ARCHIVUM MATHEMATICUM (BRNO)
Tomus 58 (2022), 241–256

GENERALISED ATIYAH'S THEORY OF PRINCIPAL CONNECTIONS

Jiří Nárožný

Abstract. This is a condensed report from the ongoing project aimed on higher principal connections and their relation with higher differential cohomology theories and generalised short exact sequences of L_∞ algebroids. A historical stem for our project is a paper from sir M. Atiyah who observed a bijective correspondence between data for a horizontal distribution on a fibre bundle and a set of sections for a certain splitting short exact sequence of Lie algebroids, nowadays called the *Atiyah sequence*. In a meantime there was developed quite firm understanding of the category theory and in the last two decades also the higher category/topos theory. This conceptual framework allows us to examine principal connections and higher principal connections in a prism of differential cohomology theories. In this text we cover mostly the motivational part of the project which resides in searching for a common language of these two successful approaches to connections. From the reasons of conciseness and compactness we have not included computations and several lengthy proofs.

1. Higher connections and their presentation

In this chapter we establish a framework and motivate our further steps. The chapter is considered to be detailed and provides an almost self-consistent transition from a differential cohomology theory with the coefficient object $\flat\mathbf{B}\mathsf{G}$ to a space of morphisms so called Atiyah connections. Throughout the whole chapter we consider a cohesive $(\infty,1)$-topos $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ of $(\infty,1)$-sheaves over an 1-site of smooth finite-dimensional manifolds Mfd and its presentation given by an 1-category of simplicial presheaves $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op},\mathsf{sSet})$ together with a local projective model structure [11].

As a small reminder we start with recalling few basic facts and definitions. Our terminology mostly coincides with the one used in [9].

2020 *Mathematics Subject Classification*: primary 18N60.
Key words and phrases: higher connections, higher parallel transport, generalised Atiyah groupoid, generalised Atiyah sequence, orthogonal factorisation systems.
Received May 11, 2022, revised September 2022. Editor M. Čadek.
DOI: 10.5817/AM2022-4-241

1.1. **What is the playground.** *Warning:* There are two conceptually different objects which will be denoted very similarly. The symbol $\mathbf{\Delta}$ stands for the simplex category [5] unlike the symbol $\mathbf{\Delta}^n$ which stands for the standard simplex as a representable simplicial set, represented by $[n] \in \mathbf{\Delta}$.

Definition 1.1. A simplicial category is any category (strictly) enriched in the closed monoidal category of simplicial sets sSet. We denote it as $\mathsf{Cat}_{\mathsf{sSet}}$.

Observation: There is a natural way how to turn the closed monoidal 1-category sSet into a simplicial category by defining hom-objects to be internal hom-objects, identity as a canonical morphism $\mathrm{id}_X \colon \mathbf{\Delta}^0 \to \underline{\mathsf{hom}}_{\mathsf{sSet}}(X, X)$ for all $X \in \mathsf{sSet}$, and the composition as a morphism of simplicial sets

$$\mu \colon \underline{\mathsf{hom}}_{\mathsf{sSet}}(X, Y) \times \underline{\mathsf{hom}}_{\mathsf{sSet}}(Y, Z) \to \underline{\mathsf{hom}}_{\mathsf{sSet}}(X, Z)$$

which is defined in components by the prescription:

$$\mu_n \colon (f, g) \mapsto g \circ (f \times p_1)\,,$$

where $f \in \mathsf{hom}_{\mathsf{sSet}}(X \times \mathbf{\Delta}^n, Y)$, $g \in \mathsf{hom}_{\mathsf{sSet}}(Y \times \mathbf{\Delta}^n, Z)$ and $p_1 \colon X \times \mathbf{\Delta}^n \to \mathbf{\Delta}^n$ is the projection onto the first component. We call this *natural simplicial enrichment.*

In general we are able to do this sSet-enriching yoga with any category of simplicial objects with values in a complete and cocomplete category $\mathcal{C}$. An explicit description of the sSet enrichment of $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathcal{C}^{op}, \mathsf{Set})$ is given in [6], [20]. Moreover, when we work with a model category $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathcal{C}^{op}, \mathsf{Set})_{\mathsf{lp}}$ with a local projective model structure (which we briefly introduce below) then a resulting simplicial category is a combinatorial simplicial model category. This fact is one of the cornerstones for the proof of a Proposition 1.6. Whenever the natural simplicial enrichment of a category $\mathcal{A}$ exists we denote it $\mathcal{A}^{\uparrow}$.

Further note that an internal hom-object between an arbitrary simplicial set (as a source) and a Kan complex (as a target) is also a Kan complex[1]. Thus we may take a full subcategory Kan of sSet consisting of Kan complexes and naturally enrich it over itself. The resulting simplicial category will be denoted simply as $\mathsf{Kan}_{\mathsf{Kan}}$.

Remark. Whenever we use a term *Kan complex* we mean an $(\infty, 0)$-category.

Definition 1.2. A coherent nerve $\mathfrak{N} \colon \mathsf{Cat}_{\mathsf{sSet}} \to \mathsf{sSet}$ defined in [9] *of the simplicial category* $\mathsf{Kan}_{\mathsf{Kan}}$ *will be called* ∞*-cosmos*[2] *and denoted* $\mathbb{S}$.

Remark. The simplicial set $\mathbb{S}$ is an $(\infty, 1)$-category. It follows from properties of the coherent nerve [9].

[1]This statement is rather standard, for its proof see for instance [7].

[2]The term ∞-cosmos in its broader sense means any well-behaved $(\infty, 1)$-category, in the strict sense it is a (nerve of a) simplicial category of fibrant objects [24]. Note that the full subcategory of Kan complexes is exactly the full subcategory of those simplicial objects which are fibrant with respect to the Quillen model structure on sSet.

Definition 1.3. Let us consider a simplicial set $\underline{\mathsf{hom}}_{\mathsf{sSet}}(\mathfrak{C}^{op}, \mathbb{S})$, where the symbol $\mathfrak{C}$ stands for an arbitrary $(\infty, 1)$-site. This simplicial set will be called the $(\infty, 1)$-category of derived $(\infty, 1)$-presheaves over $\mathfrak{C}$. Once we specify $\mathfrak{N}(\mathsf{Mfd}) := \mathfrak{C}$, where Mfd is the 1-site of smooth finite-dimensional manifolds with a Grothendieck topology defined by differentiably good open covers we result with a simplicial set called the $(\infty, 1)$-category of $(\infty, 1)$-prestacks over Mfd. If we restrict the attention to the $(\infty, 1)$-subcategory of $(\infty, 1)$-prestacks which satisfy the descent condition we end up with the $(\infty, 1)$-category of $(\infty, 1)$-stacks over Mfd which is often referred to as the $(\infty, 1)$-category of smooth ∞-groupoids. If we restrict our attention even more and consider only differentiable $(\infty, 1)$-stacks over Mfd we obtain a full $(\infty, 1)$-subcategory of Lie ∞-groupoids, denoted as $\mathsf{Lie\infty Grpd}$. These concepts can be found in [12], [15], [21], [25].

Remark. There is a filtered sequence of full $(\infty, 1)$-subtoposes $\mathbf{cosk}_n : \mathsf{Sh}_{(n,1)}(\mathsf{Mfd}) \hookrightarrow \mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$, where $\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}) := \underline{\mathsf{hom}}_{\mathsf{sSet}}(\mathfrak{N}(\mathsf{Mfd})^{op}, \mathbb{S}_n)$ and $\mathbb{S}_n$ is an $(n-1)$-truncation of $\mathbb{S}$[3]. Since $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ is a presentable $(\infty, 1)$-category we are guaranteed (from [9]) that for each $n \in \mathbb{N}$ (called level) there exists a left adjoint $\mathbf{tr}_n : \mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}) \to \mathsf{Sh}_{(n-1,1)}(\mathsf{Mfd})$ called n-truncation functor.

Remark. We call $\mathsf{Sh}_{(n,1)}(\mathsf{Mfd})$ as $(n, 1)$-category of $(n, 1)$-sheaves over the site Mfd.

Remark. The $(\infty, 1)$-category of Lie ∞-groupoids is an interesting category for us since it is presented by an 1-category of Kan simplicial manifolds KanMfd. In the literature we may find n-truncated Kan simplicial manifold under the name n-hypergroupoid [22].

Remark. It is a remarkable fact that almost all nice properties of the $(\infty, 1)$-category of smooth ∞-groupoids is summed up by the claim that it is $(\infty, 1)$-topos equipped with a differential cohesion [25].

When one considers an 1-categorical presentation of $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ by model categories there is a reasonable candidate – simplicial presheaves over Mfd together with a local projective model structure. In the next we say a couple of words on that account.

Remark. Because the category Cat of locally small categories is closed monoidal it naturally has the notion of an internal hom-space. This is nothing else than a functor category. From this reason we can define the following.

Definition 1.4. The category of simplicial presheaves is a category $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})$. Because of the symmetrical monoidal product we can equally define it as $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \mathsf{PSh}(\mathsf{Mfd}))$.

This 1-category naturally offers two main model structures with their local counterparts. For some technical reasons (computing homotopy colimits) it is

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[3] We call a simplicial set S_k to be $(k-1)$-truncation of a simplicial set S if $S_k \cong \mathbf{cosk}_k \circ \mathbf{tr}_k(S)$. It models $(k-1)$-homotopy type and generalises the notion of $(k-1)$-groupoid, meaning that all categorial homotopy groups above $k-1$ are zero.

better to take a projective model structure or more precisely its left Bousfield localization – the local projective model structure. Being equipped with this structure it forms an example of a model topos [23]. We will denote this model category as $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})_{\mathsf{lp}}$ or equivalently $\mathsf{sPSh}(\mathsf{Mfd})_{\mathsf{lp}}$.

Remark. Let us remark that $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})$ is actually big enough to accommodate either presheaves on smooth finite-dimensional manifolds or simplicial manifolds. And of course, there is an embedding of the category of the smooth finite-dimensional manifolds. For a better transparency let us pictorially sketch the situation by the following commutative diagram:

$$\begin{tikzcd} \mathsf{Mfd} \arrow[r, hook, "\mathcal{Y}"] \arrow[d, hook, "\mathbf{c}"'] & \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{Set}) \arrow[d, hook, "\mathbf{c}"'] \\ \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \mathsf{Mfd}) \arrow[r, hook, "\mathcal{Y}^{\mathbf{\Delta}^{op}}"'] & \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{Set})), \end{tikzcd} \tag{1}$$

where $\mathcal{Y}$ is the standard Yoneda embedding functor, $\mathbf{c}$ is a constant simplicial functor (also discussed below or in [3]) and $\mathcal{Y}^{\mathbf{\Delta}^{op}}$ is a simplicial Yoneda functor (computed degree-wise). Also note that we can't use ordinary Yoneda for relating the category $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \mathsf{Mfd})$ with the category $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{Set}))$ because

$$\mathcal{Y}\colon \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \mathsf{Mfd}) \hookrightarrow \underline{\mathsf{hom}}_{\mathsf{Cat}}\big((\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \mathsf{Mfd}))^{op}, \mathsf{Set}\big)$$

and the categories

$$\underline{\mathsf{hom}}_{\mathsf{Cat}}\big((\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \mathsf{Mfd}))^{op}, \mathsf{Set}\big) \quad \text{and} \quad \underline{\mathsf{hom}}_{\mathsf{Cat}}\big(\mathbf{\Delta}^{op}, \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{Set})\big)$$

are not equivalent.

Remark. There is a trivial symbolism dedicated for those categories in the literature. It reads $\mathsf{sPSh}(\mathsf{Mfd}) := \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})$, $\mathsf{PSh}(\mathsf{Mfd}) := \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{Set})$ and $\mathsf{sMfd} := \underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathbf{\Delta}^{op}, \mathsf{Mfd})$.

In the next we finally interrelate the $(\infty,1)$-category $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ with the model category $\mathsf{sPSh}(\mathsf{Mfd})_{\mathsf{lp}}$ by means of a presentation.

Definition 1.5. We say that a model 1-category $\mathcal{A}$ presents an $(\infty,1)$-category $\mathbf{A}$ if there is an $(\infty,1)$-equivalence

$$\mathbf{A} \cong \mathfrak{N}(\mathring{\mathcal{A}}^{\uparrow}),$$

where $\mathring{\mathcal{A}}^{\uparrow}$ is a full $(\infty,1)$-subcategory of fibrant-cofibant objects of $\mathcal{A}^{\uparrow}$.

Proposition 1.6 (Lurie)**.** *The model* 1*-category* $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})_{\mathsf{lp}}$ *presents* $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$.

Proof. [9]. □

Proposition 1.7. *Homotopy categories of an* ∞*-category and its model* 1*-category presentation are equivalent categories, in our case there is an* 1*-categorical equivalence*

$$\mathsf{Ho}\big(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})\big) \cong \mathsf{Ho}\big(\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})_{\mathsf{lp}}\big)\,.$$

Proof. [9]. $\square$

Remark. For a notational reasons we will denote objects of $(\infty,1)$-categories and their relevant counterparts in the model 1-categories the same.

Remark. Even though we work in one specific cohesive $(\infty,1)$-topos $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ the following applies for an arbitrary cohesive $(\infty,1)$-topos $\mathbf{H}$ thus we stick to this more general notation.

1.2. **Higher parallel transport technology.** According to a standard literature [25][4] a higher flat connection can be seen as an element of the set $\pi_0\mathsf{hom}_{\mathbf{H}}(M,\flat\mathbf{B}\mathsf{G})$[5] which can be due to the cohesive $(\infty,1)$-adjunction of $(\infty,1)$-endofunctors $\Pi\dashv\flat:\mathbf{H}\rightleftarrows\mathbf{H}$ rephrased as being an element of the set $\pi_0\mathsf{hom}_{\mathbf{H}}(\Pi(M),\mathbf{B}\mathsf{G})$.

Remark. At this point one should be careful what is meant by higher flat connection. Strictly speaking it is a datum given by a flat differential cohomology theory unlike higher curved connections which are given by a differential cohomology theory with coefficient object $\mathbf{B}\mathsf{G}_{\mathsf{conn}}$. But it still does not mean that k-truncated theories of higher flat connections must have vanishing all curvature forms. It is because for any $(\infty,1)$-functor $\nabla\colon\Pi(M)\to\mathbf{B}\mathsf{G}$ its k-truncation counterpart ∇_k (given as $\mathsf{cosk}_{k+1}\circ\mathsf{tr}_{k+1}(\nabla)$ as described above) exhibits a connection with non-vanishing curvature $(k+1)$-form since k-truncation of $\Pi(M)$ is a smooth path k-groupoid which encapsulates connections with this property. On the other hand all lower curvature forms (sometimes called *fake curvature forms*) are automatically trivial what is the distinctive property among all higher curved connections which do not require vanishing of any curvature form. From now on adjectives higher and flat will be systematically omitted for reasons of brevity because all connections in this paper are higher and flat in the sense of this remark.

Note that every connection originates as a lift of a classifying morphism of some G-principal bundle $g\colon M\to\mathbf{B}\mathsf{G}$ (where $g\in\pi_0\mathsf{hom}_{\mathbf{H}}(M,\mathbf{B}\mathsf{G})$ and G is a 1-group object in $\mathbf{H}$) along a canonical morphism $\flat\mathbf{B}\mathsf{G}\to\mathbf{B}\mathsf{G}$. Pictorially:

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, dashed, "\nabla"'] & & \mathbf{B}\mathsf{G} \\ & \flat\mathbf{B}\mathsf{G} \arrow[ur] & \end{tikzcd}$$

The next lemma deals with the question what statement is an appropriate counterpart in the perspective of mentioned adjunction $\Pi\dashv\flat$.

Lemma 1.8. *Every connection is an extension of a classifying morphism of a* G*-principal bundle* $g\colon M\to\mathbf{B}\mathsf{G}$ *along a morphism* $i\colon M\to\Pi(M)$ *which is defined as the* $(\infty,1)$*-colimit applied on a simplicial morphism* $\mathbf{i}\colon\mathsf{cM}\to\mathbf{\Pi}(M)$*, where* $\mathsf{cM}\in\underline{\mathsf{hom}}_{\mathsf{sSet}}(\mathfrak{N}(\mathbf{\Delta}^{op}),\mathbf{H})$ *is a constant simplicial diagram in* $\mathbf{H}$*,* $\mathbf{\Pi}(M)$ *is a simplicial diagram* (*degeneracies are not depicted*)

$$\ldots\underline{\mathsf{hom}}_{\mathbf{H}}(\mathbf{\Delta}^2,M)\;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\;\underline{\mathsf{hom}}_{\mathbf{H}}(\mathbf{\Delta}^1,M)\;\rightrightarrows\;\underline{\mathsf{hom}}_{\mathbf{H}}(\mathbf{\Delta}^0,M)\,,$$

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[4]Most of the terminology and notation used in this sub-chapter is defined here.

[5]We define $\pi_0\mathsf{hom}_{\mathbf{H}}(A,B):=\mathsf{hom}_{\mathsf{Ho}(\mathbf{H})}(A,B)$.

and for any $n \in \mathbb{N}$ (for $n = 0$ we define it as an identity morphism) we have a morphism $\mathbf{i}_n \colon M \cong \underline{\mathsf{hom}}_{\mathbf{H}}(\mathbf{\Delta}^0, M) \to \underline{\mathsf{hom}}_{\mathbf{H}}(\mathbf{\Delta}^n, M)$ canonically given as $\underline{\mathsf{hom}}_{\mathbf{H}}(\mathcal{Y}(\phi_0^n)^{\times n}, M)$, where the expression $\mathcal{Y}(\phi_0^n) \colon \mathbf{\Delta}^{n-1} \to \mathbf{\Delta}^n$ is just the image of the Yoneda functor applied on the zeroth coface map $\phi_0^n \colon [n-1] \to [n]$.

Proof. The first and crucial observation (which is proved in [2]) says that the morphism $i \colon M \to \Pi(M)$ is precisely a unit of the adjunction $\Pi \dashv \flat$. The second step is an application of unit/counit data from a cohesive quadruple of adjunct functors. Will be showed in the upcoming paper. $\square$

Remark. In case of $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ the space $\Pi(M)$ is commonly known as the ∞-fundamental groupoid associated with the space M.

Corollary 1.9. *Lemma 1.8 finds a common homotopy invariant which intertwines with the mentioned $(\infty, 1)$-adjunction. That is exactly the underlying principal G-bundle associated with ∇.*

1.3. **Orthogonal factorisation systems.** The previous subchapter establishes a basic dictionary between the theory of higher parallel transport and the flat differential cohomology theory approach to the connections on principal bundles. In the next we steer closer to the parallel transport theory as to find a more subtle data realizing Atiyah's approach to connections [1]. It has shown that the major role in tracking of the Atiyah's legacy has one distinguished orthogonal factorisation system in $\mathbf{H}$, called the (-1)-connected – (-1)-truncated orthogonal factorisation system.

Due to a general result (stated in [17] for instance) any morphism in $\mathbf{H}$ can be factorised on n-connected morphism followed by n-truncated one for $n \in \mathbb{N}_{0,-1,-2}$. For us the most relevant case will be $n = -1$. In this case we factorise any morphism onto $(\infty, 1)$-effective epimorphisms followed by $(\infty, 1)$-monomorphisms.

Definition 1.10. For any morphism $f \colon A \to B$ in $\mathbf{H}$ we can construct $(\infty, 1)$-colimit of the Čech nerve simplicial object $\check{\mathsf{C}}(f) \in \underline{\mathsf{hom}}_{\mathsf{sSet}}(\mathfrak{N}(\mathbf{\Delta})^{op}, \mathbf{H})$ which is built as an iterated sequence of $(\infty, 1)$-pullbacks. In the first two degrees of this simplicial diagram Čech nerve reads:

$$\ldots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; A \times_B A \times_B A \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; A \times_B A \;\rightrightarrows\; A\,. \tag{2}$$

We call this object 1-coimage[6] of $f \colon A \to B$ and denote it as $\mathsf{coim}_1(f)$.

Remark. For a definition of the Čech nerve see [9].

This object is always furnished with two canonical morphisms.

Lemma 1.11. *Let us have any morphism $f \colon A \to B$ in $\mathbf{H}$. There is a morphism $f' \colon A \to \mathsf{coim}_1(f)$ which is a morphism from colimiting cocone. It is always an $(\infty, 1)$-effective epimorphism. There is another cocone on the simplicial*

[6]Terminology says that morphisms factorise in this n-connected and n-truncated factorisation system through $(n+2)$-coimage. Many authors use a different terminology and name these objects as *images*, but we have not found it natural. Despite the both terms have distinct meaning in general, distinguishing between them in our case does not matter since they are isomorphic in $\mathbf{H}$.

diagram above which is generated by $f\colon A \to B$*. It induces by the universal property of* $(\infty,1)$*-colimits a morphism* $c_f\colon \mathsf{coim}_1(f) \to B$*. This morphism is a* $(\infty,1)$*-monomorphism.*

Proof. The statement is classical. For details look at [18]. □

Remark. Again, for the reason of brevity we call these morphisms simply as effective epimorphisms and monomorphisms.

Let us list some useful properties of this factorisation system.

Lemma 1.12. *Assume we have three morphisms* $f, g, h \in \mathbf{H}$ *which satisfy* $h = f \circ g$ *(in the homotopy category* $\mathsf{Ho}(\mathbf{H})$*). Then there are some relations between them:*
- *If* f *and* g *are effective epimorphisms then* h *is an effective epimorphisms.*
- *If* f *and* g *are monomorphisms then* h *is a monomorphism.*
- *If* h *is an effective epimorphism then* f *is an effective epimorphism.*
- *If* h *and* f *are monomorphisms then* g *is a monomorphism.*

Proof. [9]. □

Lemma 1.13. *The intersection of the effective epimorphisms class with the monomorphisms class is exactly the class of isomorphisms.*

Proof. [19]. □

Lemma 1.14. *The factorisation system described above is orthogonal. In detail, for any effective epimorphism* $f\colon A \twoheadrightarrow C$ *and any monomorphism* $g\colon B \hookrightarrow D$ *in* $\mathbf{H}$ *the mapping space (defined in* [9]*)* $\mathsf{Map}_{\mathbf{H}_{A//D}}(B, C)$ *is a contractible simplicial set.*

Proof. [9]. □

Remark. It basically says that the space of anti-diagonal arrows $h\colon C \to B$ (which is a simplicial set) in the commutative diagram (in $\mathsf{Ho}(\mathbf{H})$)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] \arrow[ur, dashed, "h"'] & D \end{tikzcd} \tag{3}$$

which make the triangles commute (in $\mathsf{Ho}(\mathbf{H})$) is homotopy equivalent to the terminal object.

Corollary 1.15. *Let us have any morphism* $f\colon A \to B$ *in* $\mathbf{H}$*. There is up to an isomorphism only one triple* (C, a, b) *where* $a\colon A \to C$ *is an effective epimorphism and* $b\colon C \to B$ *is a monomorphism such that* $f = b \circ a$ *in* $\mathsf{Ho}(\mathbf{H})$*.*

Proof. Assume we have two such triples (C_1, a_1, b_1) and (C_2, a_2, b_2). From the lifting property we get a unique morphism $d\colon C_1 \to C_2$. From the Lemma 1.12 we might deduce that d has to be an effective epimorphism and also a monomorphism. From the Lemma 1.13 we know that it has to be an isomorphism. □

After this short discussion about some general properties of (-1)-connected – (-1)-truncated orthogonal factorisation system in $\mathbf{H}$ we may apply the knowledge to our subject of concern.

Let us have morphism $\nabla\colon \Pi(M) \to \mathbf{B}\mathsf{G}$ and suppose that its underlying principal G-bundle is given by $g\colon M \to \mathbf{B}\mathsf{G}$, meaning that we have $g = \nabla \circ i$ in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$. Then we have a sequence

$$M \xrightarrow{i} \Pi(M) \xrightarrow{\nabla} \mathbf{B}\mathsf{G}$$

which can be due to the orthogonal factorisation technology decomposed as follows:

$$\begin{tikzcd}
M \arrow[rr, "i"] \arrow[dr, two heads] & & \Pi(M) \arrow[rr, "\nabla"] \arrow[dr, two heads] & & \mathbf{B}\mathsf{G} \\
& \mathsf{coim}_1(i) \arrow[ur, hook] \arrow[dr, two heads] & & \mathsf{coim}_1(\nabla) \arrow[ur, hook] & \\
& & \mathsf{coim}_1(g) \arrow[ur, hook] & &
\end{tikzcd} \tag{4}$$

Now we can draw the picture comparing factorisations of two different $\nabla_1, \nabla_2\colon \Pi(M) \to \mathbf{B}\mathsf{G}$ which are above the same $g\colon M \to \mathbf{B}\mathsf{G}$

$$\begin{tikzcd}
& & \mathsf{coim}_1(g) \arrow[dr, hook] & & \\
& \mathsf{coim}_1(i) \arrow[ur, two heads] \arrow[dr, hook] & & \mathsf{coim}_1(\nabla_1) \arrow[dr, hook] & \\
M \arrow[ur, two heads] \arrow[rr, "i"] \arrow[dr, two heads] & & \Pi(M) \arrow[ur, two heads] \arrow[rr, shift left, "\nabla_1"] \arrow[rr, shift right, "\nabla_2"'] \arrow[dr, two heads] & & \mathbf{B}\mathsf{G} \\
& \mathsf{coim}_1(i) \arrow[ur, hook] \arrow[dr, two heads] & & \mathsf{coim}_1(\nabla_2) \arrow[ur, hook] & \\
& & \mathsf{coim}_1(g) \arrow[ur, hook] & &
\end{tikzcd}$$

Due to the Corollary 1.15 we can easily see that any two $\nabla_1, \nabla_2\colon \Pi(M) \to \mathbf{B}\mathsf{G}$ which are above the same $g\colon M \to \mathbf{B}\mathsf{G}$ give birth to the diagram

$$\begin{tikzcd}
\mathsf{coim}_1(g) \arrow[r, hook] \arrow[d, hook] & \mathsf{coim}_1(\nabla_1) \arrow[d, hook, "c_{\nabla_1}"'] \\
\mathsf{coim}_1(\nabla_2) \arrow[r, hook, "c_{\nabla_2}"'] & \mathbf{B}\mathsf{G}
\end{tikzcd} \tag{5}$$

which commutes in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$.

With this knowledge we can define a keynote of this paper – the generalised Atiyah space.

Definition 1.16 (Generalised Atiyah space)**.** Let us have a classifying morphism $g\colon M \to \mathbf{B}\mathsf{G}$ for a principal G-bundle and consider all morphisms $\nabla\colon \Pi(M) \to \mathbf{B}\mathsf{G}$ which satisfy $g = \nabla \circ i$ in the homotopy category $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$. Any factorisation for g and ∇ produces a monomorphism $\mathsf{coim}_1(g) \hookrightarrow \mathsf{coim}_1(\nabla)$ as we have seen on the diagram (5). All such monomorphisms form a diagram (or better say – its coherent nerve), let us denote the diagram $\mathcal{D}$. If we present this diagram in the model 1-topos $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})_{\mathsf{lp}}$ we get an object in $\underline{\mathsf{hom}}_{\mathsf{Cat}}(\mathsf{Mfd}^{op}, \mathsf{sSet})_{\mathsf{lp}}^{\mathcal{D}}$ with fibrant-cofibrant vertices. Finally we take a wide local [7] homotopy-pushout on this and call the colimiting cocone as the generalised Atiyah space associated to g. We will denote the colimiting object as $\mathsf{At}(g)$.

Remark. For the construction and another motivation behind considering $\mathsf{coim}_1(g)$ space (called *higher Atiyah groupoid* therein) see [13]. In a different perspective but also considering the object $\mathsf{coim}_1(g)$ there is given in [4] a construction of an Atiyah sequence from integrated Kostant-Souriau ∞-group extensions.

The construction described above has indeed several important properties which are shown in the next.

Lemma 1.17. *For each choice of a classifying morphism $g\colon M \to \mathbf{B}\mathsf{G}$ we have canonically given pair of morphisms*

$$\mathsf{coim}_1(g) \xrightarrow{\alpha_g} \mathsf{At}(g) \longrightarrow \mathbf{B}\mathsf{G}.$$

Composition of these (in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$) gives a monomorphism $\mathsf{c}_g\colon \mathsf{coim}_1(g) \hookrightarrow \mathbf{B}\mathsf{G}$ coming from a factorisation of $g\colon M \to \mathbf{B}\mathsf{G}$.

Proof. The first morphism is simply given by a composition (a morphism α_∇ is mono due to the stability of monomorphisms under pushouts in $\mathbf{H}$)

$$\mathsf{coim}_1(g) \hookrightarrow \mathsf{coim}_1(\nabla) \xhookrightarrow{\alpha_\nabla} \mathsf{At}(g)$$

for an arbitrary ∇ over g, which is well defined since the wide local homotopy-pushout square commutes (in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$) directly from the definition. Because of the universal property of local homotopy-pushout and because of homotopical commutativity of (the wide version of) the diagram (5) we get a unique (in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$) morphism $\mathsf{c}_{\mathsf{At}(g)} : \mathsf{At}(g) \to \mathbf{B}\mathsf{G}$. The second part of this lemma is an easy consequence of commutativity of diagrams. $\square$

Lemma 1.18. *Every $\nabla\colon \Pi(M) \to \mathbf{B}\mathsf{G}$ over $g\colon M \to \mathbf{B}\mathsf{G}$ uniquely (in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}$ $(\mathsf{Mfd}))$) factorises through $\mathsf{At}(g)$.*

Proof. Suppose we are given any $\nabla\colon \Pi(M) \to \mathbf{B}\mathsf{G}$ over $g\colon M \to \mathbf{B}\mathsf{G}$. When we factorise it $\nabla = \mathsf{c}_\nabla \circ \nabla'$ and make a substitution for c_∇ coming from the equality $\mathsf{c}_\nabla = \mathsf{c}_{\mathsf{At(g)}} \circ \alpha_\nabla$, where $\alpha_\nabla : \mathsf{coim}_1(\nabla) \to \mathsf{At}(g)$ is the morphism from the colimiting cocone of the generalised Atiyah space, we immediately observe that $\nabla = \mathsf{c}_{\mathsf{At}(g)} \circ \nabla^g$, where $\nabla^g\colon \Pi(M) \to \mathsf{At}(g)$ is the desired morphism. $\square$

[7]There are always two in general non-equivalent definitions of homotopy universal constructions, for our purposes we use the local one. However, both constructions (local and global) on diagrams with fibrant-cofibrant vertices give the same answer up to a weak equivalence. [14]

Definition 1.19. We call such morphisms $\nabla^g\colon \Pi(M) \to \mathsf{At}(g)$ (viewed as elements of $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$) as the <u>*Atiyah connections*</u> associated to $g\colon M \to \mathbf{B}\mathsf{G}$.

As we will state in the next there is a necessary and at the same time sufficient condition for all Atiyah connections to encode some flat connection over g.

Lemma 1.20. *Every morphism $\nabla^g\colon \Pi(M) \to \mathsf{At}(g)$ which originates from the factorisation of some $\nabla : \Pi(M) \to \mathbf{B}\mathsf{G}$ over $g\colon M \to \mathbf{B}\mathsf{G}$ makes the diagram*

$$\begin{tikzcd} M \arrow[rr, "\alpha_g \circ g'"] \arrow[dr, "i"] & & \mathsf{At}(g) \\ & \Pi(M) \arrow[ur] & \end{tikzcd} \tag{6}$$

commute in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$ and every morphism $f\colon \Pi(M) \to \mathsf{At}(g)$ rendering the diagram above commutative in $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$ is an Atiyah connection associated to $g\colon M \to \mathbf{B}\mathsf{G}$.

Proof. Will be showed in the upcoming paper. □

Definition 1.21. We call the property above as the <u>*section property*</u> of the Atiyah connections.

Remark. There is a paper in preparation where we discuss how exactly this property is equivalent to the splitting of a generalised Atiyah sequence of L_∞-algebroids.

2. On the construction of a generalised Atiyah sequence

Due to the additional structure – a quadruple of adjoint $(\infty,1)$-functors between $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ and $\widehat{\mathsf{Sh}}_{(\infty,1)}(\mathsf{Mfd})$, the $(\infty,1)$-topos of formal smooth ∞-groupoids, which is called a differential cohesive structure [25] we are allowed to speak about a relationship between constructions in $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ and its infinitesimal neighbourhood. More specifically we recall a presentation of the higher Lie differentiation functor – the one-jet functor as defined in [26] and apply its derived version on Atiyah connections $\nabla^g\colon \Pi(M) \to \mathsf{At}(g)$ viewed as elements of a certain hom-set in the homotopy category $\mathsf{Ho}(\mathsf{sPSh}(\mathsf{Mfd})_{\mathsf{lp}})$. This is the route which might lead us to the desired correspondence between higher differential cohomology theory with the flat coefficient object $\flat\mathbf{B}\mathsf{G}$ and the space of sections of the generalised Atiyah sequence.

2.1. **Introduction of a derived one-jet functor.** For a recollection of an NQMfd, an 1-category of differential non-negatively graded manifolds we refer the reader to [8]. As explained, for instance, in [16] the category NQMfd is a full subcategory of $\mathsf{dgcAlg}_{\mathbb{R}}^{op}$ the opposite category of cochain dg-algebras over $\mathbb{R}$. There (*Ibid.*) is also explained how to view NQMfd as a subcategory of $\mathsf{sPSh}(\mathsf{Mfd}_{\mathsf{synth}})$ of simplicial presheaves over formal smooth manifolds $\mathsf{Mfd}_{\mathsf{synth}}$. For the definition and further discussion of formal manifolds see [25].

Remark. Note that $\mathsf{sPSh}(\mathsf{Mfd}_{\mathsf{synth}})$ has analogously to $\mathsf{sPSh}(\mathsf{Mfd})$ its own local projective model structure. Then one less trivial observation is that the presentation of this model category is weakly equivalent to $\widehat{\mathsf{Sh}}_{(\infty,1)}(\mathsf{Mfd})$.

Now let us invite the one-jet functor!

Remark. Even though this functor can be extended to the category of all simplicial manifolds we do well without this generality and focus on (finitely truncated) Kan simplicial manifolds only.

Theorem 2.1 (Ševera '06)**.** *Let us have $\mathscr{K}$ (finitely truncated) Kan simplicial manifold. Then the functor* $\mathsf{hom}_{\mathsf{KanMfd}^{\mathsf{fin}}_{\mathbb{Z}_2}}(\mathsf{hocolim}\,\check{\mathsf{C}}(\bullet), \mathscr{K})\colon \mathsf{SSM}^{op} \to \mathsf{Set}$ *is a representable presheaf on the category* SSM *of surjective submersions $Y \to X$ in* SMfd*.*

Remark. The symbol $\check{\mathsf{C}}(Y \to X)$ means nothing but the Čech simplicial resolution of the morphism $Y \to X$ as we know it from the previous chapter. Under the symbol SMfd we understand a category of $\mathbb{Z}_2$-graded finite-dimensional smooth manifolds. The symbol $\mathsf{KanMfd}^{\mathsf{fin}}_{\mathbb{Z}_2}$ stands for the category of finitely truncated Kan simplical $\mathbb{Z}_2$-graded smooth manifolds. And $\mathscr{K}$ can be seen as an object of $\mathsf{KanMfd}^{\mathsf{fin}}_{\mathbb{Z}_2}$ under the embedding $\mathsf{KanMfd}^{\mathsf{fin}} \hookrightarrow \mathsf{KanMfd}^{\mathsf{fin}}_{\mathbb{Z}_2}$.

Observation (Ševera): If we moreover restrict this functor on the subcategory SSM_1 of surjective submersions of type $\mathbb{R}^{0|1} \times X \to X$ (the projection on the second component), it is representable as the presheaf on SMfd.

Remark. This observation is enough for getting a structure of NQ manifold. Indeed, we may restore a homological vector field and extend a $\mathbb{Z}_2$-gradation to an $\mathbb{N}_0$-gradation as follows. We observe that this representative is naturally furnished with an action of a monoid $\underline{\mathsf{hom}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$. Then two generating fundamental vector fields of this action provide the data. Hence, as a result we obtain a super-smooth (finite-dimensional) manifold furnished with some homological vector field (giving the bracket structure) and some Euler vector field (defining $\mathbb{N}_0$-gradation), in other words NQ manifold in its standard definition. This is the NQ manifold which we associate with the Kan simplicial manifold $\mathscr{K}$ when we take the one-jet of $\mathscr{K}$.

Definition 2.2. The functor $J^1\colon \mathsf{KanMfd}^{\mathsf{fin}} \to \mathsf{NQMfd}$ described above[8] will be called <u>one-jet functor</u>.

Remark. We stress that we are working with finitely truncated Kan simplicial manifolds by decorating KanMfd with a superscript fin.

For both categories we know their relation to the respective model categories, namely due to the sequence $\mathsf{KanMfd}^{\mathsf{fin}} \hookrightarrow \mathsf{sMfd} \to \mathsf{sPSh}(\mathsf{Mfd})_{\mathsf{lp}}$ (from the diagram (1)) and $\mathsf{NQMfd} \to \mathsf{sPSh}(\mathsf{Mfd}_{\mathsf{synth}})_{\mathsf{lp}}$. These are faithful (but not full) functors. Let us assign fibrancy and cofibrancy property to objects of $\mathsf{KanMfd}^{\mathsf{fin}}$ and NQMfd

[8]An action on morphisms is fixed by functoriality of $\mathsf{hom}_{\mathsf{KanMfd}^{\mathsf{fin}}_{\mathbb{Z}_2}}(\mathsf{hocolim}\,\check{\mathsf{C}}(\bullet), \mathscr{K})$.

which are in the image of these functors also fibrant and cofibrant objects. Moreover assign for all pairs of parallel morphisms a property of being homotopic whenever they are so in the image of these functors. Now we are prepared for the following:

Lemma 2.3. *The one-jet functor* $J^1\colon \mathsf{KanMfd}^{\mathsf{fin}} \to \mathsf{NQMfd}$ *preserves fibrant objects, cofibrant objects and the homotopic relation on the pairs of parallel morphisms.*

Remark. Let us denote the full subcategory of the homotopy category $\mathsf{Ho}(\mathsf{sPSh}(\mathsf{Mfd})_{\mathsf{lp}})$ on objects of $\mathsf{KanMfd}^{\mathsf{fin}}$ as $\mathsf{Ho}(\mathsf{KanMfd}^{\mathsf{fin}})$ and the full subcategory of the homotopy category $\mathsf{Ho}(\mathsf{sPSh}(\mathsf{Mfd}_{\mathsf{synth}})_{\mathsf{lp}})$ on objects of NQMfd as $\mathsf{Ho}(\mathsf{NQMfd})$.

Definition 2.4. Let us consider a functor $\mathsf{Ho}(J^1)\colon \mathsf{Ho}(\mathsf{KanMfd}^{\mathsf{fin}}) \to \mathsf{Ho}(\mathsf{NQMfd})$ defined as a map $\mathsf{Ho}(J^1)\colon X \mapsto J^1(X)$ on objects and as a map $\mathsf{Ho}(J^1)\colon \mathsf{Ho}(f) \mapsto \mathsf{Ho}(J^1(f))$ on morphisms of $\mathsf{Ho}(\mathsf{KanMfd}^{\mathsf{fin}})$. We call this functor a <u>derived one-jet functor</u>.

Remark. This definition is well posed due to the Lemma 2.3.

2.2. **Notion of a generalised Atiyah sequence.**

Definition 2.5. A short exact sequence in the category NQMfd is a functor $\mathfrak{S}\colon \mathcal{B} \to \mathsf{NQMfd}$ from a short sequence diagram category $\mathcal{B}$ such that the homotopy functor $\mathsf{Ho}(\mathfrak{S})\colon \mathcal{B} \to \mathsf{Ho}(\mathsf{NQMfd})$ is an ordinary (1-categorical) short exact sequence.

Remark. The homotopy functor above is again defined simply as a map $\mathsf{Ho}(\mathfrak{S})$: $X \mapsto \mathfrak{S}(X)$ on objects and as a map $\mathsf{Ho}(\mathfrak{S})\colon f \mapsto \mathsf{Ho}(\mathfrak{S}(f))$ on morphisms.

Remark. For a notational reason we denote objects and morphisms (besides identities and compositions) in the short sequence diagram category $\mathcal{B}$ as

$$A \xrightarrow{\quad a \quad} B \xrightarrow{\quad b \quad} C \xrightarrow{\quad c \quad} D \xrightarrow{\quad d \quad} E$$

This concept is naturally accompanied with the notion of a section.

Definition 2.6. A (right) section of a short exact sequence $\mathfrak{S}\colon \mathcal{B} \to \mathsf{NQMfd}$ is a morphism $\phi\colon \mathcal{B}(E) \to \mathcal{B}(D)$ such that $\mathsf{Ho}(\mathfrak{S}(d) \circ \phi) = \mathrm{id}_{\mathbf{Ho}(\mathcal{B}(E))}$.

Remark. All our sections will be right sections hence we will drop this adjective.

Now we can define the main subject of this chapter – a generalised Atiyah sequence.

Definition 2.7. A <u>generalised Atiyah sequence</u> associated with $g\colon M \to \mathbf{B}\mathsf{G}$ is a short exact sequence in the category NQMfd whose set of sections bijectively corresponds to the set of Atiyah connections for $g : M \to \mathbf{B}\mathsf{G}$.

Under certain circumstances there is a relatively simple way how to search for a generalised Atiyah sequence. Even though we spoil the full generality the remaining case still enjoy a decent favour of theoretical and mathematical physicists.

These two extra assumptions are stated and justified by the following two lemmas.

Lemma 2.8. *If M is a n-connected smooth finite-dimensional manifold then*

$$\mathbf{tr}_n \Pi(M) \cong *$$

in $\mathsf{Ho}(\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}))$.

Proof. It suffices to show $\mathbf{tr}_n \Pi(M) \cong \mathsf{coim}_1(M \to *)$, because we know (as discussed in [25] for instance) that $\mathsf{coim}_1(M \to *) \cong *$ in $\mathsf{Ho}(\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}))$. Details are omitted in this paper. □

Remark. Let us facilitate the notation and write $\Pi(M,n)$ instead of $\mathbf{tr}_n \Pi(M)$ for the $(n-1)$-truncation of $\Pi(M)$. Moreover we denote $(n-1)$-truncated objects in $\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ and their images under the application of $\mathbf{tr}_n$ by the same symbol.

Lemma 2.9. *If a classifying space* $\mathbf{B}\mathsf{G}$ *is n-coskeletal, then*

$$\pi_0 \mathsf{hom}_{\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})}(\Pi(M), \mathbf{B}\mathsf{G}) \cong \pi_0 \mathsf{hom}_{\mathsf{Sh}_{(n,1)}(\mathsf{Mfd})}(\Pi(M,n), \mathbf{B}\mathsf{G}) \,.$$

Proof. This is verified immediately when one considers the adjunction $\mathbf{tr}_n \dashv \mathbf{cosk}_n$ and the fact that the adjunction unit[9] $X \to \mathbf{cosk}_n \circ \mathbf{tr}_n(X)$ is an isomorphism in homotopy category for X being n-coskeletal. □

Remark. It is the matter of fact that $\mathbf{B}\mathsf{G}$ is an n-coskeletal object in $\mathbf{H}$ whenever G is an $(n-1)$-group object in $\mathbf{H}$.

As a direct consequence of 2.8 and 2.9 we have:

Corollary 2.10. *If M is an n-connected smooth finite-dimensional manifold and* $\mathbf{B}\mathsf{G}$ *is an n-coskeletal classifying space then the generalised Atiyah space* $\mathsf{At}(g)$ *is isomorphic to* $\mathsf{coim}_1(g)$ *in the homotopy category* $\mathsf{Ho}(\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}))$.

Proof. Due to the Lemma 2.9 we know this is sufficient for the conclusion in the homotopy category $\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))$ to work in $\mathsf{Ho}(\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}))$ so we further proceed in $\mathsf{Ho}(\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}))$. Because for any non-empty object $M \in \mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})$ the terminal morphism is an effective epimorphism, we have that the $(n-1)$-truncated unit $i_n : M \to \Pi(M,n)$ is an effective epimorphism. From the decomposition diagram (4) it then immediately follows that $\mathsf{coim}(g) \hookrightarrow \mathsf{coim}(\nabla)$ is actual iso in $\mathsf{Ho}(\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}))$. And from the very definition of $\mathsf{At}(g)$ we can conclude that $(\alpha_g)_n$ is an isomorphism in the homotopy category $\mathsf{Ho}(\mathsf{Sh}_{(n,1)}(\mathsf{Mfd}))$. □

All these imply one important statement:

Corollary 2.11. *If M is an n-connected smooth finite-dimensional manifold and* $\mathbf{B}\mathsf{G}$ *is n-coskeletal then the section property 1.20 implies commutativity of this triangle*

$$\begin{tikzcd} TM \arrow[rr, "\mathsf{Ho}(J^1)\nabla^g"] \arrow[dr, "\sim"'] & & \mathsf{Ho}(J^1)\mathsf{coim}_1(g) \arrow[dl, two heads, "\mathsf{Ho}(J^1)\chi"] \\ & TM & \end{tikzcd} \tag{7}$$

where TM is a tangent Lie algebroid associated to M.

[9] See [27] for details.

Proof. The proof resides in a computation of the derived one-jet of a groupoid object equivalently described[10] by an effective epimorphism $M \twoheadrightarrow *$. Details are omitted in this paper. $\square$

From this argumentation we immediately see that any short exact sequence which in the homotopy category $\mathsf{Ho}(\mathsf{NQMfd})$ looks as

$$0 \longrightarrow \mathsf{Ho}(J^1)\ker(\chi) \longrightarrow \mathsf{Ho}(J^1)\mathsf{coim}_1(g) \xrightarrow{\mathsf{Ho}(J^1)\chi} TM \longrightarrow 0$$

is a potential candidate for a generalised Atiyah sequence associated to $g : M \to \mathbf{B}\mathsf{G}$ because sections of such short exact sequence (as defined in 2.6) exactly corresponds to commutative triangles (7) and thus to one-jets of Atiyah connections.

However, to get a proper (that is bijective) dictionary between Atiyah connections and sections of such short exact sequence there is needed further investigation.

2.3. **Further work.** This last subsection comprises a conjecture which is being currently investigated and lacks of full proof yet. It rather shows directions of our further research and anticipates those steps which would lead us to the desired outcomes of this project.

Even though the triangle (6) represents necessary and sufficient condition, its derived one-jet counterpart of their associated groupoids (7) serves only as a necessary requirement[11]. It ensure us that the following conjecture is at least well posed.

Conjecture 2.12. If M is a n-connected smooth finite-dimensional manifold and $\mathbf{B}\mathsf{G}$ is n-coskeletal then the derived one-jet provides a bijection

$$\mathfrak{B} \colon \mathsf{hom}^{@}_{\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))}(\Pi(M), \mathsf{At}(g)) \to \mathsf{hom}^{\Delta}_{\mathsf{Ho}(\mathsf{NQMfd})}(TM, \mathsf{Ho}(J^1)\mathsf{coim}_1(g))\,.$$

Remark. The symbol @ above hom indicates that we take only a subset consisting of Atiyah connections, similarly Δ above hom indicates that we take a subset consisting of morphisms which make the triangle (7) commute.

Remark. This conjecture would be solved with tools of a homotopy higher Lie theory which is still under current development and thus might take a longer time till being answered.

As we can anticipate despite being hard to prove this conjecture 2.12 gives the key for finding a bridge between the higher differential cohomology theory with the coefficient object $\flat\mathbf{B}\mathsf{G}$ and generalised Atiyah sequences since (as we already know from the previous chapter):

$$\pi_0 \mathsf{hom}^{(g)}_{\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd})}\big(\Pi(M), \mathbf{B}\mathsf{G}\big) \cong \mathsf{hom}^{@}_{\mathsf{Ho}(\mathsf{Sh}_{(\infty,1)}(\mathsf{Mfd}))}\big(\Pi(M), \mathsf{At}(g)\big)\,,$$

where the symbol (g) above hom means that we consider only connections over g.

[10] In any $(\infty,1)$-topos every groupoid object is effective which means that $\mathsf{Grpd}(\mathbf{H}) \cong \mathbf{H}^{\Delta^1}_{\mathsf{eff-epi}}$.

[11] This is because commutative squares are sent to commutative squares by functors.

Acknowledgement. My sincere thanks go to my supervisor Branislav Jurčo for his insightful and pleasing guidance, to Christian Sämann and Martin Wolf for sparking our work on this project, to Daniel Stevenson, Dmitry Roytenberg, Urs Schreiber, Jan Šťovíček and Michal Kamba for inspirational discussions. I am also very grateful to all contributors to *nLab* web page.

The research was supported by grant GAUK No. 544218, grant SVV 301-09/260589 and GAČR grant EXPRO No. 19-28628x.

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[14] nLab, *homotopy limit.*

[15] nLab, *infinity-stack.*

[16] nLab, *Lie infinity algebroid.*

[17] nLab, *(n-connected, n-truncated) factorization system.*

[18] nLab, *n-image.*

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Faculty of Mathematics and Physics,
Charles University,
Sokolovská 83, 18000 Praha 8, Czech Republic
E-mail: narozny@karlin.mff.cuni.cz