\lambda\wedge dx_5^\lambda\wedge dx_6^\lambda + dx_3^\lambda\wedge dx_6^\lambda\wedge dx_4^\lambda\,.$$

On each U^λ we take a connection ∇^λ defined by

$$\nabla^\lambda_{\partial/\partial x_i^\lambda}(\partial/\partial x_j^\lambda)=0\,,\quad i,j=1,\dots,6\,.$$

It is obvious that this connection is symmetric and satisfies $\nabla^\lambda \omega = 0$. Now it remains to glue these connections together. We take a partition of unity $\{a^\lambda\}_{\lambda \in I}$ subordinate to the covering $\{U^\lambda\}_{\lambda \in I}$. Then it suffices to define

$$\nabla = \sum_{\lambda \in I} a^\lambda \nabla^\lambda \,,$$

and we have on M a symmetric connection satisfying $\nabla \omega = 0$. $\square$

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