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# A NOTE ON THE NONEXISTENCE OF SPACELIKE HYPERSURFACES WITH POLYNOMIAL VOLUME GROWTH IMMERSED IN A LORENTZIAN SPACE FORM

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Abstract. We obtain nonexistence results concerning complete noncompact spacelike hypersurfaces with polynomial volume growth immersed in a Lorentzian space form, under the assumption that the support functions with respect to a fixed nonzero vector are linearly related. Our approach is based on a suitable maximum principle recently established by Alías, Caminha and do Nascimento [3].

## 1. Introduction

The study of the geometry of hypersurface immersed in a space form constitutes a classical but still fruitful thematic into the scope of Differential Geometry. In this branch, Alías, Brasil and Perdomo [2] dealt with complete hypersurfaces immersed in the unit Euclidean sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, whose support functions with respect to a fixed nonzero vector of the Euclidean space $\mathbb{R}^{n+2}$ are linearly related. In this setting, they showed that such a hypersurface having constant mean curvature must be either totally umbilical or isometric to a Clifford torus.

Later on, working with a different approach of that used in [2], the author jointly with Aquino [5, 6] characterized the totally umbilical hypersurfaces and the hyperbolic cylinders of the hyperbolic space $\mathbb{H}^{n+1}$ as the only complete hypersurfaces with constant mean curvature and whose support functions with respect to a fixed nonzero vector of the Lorentz-Minkowski space $\mathbb{L}^{n+2}$ are linearly related. When the ambient space is a Lorentzian space form, the author jointly with Aquino and dos Santos [8] extended the techniques developed in [5, 6] characterizing constant mean curvature spacelike hypersurfaces, whose support functions are linearly related.

Motivated by these works, here we obtain nonexistence results concerning complete noncompact spacelike hypersurfaces with polynomial volume growth immersed in a Lorentzian space form, under the assumption that the support functions with respect to a fixed nonzero vector are linearly related. Our approach

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is based on a suitable maximum principle recently established by Alías, Caminha and do Nascimento [3] (see Lemma 1).

It is also worth to point out that the study of the existence of spacelike hypersurfaces in a Lorentz space is a key requirement in the original formulation of the singularity theorems proved by Penrose [15] and Hawking [12], as well as in their more recent generalizations (for a throughout discussion about this topic, we recommend seeing [11, 16]).

This manuscript is organized in the following way: In Section 2 we recall some basic facts related to spacelike hypersurfaces of a Lorentzian space form. Afterwards, in Section 3 we establish our nonexistence results concerning these hypersurfaces under the assumptions that they have polynomial volume growth, that their support functions with respect to a fixed nonzero vector are linearly related and with suitable constraints on their mean curvature functions (see Theorems 1, 2 and 3).

## 2. Preliminaries

For an integer $\nu$ with $\nu \in \{1,2\}$, let $\mathbb{R}^{n+2}_\nu$ be denoting the $(n+2)$-dimensional semi-Euclidean space endowed with the following metric of index $\nu$

$$\langle u, v\rangle = -\sum_{i=1}^{\nu} u_i v_i + \sum_{i=\nu+1}^{n+2} u_i v_i\,,$$

for all $u,\ v \in \mathbb{R}^{n+2}$. In particular, when $\nu = 1$, $\mathbb{R}^{n+2}_1 = \mathbb{L}^{n+2}$ is the so-called Lorentz-Minkowski space $\mathbb{L}^{n+2}$.

The de Sitter space $\mathbb{S}^{n+1}_1$ is the hyperquadric of $\mathbb{L}^{n+2}$ defined in the following way

$$\mathbb{S}^{n+1}_1 = \{x \in \mathbb{L}^{n+2}\,;\, \langle x, x\rangle = 1\}\,.$$

Endowed with the induced metric from $\mathbb{L}^{n+2}$, $n \geq 2$, the de Sitter space is complete simply connected $(n+1)$-dimensional Lorentzian manifold with constant sectional curvature one.

On the other hand, when $\nu = 2$, we define the anti-de Sitter space $\mathbb{H}^{n+1}_1$ as being the hyperquadric of $\mathbb{R}^{n+2}_2$ given by

$$\mathbb{H}^{n+1}_1 = \{x \in \mathbb{R}^{n+2}_2\,;\, \langle x, x\rangle = -1\}\,.$$

Topologically, $\mathbb{H}^{n+1}_1$ is equivalent to the product $\mathbb{S}^1 \times \mathbb{R}^n$ and the semi-Euclidean metric on $\mathbb{R}^{n+2}_2$ induces a Lorentzian metric of constant sectional curvature $-1$ on $\mathbb{H}^{n+1}_1$. Moreover, the universal covering manifold $\widetilde{\mathbb{H}}^{n+1}_1$ of $\mathbb{H}^{n+1}_1$ is topologically $\mathbb{R}^{n+1}$ (that is, $\widetilde{\mathbb{H}}^{n+1}_1$ is simply connected) and is thus a Lorentzian analogue of the usual Riemannian hyperbolic space $\mathbb{H}^{n+1}$ of negative curvature $-1$, which is called the *universal anti-de Sitter spacetime* (see, for instance, Section 5.3 of [10] or Section 8.6 of [14]).

From now on, we consider $L^{n+1}_1(c)$, where $c \in \{-1,0,1\}$, denoting the Lorentz--Minkowski space if $c = 0$, the de Sitter space if $c = 1$ and the anti-de Sitter space if $c = -1$. A smooth immersion $x\colon \Sigma^n \to L^{n+1}_1(c)$ of an $n$-dimensional connected manifold $\Sigma^n$ is said a *spacelike hypersurface* if the induced metric via $x$ is a Riemannian metric on $\Sigma^n$, which is also denoted by $\langle\ ,\rangle$. Since $L^{n+1}_1(c)$ is time-oriented, we can choose an unique unit normal vector field $N$ on $\Sigma^n$ which is a future-directed timelike vector field in $L_1^{n+1}(c)$ and, hence, we may assume that $\Sigma^n$ is oriented by $N$. In this setting, we will denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connections of $L_1^{n+1}(c)$ and $\Sigma^n$, respectively.

Fixed a nonzero vector $a \in \mathbb{R}_\nu^{n+2}$, let us consider the support functions $l_a : \Sigma^n \to \mathbb{R}$, given by $l_a(p) = \langle x(p), a\rangle$, and $f_a : \Sigma^n \to \mathbb{R}$, given by $f_a(p) = \langle N(p), a\rangle$. Then, we can write

$$a = a^\top - f_a N + c l_a x \,,$$

and a direct calculation allows us to conclude that

$$\nabla l_a = a^\top \quad \text{and} \quad \nabla f_a = -A(a^\top)\,.$$

So, for all $X \in \mathfrak{X}(\Sigma)$,

$$\nabla_X \nabla l_a = \nabla_X a^\top = -f_a AX - c l_a X\,. \tag{2.1}$$

Taking the trace in (2.1), we get

$$\Delta l_a = nHf_a - cnl_a. \tag{2.2}$$

To close this section, we recall the standard description of totally umbilical hypersurfaces of $L_1^{n+1}(c)$. Let $a \in \mathbb{R}_\nu^{n+2}$ be a fixed nonzero vector with $\langle a, a\rangle \in \{-1, 0, 1\}$, and take the smooth function $g : L_1^{n+1}(c) \to \mathbb{R}$ defined by $g(x) = \langle a, x\rangle$. It is not difficult to see that, for every $\tau \in \mathbb{R}$ with $\langle a, a\rangle - c\tau^2 \neq 0$, the set

$$L_\tau = g^{-1}(\tau) = \{x \in L_1^{n+1}(c)\,;\, \langle a, x\rangle = \tau\}$$

is a totally umbilical hypersurface in $L_1^{n+1}(c)$, with Gauss map given by

$$N_\tau(p) = \frac{1}{\sqrt{|\langle a, a\rangle - c\tau^2|}}(a - c\tau x)\,.$$

With respect to this orientation, the shape operator and mean curvature are, respectively,

$$Av = \frac{c\tau}{\sqrt{|\langle a, a\rangle - c\tau^2|}}v\,, \quad \text{and} \quad H^2 = \frac{\tau^2}{|\langle a, a\rangle - c\tau^2|}\,.$$

Hence, with a straightforward computation we get

$$l_a = \frac{|\tau|}{\sqrt{|\langle a, a\rangle - c\tau^2|}} f_a = |H| f_a\,. \tag{2.3}$$

## 3. Main results

For our purpose, we will need to quote a suitable maximum principle that will be used to prove our nonexistence results. For this, let $\Sigma^n$ be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p, t)$ the geodesic ball centered at $p$ and with radius $t$. Given a polynomial function $\sigma : (0, +\infty) \to (0, +\infty)$, we say that $\Sigma^n$ has *polynomial volume growth* like $\sigma(t)$ if there exists $p \in \Sigma^n$ such that

$$\mathrm{vol}(B(p, t)) = \mathcal{O}(\sigma(t))\,,$$

as $t \to +\infty$, where vol denotes the standard Riemannian volume. As it was already observed in the beginning of Section 2 in [3], if $p, q \in \Sigma^n$ are at distance $d$ from each other, we can verify that

$$\frac{\mathrm{vol}(B(p,t))}{\sigma(t)} \geq \frac{\mathrm{vol}(B(q,t-d))}{\sigma(t-d)} . \frac{\sigma(t-d)}{\sigma(t)} .$$

So, the choice of $p$ in the notion of volume growth is immaterial. For this reason, we will just say that $\Sigma^n$ has polynomial volume growth.

Keeping in mind this previous digression, we quote the following key lemma which corresponds to a particular case of a new maximum principle due to Alías, Caminha and do Nascimento (see Theorem 2.1 of [3]).

**Lemma 1.** *Let $\Sigma^n$ be a connected, oriented, complete noncompact Riemannian manifold, and let $u \in \mathcal{C}^\infty(\Sigma)$ be a nonnegative smooth function such that $\Delta u \geq au$ on $\Sigma^n$, for some positive constant $a \in \mathbb{R}$. If $\Sigma^n$ has polynomial volume growth and $|\nabla u|$ is bounded on $\Sigma^n$, then $u$ vanishes identically on $\Sigma^n$.*

Returning to the context of the previous section, motivated by (2.3) we will deal with complete spacelike hypersurfaces immersed in $L_1^{n+1}(c)$ whose support functions $l_a$ and $f_a$ are linearly related, that is,

$$l_a = \lambda f_a \, , \tag{3.1}$$

for some smooth function $\lambda\colon \Sigma^n \to \mathbb{R}$. For this, we will extend a nomenclature of Section 4 in [13] and consider the following open sets

$$\mathcal{A} = \{p \in \mathbb{H}_1^{n+1} \, ; \, \langle p, a\rangle \neq 0\} \subset \mathbb{H}_1^{n+1}$$

and

$$\mathcal{O} = \{p \in \mathbb{S}_1^{n+1} \, ; \, \langle p, a\rangle \neq 0\} \subset \mathbb{S}_1^{n+1} \, .$$

In this setting, we present our first nonexistence result.

**Theorem 1.** *There does not exist a complete spacelike hypersurface $x\colon \Sigma^n \to \mathcal{A} \subset \mathbb{H}_1^{n+1}$ with polynomial volume growth, satisfying* (3.1) *for some a nonzero timelike vector $a \in \mathbb{R}_2^{n+2}$, lying between two totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$ determined by $a$, with $|a^\top|$ bounded on $\Sigma^n$ and such that its mean curvature $H$ has the same sign of the function $\lambda$ at each point of $\Sigma^n$.*

**Proof.** Suppose by contradiction the existence of such a spacelike hypersurface $x\colon \Sigma^n \to \mathcal{A} \subset \mathbb{H}_1^{n+1}$. Since $x(\Sigma) \subset \mathcal{A}$, we have that the function $l_a$ has strict sign and, consequently, from relation (3.1) we get that $\lambda$ also has strict sign on $\Sigma^n$. Moreover, as we are supposing that $\Sigma^n$ lies between two totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$ determined by $a$, there exists $\rho \in \mathbb{R}$ such that $|l_a| \leq \rho$.

Taking $c = -1$ in (2.2), we obtain

$$\Delta l_a = nHf_a + nl_a \, . \tag{3.2}$$

Computing the Laplacian of function $l_a^2$, using (3.1) and (3.2), we have

$$\frac{1}{2}\Delta l_a^2 = l_a \Delta l_a + |\nabla l_a|^2 = nHf_a l_a + nl_a^2 + |\nabla l_a|^2 = n\left(\frac{H}{\lambda} + 1\right) l_a^2 + |\nabla l_a|^2 \, . \tag{3.3}$$

Thus, since $H$ has the same sign of the function $\lambda$, from (3.3) we conclude that

$$\Delta l_a^2 \geq 2n l_a^2 \,. \tag{3.4}$$

Moreover, we have that

$$|\nabla l_a^2| = 2|l_a||\nabla l_a| \leq 2\rho |a^\top| \,. \tag{3.5}$$

Thus, since we are assuming $|a^\top|$ bounded on $\Sigma^n$, from (3.5) we get that $|\nabla l_a^2|$ must be also bounded on $\Sigma^n$.

On the other hand, as it was pointed out on the beginning of Section 3 of [9], there is no compact (without boundary) spacelike hypersurfaces immersed in $\mathbb{H}_1^{n+1}$ (see also Section 4 in [1]). So, $x\colon \Sigma^n \to \mathcal{A} \subset \mathbb{H}_1^{n+1}$ must be complete noncompact.

Therefore, we are in position to apply Lemma 1 and conclude that $l_a$ vanishes identically on $\Sigma^n$, leading us to a contradiction to the fact that $l_a$ has strict sign on $\Sigma^n$. □

According to Section 4 of Montiel [13], when $a \in \mathbb{L}^{n+2}$ is a timelike vector, the Lorentz-Minkowski space $\mathbb{L}^{n+2}$ can be foliated by means of parallel spacelike hyperplanes orthogonal to $a$. In this setting, we establish our second nonexistence result.

**Theorem 2.** *There does not exist a complete spacelike hypersurface* $x\colon \Sigma^n \to \mathbb{R}_1^{n+1} \setminus \{0\}$ *with polynomial volume growth, satisfying* (3.1) *for some nonzero timelike vector* $a \in \mathbb{R}_1^{n+2}$*, lying between two spacelike hyperplanes orthogonal to* $a$*, with* $|a^\top|$ *bounded on* $\Sigma^n$ *and such that its mean curvature* $H$ *and the function* $\lambda$ *satisfy* $\frac{H}{\lambda} \geq \beta$ *for some constant* $\beta > 0$*.*

**Proof.** Supposing by contradiction the existence of such a spacelike hypersurface, since $x\colon \Sigma^n \to \mathbb{R}_1^{n+1}$ lies between two spacelike hyperplanes orthogonal to $a$, the function $l_a$ is bounded. Consequently, from our assumption that $|a^\top|$ is bounded on $\Sigma^n$, we have that the same holds for $\nabla l_a^2 = 2l_a \nabla l_a = 2l_a a^\top$.

Moreover, taking $c = 0$ in equation (2.2) and using (3.1), we get

$$\Delta l_a^2 = \frac{2nH}{\lambda} l_a^2 + 2|\nabla l_a|^2 \geq \frac{2nH}{\lambda} l_a^2 \geq 2\beta l_a^2 \,.$$

Hence, since $\Sigma^n$ cannot be compact (see the last part of Section 2 of [4]), we can apply once more Lemma 1 obtaining that $l_a$ is identically zero on $\Sigma^n$, that is, $\Sigma^n$ is a spacelike hyperplane passing through the origin of $\mathbb{R}_1^{n+1}$. Therefore, we reach at a contradiction with the hypothesis that the immersion $x$ takes values in $\mathbb{R}_1^{n+1} \setminus \{0\}$. □

Finally, when the ambient space is the de Sitter space, we obtain our last nonexistence result.

**Theorem 3.** *There does not exist a complete noncompact spacelike hypersurface* $x\colon \Sigma^n \to \mathcal{O} \subset \mathbb{S}_1^{n+1}$ *with polynomial volume growth, satisfying* (3.1) *for some nonzero vector* $a \in \mathbb{L}^{n+2}$*, lying between two totally umbilical hypersurfaces determined by* $a$*, with* $|a^\top|$ *bounded on* $\Sigma^n$ *and such that its mean curvature* $H$ *and the function* $\lambda$ *satisfy* $\frac{H}{\lambda} \geq \alpha$ *for some constant* $\alpha > 1$*.*

**Proof.** Assuming the existence of such a spacelike hypersurface $x\colon \Sigma^n \to \mathcal{O} \subset \mathbb{S}_1^{n+1}$, the condition $x(\Sigma) \subset \mathcal{O}$ guarantees us that the function $l_a$ has strict sign and, hence, from (3.1) we get once more that $\lambda$ also has strict sign on $\Sigma^n$. So, the ration $\frac{H}{\lambda}$ is, indeed, well defined.

On the other hand, taking $c = 1$ in (2.2) and using (3.1), we have

$$\Delta l_a^2 = 2n\left(\frac{H}{\lambda} - 1\right) l_a^2 + 2|\nabla l_a|^2 \geq 2n\left(\frac{H}{\lambda} - 1\right) l_a^2 \geq 2n\,(\alpha - 1)\, l_a^2 \,.$$

Therefore, applying again Lemma 1, we get that $l_a$ vanishes on $\Sigma^n$, which is a contradiction to the hypothesis that $x(\Sigma) \subset \mathcal{O}$. □

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