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EXISTENCE OF BLOW-UP SOLUTIONS FOR A DEGENERATE PARABOLIC-ELLIPTIC KELLER–SEGEL SYSTEM WITH LOGISTIC SOURCE

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Abstract. This paper deals with existence of finite-time blow-up solutions to a degenerate parabolic–elliptic Keller–Segel system with logistic source. Recently, finite-time blow-up was established for a degenerate Jäger–Luckhaus system with logistic source. However, blow-up solutions of the aforementioned system have not been obtained. The purpose of this paper is to construct blow-up solutions of a degenerate Keller–Segel system with logistic source.

1. Introduction and main result

In this paper we consider the quasilinear degenerate Keller–Segel system with logistic source,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa \,, & x \in \Omega,\ t > 0\,, \\ 0 = \Delta v - v + u\,, & x \in \Omega,\ t > 0\,, \\ \dfrac{\partial u^m}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0\,, & x \in \partial\Omega,\ t > 0\,, \\ u(x,0) = u_0(x)\,, & x \in \Omega\,, \end{cases} \tag{1.1}$$

where $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball with some $R > 0$; $m \geq 1$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ and $\kappa > 1$; ν is the outward normal vector to $\partial\Omega$; $u_0 \in L^\infty(\Omega)$ is nonnegative and radially symmetric. This system describes a situation such that a cellular slime moves towards higher concentrations of the chemical substance.

In the case $m = 1$, Winkler [10] obtained initial data leading to finite-time blow-up under a smallness condition for $\kappa > 1$ in three- or higher-dimensional cases. In the case $m \in \left[1, 2 - \frac{2}{n}\right)$, for the system such that the diffusion term is replaced with $\nabla \cdot ((u+1)^{m-1} \nabla u)$, Black, Fuest and Lankeit showed that solutions blow up in finite time under the condition that $\kappa < 1 + \min\left\{\frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)}\right\}$ in [1, Theorem 1.2 (ii)]. On the other hand, a difficulty is caused in (1.1) by the degenerate diffusion term Δu^m because in the case of nondegenerate diffusion

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classical solutions can be considered, whereas in the case of degenerate diffusion classical solutions are not always obtained. In such circumstances, it had not been clear whether blow-up of solutions to (1.1) occurs.

Regarding this difficulty, existence of blow-up solutions was recently established in [8] for the following Jäger–Luckhaus system with $\varepsilon = 0$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(u+\varepsilon)^m - \chi\nabla\cdot(u\nabla v) + \lambda u - \mu u^\kappa\,, & x\in\Omega,\ t>0\,,\\ 0 = \Delta v - \overline{M}(t) + u\,, & x\in\Omega,\ t>0\,, \end{cases}$$

where $\overline{M}(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx$. This system was studied in [1,3,7,9]; in the case $m = 1$ and $\varepsilon = 0$, finite-time blow-up was shown under smallness conditions for κ in the three- and higher-dimensional cases in [1,9] (in the case $\overline{M}(t) = v$, see [10]); these conditions were improved in [3]; in the case $m \neq 1$, the condition $\kappa < \min\left\{2, \frac{n}{2}\right\}$ in [3] was generalized to the condition that $\kappa < \min\left\{2, (2-m)\frac{n}{2}\right\}$ if $m \geq 0$ or $\kappa < \min\{2, n\}$ if $m < 0$ in [7]. After that, in the case of degenerate diffusion ($\varepsilon = 0$), finite-time blow-up solutions was constructed in a framework of weak solutions in [8].

In contrast, for the degenerate Keller–Segel system with logistic source there is no result on blow-up. The purpose is to prove existence of blow-up solutions to (1.1) in a framework of weak solutions under the same condition as in [1, Theorem 1.2 (ii)]. Referring to the method in [8], we introduce *moment solutions* as follows.

Definition 1.1. Let $T \in (0,\infty]$. A pair (u,v) of nonnegative and radially symmetric functions defined on $\Omega\times(0,T)$ is called a *moment solution* of (1.1) on $[0,T)$ if

(i) $u \in C^0_{\mathrm{w}-\star}([0,T);L^\infty(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,T);L^\infty(\Omega))$,
$u^m \in L^2(0,T;H^1(\Omega))$ if $T<\infty$; $u^m \in L^2_{\mathrm{loc}}([0,T);H^1(\Omega))$ if $T=\infty$,
$v \in L^\infty_{\mathrm{loc}}([0,T);H^1(\Omega))$,

(ii) for all $\varphi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ (a.a. $x\in\Omega$),

$$\begin{aligned} &\int_0^T\int_\Omega (\nabla u^m\cdot\nabla\varphi - \chi u\nabla v\cdot\nabla\varphi - (\lambda u - \mu u^\kappa)\varphi - u\varphi_t)\,dxdt \\ &\quad = \int_\Omega u_0(x)\varphi(x,0)\,dx, \\ &\int_0^T\int_\Omega (\nabla v\cdot\nabla\varphi + v\varphi - u\varphi)\,dxdt = 0, \end{aligned}$$

(iii) (u,v) satisfies the following moment inequality:

$$\phi(t) - \phi(0) \geq K\int_0^t \phi^2(\tau)\,d\tau \quad \text{for all } t\in(0,T),$$

where

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)\,ds \quad \text{for } t \in (0,T),$$

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\,d\rho \quad \text{for } s \in [0,R^n] \text{ and } t \in (0,T)$$

with some $s_0 \in (0,R^n)$, $\gamma \in (0,1)$ and $K = K(R,m,\chi,\mu,\kappa,\gamma,s_0) > 0$.

We next define *maximal moment solutions*, which are ensured by Zorn's lemma as in the proof of [6, Lemma 2.4].

Definition 1.2. Define the set $\mathcal{S}$ as

$$\mathcal{S} := \{(T,u,v) \mid T \in (0,\infty],\ (u,v) \text{ is a moment solution of (1.1) on } [0,T)\},$$

which is not empty as shown in the proof of Theorem 1.3, with the order relation $\preceq$ given by

$$(T_1,u_1,v_1) \preceq (T_2,u_2,v_2) \ :\Longleftrightarrow\ T_1 \le T_2,\ u_2|_{(0,T_1)} = u_1,\ v_2|_{(0,T_1)} = v_1\,.$$

Then Zorn's lemma assures some maximal element $(T_{\max},u,v) \in \mathcal{S}$, and (u,v) is called a *maximal moment solution* of (1.1) on $[0,T_{\max})$.

Now we state the main theorem, in which (1.2) is the same condition in [1, Theorem 1.2 (ii)].

Theorem 1.3. *Let $m \in \left[1, 2-\frac{2}{n}\right)$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Assume that*

$$\kappa < 1 + \min\left\{\frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)}\right\}. \tag{1.2}$$

Then for all $M_0 > 0$ and $L > 0$ there exist $\sigma_0 > 0$, $\eta_0 \in (0,M_0)$ and $r_\star \in (0,R)$ with the following property: If

$$u_0 \in L^\infty(\Omega) \text{ is nonnegative and radially symmetric} \tag{1.3}$$

and

$$\int_\Omega u_0(x)\,dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\,dx \ge M_0 - \eta_0 \tag{1.4}$$

as well as

$$u_0(x) \le L|x|^{-p} \quad \text{for a.a. } x \in \Omega\,, \tag{1.5}$$

where $p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0$, then there exists a moment solution of (1.1) on $[0,T_{\max})$ which blows up at $T_{\max} < \infty$ in the sense that

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty\,.$$

In order to prove Theorem 1.3, we will construct a moment solution. To this end, we derive a moment inequality for a solution of a problem approximate to (1.1). The key to obtaining the inequality is to establish a pointwise estimate for an approximate solution (Lemma 2.1).

2. Proof of Theorem 1.3

To show finite-time blow-up of solutions to (1.1), for the present we focus on the following approximate problem:

$$
\begin{cases}
\dfrac{\partial u_\varepsilon}{\partial t} = \Delta(u_\varepsilon+\varepsilon)^m - \chi\nabla\cdot(u_\varepsilon\nabla v_\varepsilon) + \lambda u_\varepsilon - \mu u_\varepsilon^\kappa\,, & x\in\Omega,\ t>0\,,\\
0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon\,, & x\in\Omega,\ t>0\,,\\
\dfrac{\partial u_\varepsilon}{\partial\nu} = \dfrac{\partial v_\varepsilon}{\partial\nu} = 0\,, & x\in\partial\Omega,\ t>0\,,\\
u_\varepsilon(x,0) = u_{0\varepsilon}(x)\,, & x\in\Omega\,,
\end{cases}
\tag{2.1}
$$

where $\varepsilon\in(0,1)$, and $u_{0\varepsilon} := (\rho_\varepsilon * \overline{u_0})|_{\overline{\Omega}}$ with

$$
\overline{u_0}(x) := \begin{cases} u_0(x) & \text{if } x\in\Omega\,,\\ 0 & \text{otherwise,}\end{cases}
$$

$$
\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\left(\int_{\mathbb{R}^n}\rho(y)\,dy\right)^{-1}\rho\left(\frac{x}{\varepsilon}\right)\,,\quad \rho(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x|<1\,,\\ 0 & \text{if } |x|\ge 1\,.\end{cases}
$$

We note that the solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) on $[0,T_\varepsilon)$ is obtained by a standard fixed point argument (see e.g. [11]), where T_ε is the maximal existence time for the solution $(u_\varepsilon, v_\varepsilon)$. We know that ρ_ε is nonnegative and radially symmetric. Thus, for the initial data u_0 satisfying (1.3), $u_{0\varepsilon}$ is nonnegative and radially symmetric. Moreover, we see that $u_{0,\varepsilon}\to u_0$ in $L^1(\Omega)$ as $\varepsilon\searrow 0$ and that on passing to a subsequence if necessary, $u_{0,\varepsilon}\to u_0$ a.a. $x\in\Omega$ as $\varepsilon\searrow 0$. Furthermore, as in [5, Section 2.2] and [8, Lemmas 2.2 and 2.3], we can find $T_0>0$ and $K_0>0$ such that for all $\varepsilon\in(0,1)$,

$$
T_0\le T_\varepsilon \quad\text{and}\quad \sup_{t\in(0,T_0)}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le K_0\,.
\tag{2.2}
$$

In order to establish a moment inequality, an estimate for u_ε is a cornerstone. In a degenerate Jäger–Luckhaus system with logistic source the key is radial monotonicity of an approximate solution (see [8, Lemma 2.7]). However, in our case it is difficult to obtain this property due to the structure of the second equation in (2.1). For this reason, instead of monotonicity, based on [10, Lemma 3.3] and [1, lemma 5.2], we show a pointwise estimate for u_ε.

Lemma 2.1. *Let $m\in\left[1,2-\frac{2}{n}\right)$, $\chi>0$, $\lambda>0$, $\mu>0$, $\kappa>1$, $M_0>0$ and $L>0$. Moreover, for any $\sigma_0>0$, set $p:=\frac{n(n-1)}{(m-1)n+1}+\sigma_0$ and assume that u_0 satisfies (1.3), (1.5) and $\int_\Omega u_0(x)\,dx = M_0$ and that there exist $T_0>0$ and $K_0>0$ fulfilling (2.2). Then there exist $\varepsilon_0\in(0,1)$ and $L_1>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$,*

$$
u_\varepsilon(x,t)\le L_1|x|^{-p}
\tag{2.3}
$$

for all $x\in\Omega$ and $t\in(0,T_0)$.

Proof. Putting $\widetilde{u}_\varepsilon(x,t) := e^{-\lambda t}u_\varepsilon(x,t)$, we can derive from (2.1) that

$$\begin{cases} \dfrac{\partial \widetilde{u}_\varepsilon}{\partial t} \le \nabla\cdot(m(e^{\lambda t}\widetilde{u}_\varepsilon+\varepsilon)^{m-1}\nabla\widetilde{u}_\varepsilon - \chi\widetilde{u}_\varepsilon\nabla v_\varepsilon)\,, & x\in\Omega,\ t>0,\\ (m(e^{\lambda t}\widetilde{u}_\varepsilon+\varepsilon)^{m-1}\nabla\widetilde{u}_\varepsilon - \chi\widetilde{u}_\varepsilon\nabla v_\varepsilon)\cdot\nu = 0\,, & x\in\partial\Omega,\ t>0\,,\\ \widetilde{u}_\varepsilon(x,0)=u_{0\varepsilon}(x)\,, & x\in\Omega\,. \end{cases} \tag{2.4}$$

Next, let $\sigma_0>0$. We can take $\xi>0$ small enough and $\varepsilon_0\in(0,1)$ such that $u_{0,\varepsilon}\le u_0+\xi$ for a.a. $x\in\Omega$ and all $\varepsilon\in(0,\varepsilon_0)$. By virtue of this inequality, (1.5) and the fact that $|x|\le R$, it follows that

$$u_{0,\varepsilon}\le L|x|^{-p}+\xi R^p|x|^{-p}=(L+\xi R^p)|x|^{-p} \tag{2.5}$$

for all $x\in\Omega$ and $\varepsilon\in(0,\varepsilon_0)$. Also, from the condition $\int_\Omega u_0\,dx=M_0$, we obtain that

$$\int_\Omega u_{0,\varepsilon}\,dx\le M_0+\xi|\Omega| =: \widetilde{M_0} \tag{2.6}$$

for all $\varepsilon\in(0,\varepsilon_0)$. On the other hand, integrating the first equation in (2.1) over Ω, we infer that

$$\frac{d}{dt}\int_\Omega u_\varepsilon\,dx=\lambda\int_\Omega u_\varepsilon\,dx-\mu\int_\Omega u_\varepsilon^\kappa\,dx\le\lambda\int_\Omega u_\varepsilon\,dx\,,$$

which ensures that

$$\int_\Omega u_\varepsilon\,dx\le e^{\lambda t}\int_\Omega u_{0,\varepsilon}\,dx\le e^{\lambda T_0}\widetilde{M_0} \tag{2.7}$$

for all $t\in(0,T_0)$. Moreover, we see from the second equation in (2.1) that

$$r^{n-1}(v_\varepsilon)_r=\int_0^r\rho^{n-1}v_\varepsilon\,d\rho-\int_0^r\rho^{n-1}u_\varepsilon\,d\rho\le\frac{1}{\omega_n}\left(\int_\Omega v_\varepsilon\,dx+\int_\Omega u_\varepsilon\,dx\right)$$

for all $r\in(0,R)$ and $t\in(0,T_\varepsilon)$, where $\omega_n:=n|B_1(0)|$. Here, since we integrate the second equation in (2.1) over Ω to guarantee that

$$\int_\Omega u_\varepsilon\,dx=\int_\Omega v_\varepsilon\,dx\,,$$

the above inequality and (2.7) yields

$$r^{n-1}(v_\varepsilon)_r\le\frac{2}{\omega_n}e^{\lambda T_0}\widetilde{M_0}=:c_1$$

for all $r\in(0,R)$ and $t\in(0,T_0)$. Picking $\theta_0>n$ so large satisfying $m-1>\frac{1}{\theta_0}-\frac{1}{n}$ and $p=\frac{n(n-1)}{(m-1)n+1}+\sigma_0>\frac{(n-1)}{(m-1)+\frac{1}{n}-\frac{1}{\theta_0}}$, we have

$$\begin{aligned}\int_\Omega|x|^{\theta_0(n-1)}|\nabla v_\varepsilon(x,t)|^{\theta_0}\,dx&=\omega_n\int_0^R r^{(\theta_0+1)(n-1)}|(v_\varepsilon)_r(\rho,t)|^{\theta_0}\,d\rho\\&\le\frac{1}{n}\omega_n c_1^{\theta_0}R^n\end{aligned}$$

for all $t\in(0,T_0)$. From this inequality and (2.4)–(2.6) we therefore can apply [2, Theorem 1.1] to obtain (2.3). □

We next derive a moment inequality for an approximate solution of (2.1).

Lemma 2.2. *Let* $m \in \left[1, 2 - \frac{2}{n}\right)$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ *and* $\kappa > 1$. *Assume that* (1.2) *is satisfied and that there exist* $T_0 > 0$ *and* $K_0 > 0$ *fulfilling* (2.2). *Then for all* $M_0 > 0$ *and* $L > 0$ *there exist* $\eta_0 \in (0, M_0)$ *and* $r_\star \in (0, R)$ *which satisfy the following property*: *If* u_0 *satisfies* (1.3)–(1.5) *with some* $\sigma_0 > 0$, *then there exist* $\varepsilon_0 \in (0,1)$ *and* $K > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$,

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \geq K \int_0^t \phi_\varepsilon^2(\tau)\, d\tau \tag{2.8}$$

for all $t \in (0, T_0)$, *where*

$$\phi_\varepsilon(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w_\varepsilon(s,t)\, ds \quad \textit{for } t \in (0, T_\varepsilon),$$

$$w_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_\varepsilon(\rho, t)\, d\rho \quad \textit{for } s \in [0, R^n] \textit{ and } t \in (0, T_\varepsilon)$$

with some $s_0 \in (0, R^n)$ *and* $\gamma \in (0,1)$.

Proof. Let us first put $p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0$, where we choose $\sigma_0 > 0$ sufficiently small fulfilling that $\kappa < 1 + \min\left\{\frac{n}{2p}, \frac{n-2}{p} - (m-1)\right\}$. Furthermore, we select $\gamma \in \left(\max\left\{\frac{2p\kappa}{n}, 1 - \frac{2}{n} - \frac{p}{n}(m-1)\right\}, \min\left\{2 - \frac{4}{n} - \frac{2p}{n}(m-1), 1\right\}\right)$. Also, noting that $u_{0,\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon \searrow 0$, we fix $\xi_0 > 0$ small enough and pick $\varepsilon_0 \in (0,1)$ given by Lemma 2.1 satisfying

$$\int_\Omega u_{0,\varepsilon} \geq M_0 - \xi_0$$

for all $\varepsilon \in (0, \varepsilon_0)$. In order to obtain (2.8), we shall show that there exist $c_1 > 0$, $c_2 > 0$, $\theta \in (0,2)$ and $s_1 \in (0, R^n)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $s_0 \in (0, s_1)$,

$$\phi_\varepsilon'(t) \geq c_1 s_0^{\gamma-3} \phi_\varepsilon^2(t) - c_2 s_0^{3-\gamma-\theta} \tag{2.9}$$

for all $t \in (0, T_0)$. By straightforward computations we have from (2.1) and the definitions of w_ε and ϕ_ε that

$$\begin{aligned}
\phi_\varepsilon'(t) \geq{}& mn^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)\left(n(w_\varepsilon)_s + \varepsilon\right)^{m-1}(w_\varepsilon)_{ss}\, ds \\
&+ n \int_0^{s_0} s^{-\gamma}(s_0 - s)(w_\varepsilon)_s w_\varepsilon\, ds - n \int_0^{s_0} s^{-\gamma}(s_0 - s)(w_\varepsilon)_s z_\varepsilon\, ds \\
&- n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s)\left\{\int_0^s (w_\varepsilon)_s^\kappa\, d\sigma\right\} ds
\end{aligned}$$

for all $t \in (0, T_\varepsilon)$, where $z_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v_\varepsilon(\rho, t)\, d\rho$ for $s \in [0, R^n]$ and $t \in (0, T_\varepsilon)$. Here, we note that we can apply [1, Lemmas 3.5, 3.8 and 3.9] to the second, third and fourth terms on the right-hand side of the above inequality. Thus, in order to derive (2.9), it is sufficient to estimate the first term. To this end, we will find $c_3 > 0$ independent of ε such that

$$\left(n(w_\varepsilon)_s + \varepsilon\right)^m \leq c_3 s^{-\frac{p}{n}(m-1)}(w_\varepsilon)_s + c_3 \tag{2.10}$$

for all $s \in (0, R^n)$ and $t \in (0, T_0)$, which is used after integration by parts in estimating the first term. By means of (2.3), it follows that for any $\varepsilon \in (0, \varepsilon_0)$, $w_\varepsilon(s,t) = \frac{1}{n} u_\varepsilon(s^{\frac{1}{n}}, t) \le c_4 s^{-\frac{p}{n}}$ for all $s \in (0, R^n)$ and $t \in (0, T_0)$, where $c_4 := \frac{L_1}{n}$. From this inequality and the fact that $s \le R^n$ as well as $\varepsilon < 1$, we have

$$\begin{aligned}(n(w_\varepsilon)_s + \varepsilon)^m &\le 2^{m-1}(n^m (w_\varepsilon)_s^m + \varepsilon^m)\\ &\le 2^{m-1} n^m c_4^{m-1} s^{-\frac{p}{n}(m-1)} (w_\varepsilon)_s + 2^{m-1}\end{aligned}$$

for all $s \in (0, R^n)$ and $t \in (0, T_0)$, which means that (2.10) holds. Therefore, by [1, Lemmas 3.5, 3.6 (i), 3.8, 3.9 and 3.11] we can take $c_5 > 0$, $c_6 > 0$, $\theta \in (0, 2)$ and $s_1 \in (0, R^n)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $s_0 \in (0, s_1)$,

$$\phi'_\varepsilon(t) \ge c_5 s_0^{\gamma-3} \phi_\varepsilon^2(t) - c_6 s_0^{3-\gamma-\theta}$$

for all $t \in (0, T_0)$. Furthermore, arguing as in [8, Proof of Proposition 2], we pick $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ such that for any u_0 satisfying (1.3)–(1.5), the inequality $\phi'_\varepsilon(t) \ge \frac{c_5}{2} s_0^{\gamma-3} \phi_\varepsilon^2(t)$ holds for all $\varepsilon \in (0, \varepsilon_0)$, $s_0 \in (0, s_1)$ and $t \in (0, T_0)$, which implies (2.8). □

We are now in the position to show Theorem 1.3.

Proof of Theorem 1.3. We can derive results similar to [8, Lemmas 2.4 and 2.5] since the second equation in (2.1) entails that $\Delta v_\varepsilon = v_\varepsilon - u_\varepsilon \ge -u_\varepsilon$. Thus, as in the proof of [4, Lemma 5.3] we can choose subsequence $\{u_{\varepsilon_k}\}$, $\{v_{\varepsilon_k}\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) and nonnegative functions u, v such that $u \in L^\infty(0, T_0; L^\infty(\Omega))$, $u^m \in L^2(0, T_0; H^1(\Omega))$, $v \in L^\infty(0, T_0; W^{1,\infty}(\Omega))$ and

(2.11) $\quad u_{\varepsilon_k} \to u \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega)),$

(2.12) $\quad u_{\varepsilon_k} \to u \quad \text{in } C^0([\delta, T_0]; L^q(\Omega)) \quad \text{for all } \delta \in (0, T_0) \text{ and } q \in [1, \infty)\,,$

(2.13) $\quad \nabla(u_{\varepsilon_k} + \varepsilon)^m \to \nabla u^m \quad \text{weakly in } L^2(0, T_0; L^2(\Omega))\,,$

(2.14) $\quad \nabla v_{\varepsilon_k} \to \nabla v \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega))$

as $k \to \infty$. Moreover, thanks to Lemma 2.2, we can take the initial data u_0 leading to (2.8). Thus, by (2.11)–(2.14), we can show that (u, v) fulfills (i)–(iii) with $T = T_0$ in Definition 1.1 as in [8, Proof of Proposition 1]. Hence, from Definition 1.2 there exists a maximal moment solution (u, v) on $(0, T_{\max})$. In particular, we have

$$\phi(t) - \phi(0) \ge K \int_0^t \phi^2(\tau)\, d\tau$$

for all $t \in (0, T_{\max})$ with some $K > 0$. Putting $\Phi(t) := \int_0^t \phi^2(\tau)\, d\tau + \frac{\phi(0)}{K}$ for $t \in (0, T_{\max})$, we see that $\Phi \in C^0([0, T_{\max}) \cap C^1((0, T_{\max}))$ and from the above inequality that $\Phi'(t) \ge K^2 \Phi^2(t)$ for all $t \in (0, T_{\max})$, which yields

$$t \le \frac{1}{K^2}\left(-\frac{1}{\Phi(t)} + \frac{1}{\Phi(0)}\right) \le \frac{1}{K^2 \Phi(0)}$$

for all $t \in (0, T_{\max})$. This proves $T_{\max} \le \frac{1}{K^2\Phi(0)} < \infty$. By an extension argument as in [8, Proof of Theorem 1.1] we can obtain $\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$, which concludes the proof. □

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References

[1] Black, T., Fuest, M., Lankeit, J., *Relaxed parameter conditions for chemotactic collapse in logistic-type parabolic-elliptic Keller-Segel systems*, Z. Angew. Math. Phys. **72** (96) (2021), 23 pp.

[2] Fuest, M., *Blow-up profiles in quasilinear fully parabolic Keller-Segel systems*, Nonlinearity **33** (5) (2020), 2306–2334.

[3] Fuest, M., *Approaching optimality in blow-up results for Keller-Segel systems with logistic-type dampening*, NoDEA Nonlinear Differential Equations Appl. **28** (16) (2021), 17 pp.

[4] Ishida, S., Yokota, T., *Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type*, J. Differential Equations **252** (2) (2012), 1421–1440.

[5] Ishida, S., Yokota, T., *Blow-up in finite or infinite time for quasilinear degenerate Keller–Segel systems of parabolic–parabolic type*, Discrete Contin. Dyn. Syst. Ser. B **18** (10) (2013), 2569–2596.

[6] Lankeit, J., *Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion*, J. Differential Equations **262** (7) (2017), 4052–4084.

[7] Tanaka, Y., *Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production*, J. Math. Anal. Appl. **506** (2022), 29 pp., no. 125654.

[8] Tanaka, Y., Yokota, T., *Finite-time blow-up in a quasilinear degenerate parabolic–elliptic chemotaxis system with logistic source and nonlinear production*, Discrete Contin. Dyn. Syst. Ser. B **28** (1) (2023), 262–286.

[9] Winkler, M., *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction*, J. Math. Anal. Appl. **384** (2) (2011), 261–272.

[10] Winkler, M., *Finite-time blow-up in low-dimensional Keller–Segel systems with logistic-type superlinear degradation*, Z. Angew. Math. Phys. **69** (69) (2018), 40 pp.

[11] Winkler, M., Djie, K.C., *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect*, Nonlinear Anal. **72** (2) (2010), 1044–1064.

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