Examples of liftings of Nichols algebras over racks *

Nicolás Andruskiewitsch and Matías Graña

September 4, 2003

Abstract

We review known examples of finite-dimensional Nichols algebras associated to racks. We discuss pointed Hopf algebras whose infinitesimal Yetter-Drinfeld module is a realization of the rack of transpositions in $\mathbb{S}_n$ with constant cocycle -1.

2000 AMS Subject Classification: 16W30
Keywords: pointed Hopf algebras

1 Introduction

A general method for the classification of Hopf algebras whose coradical is a Hopf subalgebra was proposed in [AS1], see also [AS3]. In the case of pointed Hopf algebras of finite dimension, by [G1, Lemma 3.8] (see also [AG, Thm. 4.14]), we are first faced to the following question.

Question 1.1. *For any finite rack X and for any 2-cocycle $q : X \times X \to \mathbb{C}^\times$, determine if the Nichols algebra of the braided vector space $(\mathbb{C}X, c^q)$ has finite dimension.*

Recall that the Nichols algebra of a braided vector space (V, c) is given by $\mathfrak{B}(V) = T(V)/J$ where $J = \oplus_{n\geq 2} J_n$, and where $J_n \subset T^n(V)$ is the kernel of the quantum symmetrizer; see *e. g.* [AS3]. Let $r \geq 2$; the rth. partial Nichols algebra of (V, c) is $\widehat{\mathfrak{B}}_r(V) = T(V)/\langle \oplus_{2\leq n\leq r} J_n\rangle$, see [AG, 6.3].

Recall also that a rack is a non-empty set endowed with an operation $\triangleright : X \times X \to X$ such that $\phi_i := i \triangleright \_$ is a bijection for all $i \in X$, and the self-distributivity axiom holds: $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$, for all $i, j, k \in X$. The braiding $c^q : \mathbb{C}X \otimes \mathbb{C}X \to \mathbb{C}X \otimes \mathbb{C}X$, on the vector space $\mathbb{C}X$ with a basis $(x_i)_{i\in X}$, is given by $c(x_i \otimes x_j) = q_{ij}\, x_{i\triangleright j} \otimes x_i$, for all $i, j \in X$; it is a solution of the braid equation by the cocycle condition $q_{i\triangleright j,i\triangleright k}q_{i,k} = q_{i,j\triangleright k}q_{j,k}$ $\forall i, j, k \in X$. See [AG, Ch. 1] for a survey on racks focused on our needs for classification of pointed Hopf algebras. Nichols algebras corresponding to a trivial rack ($\phi_i = \mathrm{id}$ $\forall i \in X$) were considered in several articles, see [AS3]. The examples of racks we consider below are:

- Subsets of a group stable under conjugation; then $a \triangleright b = aba^{-1}$.
- Particularly, affine (or Alexander) racks; these are pairs (A, g) where A is a finite abelian group and $g \in \mathrm{Aut}\, A$; then $a \triangleright b = g(b) + (\mathrm{id} - g)(a)$ defines a rack structure on A. If A is a finite ring and g is multiplication by an invertible $N \in A$, we shall denote $(A, N) := (A, g)$.

We collect the main examples of Nichols algebras which are known to be finite-dimensional; we exclude the diagonal case (= the rack is trivial). Other examples are in [AG, Prop. 6.8]; they are not of diagonal type but they arise from Nichols algebras of diagonal type by a kind of Fourier transform.

*This work was partially supported by CONICET, Agencia Córdoba Ciencia, ANPCyT and Secyt (UNC). N. A. thanks the kind invitation of R. Berger and LARAL to the University of St. Etienne, where his work was done. M. G. thanks the warm hospitality of IRMA and ULP, Strasbourg, where his work was done.

Theorem 1.2. *In all the examples below we take $q_{ij} = -1$ for all i, j.*

(i). *Let X be the set of transpositions in $\mathbb{S}_n$, $n = 3$, resp. $4, 5$. Then the Nichols algebra of the corresponding braided vector space $(V, c) = (\mathbb{C}X, c^q)$ is quadratic and has dimension 12, resp. $576, 8294400$. Its Hilbert polynomial has degree 4, resp. $12, 40$. The space J_2 of relations in degree 2 has a basis*

$$\begin{aligned} &x_\sigma^2 && \forall \sigma \in X, \\ &x_\sigma x_\tau + x_\tau x_\sigma && \forall \sigma \neq \tau \in X \ s.t.\ \sigma\tau = \tau\sigma, \\ &x_\sigma x_\tau + x_\nu x_\sigma + x_\tau x_\nu && \forall \sigma \neq \tau \neq \nu \in X \ s.t.\ \sigma\tau = \nu\sigma, \end{aligned}$$

(consider only one vector for each pair σ, τ or triple σ, τ, ν).

(ii). *Let $(A, g) = (\mathbb{F}_p, N)$ where $p = 3, 4, 5, 7$, $N = 2, \omega, 2, 3$ respectively (here $\omega^2 + \omega + 1 = 0 \in \mathbb{F}_4$). Then:*

- *The Nichols algebra of the corresponding braided vector space $(V, c) = (\mathbb{C}X, c^q)$ has dimension $p\varphi(p)(p-1)^{p-2}$ (here φ is the Euler function).*
- *Its Hilbert polynomial has degree $(p-1)^2$.*
- *The defining ideal J is generated by $J_2 + J_{v(p)(p-1)}$ (here $v(p) = 2$ for $p = 4$ and $v(p) = 1$ otherwise).*

Furthermore, J_2 has a basis (we write subindices in $\mathbb{F}_p$)

$$\begin{aligned} &x_i^2, && \textit{always} && (1) \\ &x_i x_j + x_{-i+2j} x_i + x_j x_{-i+2j} && \textit{for } p = 3, && (2) \\ &x_i x_j + x_{(\omega+1)i+\omega j} x_i + x_j x_{(\omega+1)i+\omega j} && \textit{for } p = 4, && (3) \\ &x_i x_j + x_{-i+2j} x_i + x_{3i-2j} x_{-i+2j} + x_j x_{3i-2j} && \textit{for } p = 5, && (4) \\ &x_i x_j + x_{-2i+3j} x_i + x_j x_{-2i+3j} && \textit{for } p = 7, && (5) \end{aligned}$$

(consider only one of repeated vectors). The relations in $J_{v(p)(p-1)}$ are generated by J_2 and the independent relation

$$\begin{aligned} &x_\omega x_1 x_0 x_\omega x_1 x_0 + x_1 x_0 x_\omega x_1 x_0 x_\omega + x_0 x_\omega x_1 x_0 x_\omega x_1 && \textit{for } p = 4, && (6) \\ &x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1 && \textit{for } p = 5, && (7) \\ &x_2 x_1 x_0 x_2 x_1 x_0 + x_1 x_0 x_2 x_1 x_0 x_2 + x_0 x_2 x_1 x_0 x_2 x_1 && \textit{for } p = 7. && (8) \end{aligned}$$

(iii). *The Nichols algebra of the rack defined by the faces of the cube is quadratic and has the same dimension and Hilbert series as the Nichols algebra corresponding to the transpositions in $\mathbb{S}_4$ (see [AG, Thm. 6.12]).*

Proof. The case of the transpositions in $\mathbb{S}_3$ (which is equal to $(\mathbb{F}_3, 2)$) and the transpositions in $\mathbb{S}_4$ are in [MS]. In the case of the transpositions in $\mathbb{S}_5$, the quadratic approximation $\widehat{\mathfrak{B}}_2(V)$ was computed with the help of the computer program `bergman`. The equality $\widehat{\mathfrak{B}}_2(V) = \mathfrak{B}(V)$ was established in [G2] with a program in C there called `Deriva`, using [AG, Th. 6.4]. An algebra over the same rack but with a different cocycle appeared in [FK]. Both algebras have the same Hilbert polynomial, a coincidence also found in [MS] for $\mathbb{S}_3$ and $\mathbb{S}_4$. The case of $(\mathbb{F}_4, \omega)$ is treated in [G1]. The cases of $(\mathbb{F}_5, 2)$ and the faces of the cube appear in [AG]. The Nichols algebra corresponding to $(\mathbb{F}_7, 3)$ was computed with `bergman` and `Deriva`, see [G2]. □

Remark 1.3. *The cases $X = (\mathbb{F}_p, N)$ with $N = 3, 5$ for $p = 5, 7$ respectively are analogous to those stated here. In fact, these are the duals to those in the theorem. The other Nichols algebras in the theorem are self-dual.*

In the rest of this Note, we consider the next steps of the Lifting method in the context of the Nichols algebras in 1.2.

2 Generation in degree one

Theorem 2.1. *Any finite dimensional complex pointed Hopf algebra such that its infinitesimal braiding (see [AS3, Def. 1.15]) is one of those in 1.2 is generated as an algebra by its group-like and skew-primitive elements.*

This result is in agreement with the Conjecture [AS2, Conj. 1.4].

Proof. We follow the lines of proof of an analogous statement in [AS4, Thm. 7.6]. It is equivalent to prove that a finite dimensional connected graded braided Hopf algebra $R = \oplus_{n\geq 0} R(n)$ with braiding c which is generated in degree 1, and such that $(R(1), c|_{R(1)\otimes R(1)})$ is dual to one of the braided vector spaces in 1.2, must be a Nichols algebra. Thus, let R be such an algebra with $R(1) = V$. We must prove that all the relations of $\mathfrak{B}(V)$ hold in R. It is straightforward to see that in all the examples under consideration the elements in J_2 are primitive, and if $r \in J_2$ is one of the relations in the statement of 1.2 then $c(r \otimes r) = r \otimes r$. This implies that $r = 0$, or otherwise R would be infinite dimensional. We have to deal then with the relations in J_4 and J_6. But those relations are primitive elements in the algebra $T(V)/\langle J_2\rangle$, and then it is enough to prove that in $T(V)/\langle J_2\rangle$ we have $c(r \otimes r) = r \otimes r$, r in J_4 or J_6. Again, this is straightforward. $\square$

3 Liftings

3.1 The problem

Let X be a finite rack, $q : X \times X \to \mathbb{C}^\times$ be a 2-cocycle and $(V, c) = (\mathbb{C}X, c^q)$. A *Y-D realization* of (V, c) over a group G is a structure of Yetter-Drinfeld module over the group algebra $\mathbb{C}G$ on V such that c coincides with the braiding in ${}^{k[G]}_{k[G]}\mathcal{YD}$ and the elements of X are G-homogeneous; *i. e.*, there exists a function $g : X \to G$, $i \mapsto g_i$, such that $\delta(i) = g_i \otimes i$, $i \in X$.
Let A be a pointed Hopf algebra, $G = G(A)$. The *infinitesimal Yetter-Drinfeld module* of A is the coinvariant part $(A_1/A_0)^{\mathrm{co}\,\mathbb{C}G}$ of the $\mathbb{C}G$-Hopf bimodule A_1/A_0. Another fundamental step in the lifting method is to address the following question.

Question 3.1. *Assume that the Nichols algebra of the braided vector space $(V, c) = (\mathbb{C}X, c^q)$ has finite dimension. Then describe all finite-dimensional pointed Hopf algebras A such that the infinitesimal Yetter-Drinfeld module of A is a Y-D realization of (V, c) over $G = G(A)$.*

See [AS3, AS4] for substantial partial answers in the case of trivial racks. Note that the related problem of describing all Y-D realizations of (V, c) presents hard computational aspects. The projection $\pi : A_1 \to A_1/A_0$ being a morphism of $\mathbb{C}G$-Hopf bimodules, we can choose a section σ of Hopf bimodules; set $a_i := \sigma(x_i\#1) \in A_1$. Then

$$a_i \quad \text{is a} \quad (g_i, 1)\text{-primitive and} \quad g_i a_j g_i^{-1} = q_{ij} a_{i\triangleright j}, \quad \forall i, j \in X. \tag{9}$$

To answer 3.1 we need to find the relations between the a_i's, which are deformations of the relations in the Nichols algebra.

3.2 Faithful Y-D realizations

Let X be a finite rack, $q : X \times X \to \mathbb{C}^\times$ be a 2-cocycle and let $(V, c) = (\mathbb{C}X, c^q)$ as above. We shall say that a Y-D realization is *faithful* if $g : X \to G$ is injective.

The following definition, with a different notation, appears in [MS, S. 5].

Definition 3.2. A *principal YD-realization* for X, q over a finite group G is a collection $(\cdot, g : X \to G, (\chi_i)_{i\in X})$, where $\cdot$ is an action of G on X, g is an equivariant function with respect to the conjugation in G and $\chi_i : G \to \mathbb{C}^\times$; such that the family $(\chi_i)_{i\in X}$ is a 1-cocycle: $\chi_i(ht) = \chi_i(t)\chi_{t.i}(h)$, for all $i \in X$, $h, t \in G$; $g_i.j = i \triangleright j$ and $\chi_i(g_j) = q_{ij}$ for all $i, j \in X$.

Lemma 3.3. *(a). If the rack X is faithful (that is, $\phi_i = \phi_j$ only for $i = j$) then any Y-D realization is faithful.*

(b). A principal Y-D realization over G defines a Y-D realization by

$$\delta(x_i) = g_i \otimes x_i, \ h.x_i = \chi_i(h) x_{h.i}, \ i \in X, \ h \in G.$$

(c). Any faithful Y-D realization arises from a unique principal Y-D realization as in (b); and G acts by rack automorphisms on X.

(d). If X is faithful and indecomposable, and q is constant, then the 1-cocycle $(\chi_i)_{i\in X}$ appearing in an arbitrary principal Y-D realization is constant: $\chi_i = \chi$ for all $i \in X$, and χ is a multiplicative character of G.

Proof. Since $g_i.x_j = q_{ij} x_{i\triangleright j}$ in any realization, (a) follows. To prove (b), the cocycle condition insures that the action is well-defined, whereas the equivariance of g insures the Yetter-Drinfeld compatibility. We now prove (c). If $i \in X$, $h \in G$ then $\delta(h.x_i) = hg_ih^{-1} \otimes h.x_i$; thus there exists a unique j and a scalar $\chi_i(h)$ such that $h.x_i = \chi_i(h)x_j$; set $h.i := j$. Since G acts well on V, this defines an action of G on X and χ_i is a 1-cocycle; since $hg_ih^{-1} = g_j$, g is equivariant. Let now $i, j \in X$, $h \in G$. Then $g_{h.(i\triangleright j)} = hg_{i\triangleright j}h^{-1} = hg_ig_jg_i^{-1}h^{-1} = g_{h.i}g_{h.j}g_{h.i}^{-1} = g_{(h.i)\triangleright(h.j)}$, hence $h.(i \triangleright j) = (h.i) \triangleright (h.j)$ by faithfulness. It is finally clear that $\chi_i(g_j) = q_{ij}$ for all $i, j \in X$. We prove (d), say with $q_{ij} = \omega$ $i, j \in X$. For $h \in G$, $i, j \in X$, we compute

$$\begin{aligned}\omega &= \chi_i(g_{h.j}) = \chi_i(hg_jh^{-1}) \\ &= \chi_i(h^{-1})\chi_{h^{-1}.i}(g_j)\chi_{(g_jh^{-1}).i}(h) = \chi_i(h^{-1})\omega\chi_{j\triangleright(h^{-1}.i)}(h).\end{aligned}$$

If $k = h^{-1}.i$, this implies $\chi_{j\triangleright k}(h) = \chi_{h.k}(h^{-1})^{-1} = \chi_k(h)$. Hence $\chi_{j\triangleright k} = \chi_k$ for all $j, k \in X$; since X is indecomposable the 1-cocycle is constant, and *a fortiori* a multiplicative character of G. □

3.3 Some liftings

Let X, q and $(V, c) = (\mathbb{C}X, c^q)$ be as above. Let us fix a Y-D realization of (V, c) over a finite group G. Let K be the subgroup of G generated by the image of X; K acts by rack automophisms on X. Let $\mathcal{R}$ be the set of equivalence classes in $X \times X$ for the relation generated by $(i, j) \sim (i \triangleright j, i)$; that is, the orbits of the set-theoretical solution of the braid equation associated to the rack. Then $\mathrm{Aut}_\triangleright X$ also acts on $\mathcal{R}$. If $C \in \mathcal{R}$, then $g_C := g_ig_j$ is well-defined, *i. e.* it does not depend on the choice of $(i, j) \in C$; and $\gamma g_C\gamma^{-1} = g_{\gamma\cdot C}$ for any $\gamma \in K$; same for $\gamma \in G$ if the realization is faithful, by 3.3.
Assume that $q_{ij} = -1$, $i, j \in X$. Then the set of relations $\sum_{(i,j)\in C} x_ix_j = 0$, $C \in \mathcal{R}$, is a basis of the space J_2 of degree-2 relations in $\mathfrak{B}(V)$. The lifting of these relations is given by the following result.

Lemma 3.4. *Let A be a finite dimensional pointed Hopf algebra, $G = G(A)$, whose infinitesimal Yetter-Drinfeld module is a realization of (V, c). Let (a_i) be as in (9). Then there exists $\lambda_C \in \mathbb{C}$, for all $C \in \mathcal{R}$, normalized by*

$$\lambda_C = 0 \qquad \text{if } g_C = 1, \tag{10}$$

$$\lambda_C = \lambda_{\gamma\cdot C}, \quad \forall\gamma \in K \ (\text{or } \forall\gamma \in \mathrm{Int}_\triangleright X), \tag{11}$$

such that the following relations hold in A:

$$\sum_{(i,j)\in C} a_ia_j = \lambda_C(1 - g_C), \quad C \in \mathcal{R}. \tag{12}$$

If X is faithful and indecomposable and χ is as in 3.3 (d), then

$$\chi^2(h)\lambda_C = \lambda_{h\cdot C}, \quad \forall h \in G. \tag{13}$$

Proof. Let $C \in \mathcal{R}$. It is straightforward to see that $b_C := \sum_{(i,j)\in C} a_i a_j$ is a $(g_C, 1)$-primitive. Thus $b_C = \lambda_C(g_C - 1) + \sum_{i\in X \text{ s.t. } g_i = g_C} \lambda_i a_i$. But for all $i \in X$, g_i acts on V in the basis X by -1 times a permutation matrix, while g_C acts as a permutation matrix in the same basis. Thus, $g_i \neq g_C \forall i \in X$ and we get (12). The relations (11), resp. (13), follow by applying $\operatorname{ad}\gamma$, resp. $\operatorname{ad} h$, to both sides of (12). □

Example 3.5. Let X be the rack of transpositions in $\mathbb{S}_n$, $n \geq 3$. The classes in $\mathcal{R}$ have either 1, 2 or 3 elements; $\operatorname{Int}_\triangleright X$ permutes the classes with the same cardinality, so we have only scalars $\lambda_1, \lambda_2, \lambda_3$, the index corresponding to the cardinality; no λ_2 for $\mathbb{S}_3$. This notation is used in (16), (17), (18) below.

Example 3.6. Let $X = (\mathbb{F}_3, 2)$, $q_{ij} = -1$, $i, j \in X$, and $G = \mathbb{F}_3 \rtimes \Gamma$, where Γ is a cyclic group of order $2P$, denoted multiplicatively with a generator u; and u acts on $\mathbb{F}_3$ by -1. Let t be odd, $1 \leq t \leq P$ and set $g_i := (i, u^t)$, $i \in \mathbb{F}_3$; let $\xi \in \mathbb{C}$ be such that $\xi^t = -1$ and let $\chi : G \to \mathbb{C}^\times$ be given by $\chi(i, u^s) = \xi^s$, $0 \leq i \leq 2$, $0 \leq s < 2P$. Then G acts on X and we have a principal YD-realization for X, q. Since $g_i g_j$ is never 1, if $i \neq j$, no conclusion on λ_3 arises from (10) (we use here the notation from 3.5), but:

- $g_i^2 = 1$ only if $t = P$; in this case necessarily $\lambda_1 = 0$.
- If $h = (0, u)$ then (13) implies $\lambda_1 = \xi^2\lambda_1$ and $\lambda_3 = \xi^2\lambda_3$. Thus $\lambda_1 = \lambda_3 = 0$ if $\xi \neq -1$.

Definition 3.7. Let X be the rack of transpositions in $\mathbb{S}_n$, $3 \leq n \leq 5$, $q_{ij} = -1$, $i, j \in X$. Let G be a finite group admitting a principal Y-D realization $(\cdot, g : X \to G, \chi)$ for X, q. Let $\lambda_1, \lambda_2, \lambda_3$ be scalars satisfying (10), (13). Let $A(G, \cdot, g, \chi, \lambda_i)$ be the algebra presented by generators a_σ, $\sigma \in X$ and H_t, $t \in G$; with relations

$$
\begin{aligned}
H_t H_s &= H_{ts}, \quad H_e = 1; && (14)\\
H_t a_\sigma &= \chi(t) a_{t\cdot\sigma} H_t \qquad \forall t \in G,\ \sigma \in X; && (15)\\
a_\sigma^2 &= \lambda_1(1 - g_\sigma^2) \qquad \sigma \in X; && (16)\\
a_\sigma a_\tau + a_\tau a_\sigma &= \lambda_2(1 - g_\sigma g_\tau) \qquad \textit{for } \sigma\tau = \tau\sigma, \sigma \neq \tau; && (17)\\
a_\sigma a_\tau + a_\tau a_\nu + a_\nu a_\sigma &= \lambda_3(1 - g_\sigma g_\tau) \qquad \textit{for } \sigma\tau = \nu\sigma, \sigma \neq \tau \neq \nu. && (18)
\end{aligned}
$$

Theorem 3.8. *Assume that $n = 3, 4$. Then $A = A(G, \cdot, g, \chi, \lambda_i)$ is a pointed Hopf algebra, with the comultiplication uniquely defined by declaring the elements H_t, $t \in G$ to be group-like and a_σ to be $(g_\sigma, 1)$-primitive, $\sigma \in X$. The graded Hopf algebra associated to the coradical filtration of A is isomorphic to $\mathfrak{B}(V)\#\mathbb{C}G$; in particular,* $\dim(A) = \dim \mathfrak{B}(V)|G|$. *Conversely, any pointed Hopf algebra whose infinitesimal YD-module is a realization of* $(\mathbb{C}X, c^q)$ *is isomorphic to $A(G, \cdot, g, \chi, \lambda_i)$ for some principal Y-D realization $(\cdot, g : X \to G, \chi)$ and scalars $\lambda_1, \lambda_2, \lambda_3$ satisfying* (10), (13).

Proof. A routinary computation shows that the comultiplication is well-defined and admits a counit given by $\varepsilon(H_t) = 1$, $\varepsilon(a_\sigma) = 0$. The antipode is the unique algebra map $\mathcal{S} : A \to A^{\mathrm{op}}$ such that $\mathcal{S}(H_t) = H_{t^{-1}}$, $\mathcal{S}(a_\sigma) = -g_\sigma^{-1} a_\sigma$. We next check that dim A is the desired one. With the help of a computer program we found the Gröbner basis the of ideal generated by the relations when $n = 4$; the case $n = 3$ is easier and we omit it. We write $a > b > c > d > e > f$ for $(01), (02), (03), (12), (13), (23)$ and α, β, ϵ for $\lambda_2, \lambda_3, \lambda_1$ respectively. With this order, the basis is given by the relations (15) together with

$$
\begin{aligned}
&bab - aba - \beta b + \beta a = 0, \qquad cac - aca - \beta c + \beta a = 0,\\
&cbc - bcb - \beta c + \beta b = 0, \qquad ede - ded - \beta e + \beta d = 0,\\
&cabc + bcab + abca + \alpha cb - \beta ca - \beta bc + \alpha ba + \alpha ac - \beta ab + \epsilon\beta g_f^2\\
&\qquad - (\beta - \alpha)\beta g_f^3 g_d - (\epsilon + \alpha - \beta)\beta = 0,\\
&cbac + bacb + acba - \beta cb + \alpha ca + \alpha bc - \beta ba - \beta ac + \alpha ab + \epsilon\beta g_f^2\\
&\qquad - (\beta - \alpha)\beta g_f^3 g_e - (\epsilon + \alpha - \beta)\beta = 0,
\end{aligned}
$$

$$
\begin{aligned}
&cabaca + bcabac + \epsilon^2 abg_f^4 - \beta caba - \beta bcab - \alpha bacb - \beta baca - \alpha acba \\
&\quad - \beta abca - \beta abac - (\beta-\alpha)\beta cbg_f^3 g_d - \epsilon\alpha cag_f^2 - \epsilon\alpha bcg_f^2 + \epsilon\beta acg_f^2 \\
&\quad - 2\epsilon^2 abg_f^2 + (\beta^2 - \alpha^2 + \alpha\epsilon)ca - (\alpha-\epsilon)\alpha bc + \beta^2 ba + (\epsilon^2 - \alpha^2 + \beta^2)ab \\
&\quad + (\beta-\epsilon)\beta ac + (\beta-\alpha)\beta^2 g_f^3 g_d + (\beta-\alpha)\alpha\beta g_f^3 g_e - \epsilon^2\beta g_f^4 \\
&\quad + (2\epsilon-\beta)\epsilon\beta g_f^2 - ((\epsilon\beta) + (\beta-\epsilon)^2 - \alpha^2)\beta = 0, \\
&cabacb + acabac - \beta caba - \alpha bcab - \beta bacb - \beta acba - \beta acab - \alpha abca \\
&\quad - \beta abac - (\beta-\alpha)\beta cag_f^3 g_e + \epsilon^2 bag_f^4 - \epsilon\alpha cbg_f^2 + \epsilon\beta cag_f^2 - 2\epsilon^2 bag_f^2 \\
&\quad - \epsilon\alpha acg_f^2 - (\alpha-\epsilon)\alpha cb + (\beta-\epsilon)\beta ca + (\epsilon^2 - \alpha^2 + \beta^2)ba \\
&\quad + (\beta^2 - \alpha^2 + \alpha\epsilon)ac + \beta^2 ab + (\beta-\alpha)\alpha\beta g_f^3 g_d + (\beta-\alpha)\beta^2 g_f^3 g_e \\
&\quad - \epsilon^2\beta g_f^4 + (2\epsilon-\beta)\epsilon\beta g_f^2 - ((\epsilon\beta) + (\beta-\epsilon)^2 - \alpha^2)\beta = 0.
\end{aligned}
$$

If $\epsilon = \alpha = \beta = 0$, the leading terms of the relations remain unchanged; thus $\dim A = \dim(\mathfrak{B}(V)\#\mathbb{C}G)$. One concludes that $\operatorname{gr} A \simeq \mathfrak{B}(V)\#\mathbb{C}G$. The converse follows from 2.1, 3.3 (d) and the first part of the Theorem. □

The determination of isomorphisms between different Hopf algebras of this type can be easily performed with available techniques, see *e. g.* [AS3]. Under suitable choices of YD-data, new infinite families of non-isomorphic Hopf algebras of the same dimension are obtained.

References

[AG] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math., to appear. Also in `arXiv/0202084`.

[AS1] N. Andruskiewitsch and H.-J. Schneider, *Lifting of Quantum Linear Spaces and Pointed Hopf Algebras of order p^3*, J. Algebra **209** (1998), 658–691.

[AS2] N. Andruskiewitsch and H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), 1–45.

[AS3] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf Algebras*, in "New directions in Hopf algebras", 1–68, Math. Sci. Res. Inst. Publ. **43**, Cambridge Univ. Press, Cambridge, 2002.

[AS4] N. Andruskiewitsch and H.-J. Schneider, *Finite quantum groups over abelian groups of prime exponent*, Ann. Sci. Ec. Norm. Super. **35** (2002), 1–26.

[FK] S. Fomin and A.N. Kirillov, *Quadratic algebras, Dunkl elements, and Schubert calculus. (English)*, Prog. Math. 172, 147-182 (1999).

[G1] M. Graña, *On Nichols algebras of low dimension*, in New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math. **267** (2000), 111–134.

[G2] M. Graña, *Zoo of finite dimensional Nichols algebras of nonabelian group type*, document available at `http://mate.dm.uba.ar/~matiasg/zoo.html`.

[MS] A. Milinski and H-J. Schneider, *Pointed Indecomposable Hopf Algebras over Coxeter Groups*, Contemp. Math. **267** (2000), 215–236.

N. A.: FaMAF - Univ. Nacional de Córdoba - CIEM - CONICET - (5000) Ciudad Universitaria - Córdoba, Argentina, andrus@mate.uncor.edu
M. G.: Depto de Matemática - FCEyN - Universidad de Buenos Aires - Pab. I - Ciudad Universitaria - (1428) Buenos Aires - Argentina, matiasg@dm.uba.ar

[AMA - Algebra Montpellier Announcements - 01-2003] [September 2003]