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A HARDY–LITTLEWOOD–LIKE INEQUALITY ON TWO–DIMENSIONAL COMPACT TOTALLY DISCONNECTED SPACES

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Abstract. We prove a Hardy-Littlewood type inequality with respect to a system called Vilenkin-like system (which is a common generalisation of several well-known systems) in the two-dimensional case.

1. Introduction

Let $\mathbf{P}$ denote the set of positiv integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. For any set E let E^2 the cartesian product $E \times E$. Thus $\mathbf{N}^2$ is the set of integral lattice points in the first quadrant.

Let $m = (m_0, m_1, ..., m_k, ...)$ $(2 \leq m_k \in \mathbf{N}, k \in \mathbf{N})$ be a sequence of natural numbers and denote by G_{m_k} a set of which the number of elements is m_k. A measure on G_{m_k} is given in the way that $\mu_k(\{j\}) := \frac{1}{m_k}$ $(j \in G_{m_k}, k \in \mathbf{N})$. Let the topology be the discrete topology on the set G_{m_k}. Let G_m be the complete direct product of the compact spaces G_{m_k} $(k \in \mathbf{N})$ with the product of the topologies and measures (μ). The elements of G_m are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in G_{m_k} (k \in \mathbf{N})$. G_m is called a Vilenkin space. The Vilenkin space G_m is said to be bounded Vilenkin space if the generating sequence m is a bounded one. In this paper the boundedness of G_m is supposed. A base of the neighborhoods can be given in the following way:

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m : y = (x_0, ..., x_{n-1}, y_n, ...)\} \quad (x \in G_m, n \in \mathbf{P}),$$

$I_n := I_n(0)$ for $n \in \mathbf{N}$. Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the coresponding norms) $(1 \leq p \leq \infty)$, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)$ $(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbf{N})$.

If we define the sequence $(M_k : k \in \mathbf{N})$ by $M_0 := 1$ and $M_k := m_0 m_1 ... m_{k-1}$ $(k \in \mathbf{P})$ then each $n \in \mathbf{N}$ has a uniqe representation of the form $n = \sum_{k=0}^{\infty} n_k M_k$, where $0 \leq n_k < m_k$ $(n_k \in \mathbf{N})$. Let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ (that is $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} := \sum_{j=k}^{\infty} n_k M_k$.

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Now, introduce an orthonormal system on G_m called Vilenkin-like system (see [G]). The complex valued functions $r_k^n : G_m \to \mathcal{C}$ are called generalised Rademacher functions if the following properties holds:

i. r_k^n is $\mathcal{A}_{k+1}$ measurable, $r_k^0 = 1$ for all $k, n \in \mathbf{N}$.

ii. If M_k is a divisor of n, l and $n^{k+1} = l^{k+1}$ $(n, lk \in \mathbf{N})$, then

$$E_k(r_k^n \overline{r_k^l}) = \begin{cases} 1 & \text{if } n_k = l_k \\ 0 & \text{if } n_k \neq l_k, \end{cases}$$

where $\overline{z}$ is the complex conjugate of z.

iii. If M_k is a divisor of n (that is $n = n_k M_k + n_{k+1} M_{k+1} + ... + n_{|n|} M_{|n|}$), then

$$\sum_{n_k=0}^{m_k-1} |r_k^n(x)|^2 = m_k$$

for all $x \in G_m$.

iv. There exists a $\delta > 1$ for which $\|r_k^n\|_\infty \leq \sqrt{\frac{m_k}{\delta}}$.

Now we define the Vilenkin-like system $\psi = \{\psi_n : n \in \mathbf{N}\}$ by

$$\psi_n := \prod_{k=0}^{\infty} (r_k)^{n^{(k)}} \quad (n \in \mathbf{N}).$$

The notation of Vilenkin-like systems is due to Gát [G]. We remark that ψ is an orthonormal system and some well-known systems are Vilenkin-like systems (e.g. the Vilenkin system, the Walsh system and the UDMD product system) (see [G], [SWS], [V], [SW], [GT]).

Suppose that the sequences $m = (m_0, m_1, ..., m_k, ...)$ and $\tilde{m} = (\tilde{m}_0, \tilde{m}_1, ..., \tilde{m}_k, ...)$ are bounded.

The Kronecker product $\{\psi_{n,m} : \ n, m \in \mathbf{N}\}$ of two Vilenkin-like systems $\{\psi_n : \ n \in \mathbf{N}\}$ and $\{\tilde{\psi}_n : \ n \in \mathbf{N}\}$ is said to be the two-dimensional Vilenkin-like system. Thus

$$\psi_{n,m}(x, y) := \psi_n(x) \tilde{\psi}_m(y),$$

where $x \in G_m, \ y \in G_{\tilde{m}}$.

For a function f in $L^1(G_m)$ the Fourier coefficients, the partial sums of the Fourier series, the Diriclet kernels are defined as follows.

$$\hat{f}(n) := \int_{G_m} f \overline{\psi_n} d\mu, \ S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbf{P}, \ S_0 f := 0)$$

$$D_n(y, x) := \sum_{k=0}^{n-1} \psi_k(y) \overline{\psi_k(x)} \quad (n \in \mathbf{P}, \ D_0 := 0).$$

It is well known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y,x) d\mu(x) \quad (x \in G_m,\ n \in \mathbf{N}).$$

If $f \in L^1(G_m \times G_{\tilde{m}})$ then the (n,k)-th Fourier coefficients, the (n,k)-th partial sum of double Fourier series are the following.

$$\hat{f}(n,k) := \int_{G_m \times G_{\tilde{m}}} f \overline{\psi_{n,k}},\ S_{n,k} f := \sum_{j=0}^{n-1} \sum_{l=0}^{k-1} \hat{f}(j,l) \psi_{j,l},$$

It is simple to show that, in case $f \in L^1(G_m \times G_{\tilde{m}})$,

$$S_{n,k} f(x,y) = \int_{G_m} \int_{G_{\tilde{m}}} f(t,u) D_n(x,t) \tilde{D}_k(y,u) d\mu(t) d\mu(u).$$

Let $\tilde{I}_n(x)$ $(x \in G_{\tilde{m}})$ denote the n-th intervals generated by $\tilde{m}$, that is

$$\tilde{I}_0(x) := G_{\tilde{m}},\ \tilde{I}_n(x) := \{y \in G_{\tilde{m}} : y = (x_0, ..., x_{n-1}, y_n, ...)\} \ (x \in G_{\tilde{m}},\ n \in \mathbf{P}).$$

Define $\tilde{n} = \tilde{n}(n) := \min(l \in \mathbf{N} : M_n \leq \tilde{M}_l)$. Then there exists a constant c for which $M_n \leq \tilde{M}_{\tilde{n}} < c M_n$ for all $n \in \mathbf{N}$ (c does not depend on n, but do depends on $\max_{j \in \mathbf{N}} m_j$ and $\max_{n \in \mathbf{N}} \tilde{m}_j$). The atomic decomposition is a useful characterisation of Hardy spaces, to show this let us introduce the concept of an atom.

A function $a \in L^\infty(G_m \times G_{\tilde{m}})$ is said to be an atom if there exit a rectangle $I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2)$ $(\mathbf{x} := (x^1, x^2) \in G_m \times G_{\tilde{m}},\ k \in \mathbf{N})$ such that

(i.) *supp* $a \subset I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2)$

(ii.) $\|a\|_\infty \leq M_k \tilde{M}_{\tilde{k}}$

(iii.) $\int_{I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2)} a = 0$.

We say that $f \in L^1(G_m \times G_{\tilde{m}})$ is an element of the Hardy space $H(G_m \times G_{\tilde{m}})$ (or in brief H), if there exists $\lambda_j \in \mathcal{C}$ $(j \in \mathbf{P})$ constants and a_j $(j \in \mathbf{P})$ atoms that $\sum_{j=1}^\infty |\lambda_j| < \infty$ and $f = \sum_{j=1}^\infty \lambda_j a_j$. Moreover, H is a Banach space with the norm $\|f\|_H := \inf(\sum_{j=0}^\infty |\lambda_j|)$ where the infimum is taken over all decompositions of f.

2. The Main result and the proof

Theorem. Let $\beta > 1$ be a constant, for all $f \in H(G_m \times G_{\tilde{m}})$ there exists a $C > 0$ constant that

$$\sum_{n=1}^\infty \sum_{\substack{m=1 \\ \frac{1}{\beta} \leq \frac{n}{m} \leq \beta}}^\infty \frac{|\hat{f}(n,m)|}{nm} \leq C \|f\|_H.$$

Proof of Theorem. Throughout this paper C will denote a constant which may vary at different occurances and may depend only on $\beta, \sup m_n$ and $\sup \tilde{m}_n$.

Since $f \in H(G_m \times G_{\tilde{m}})$, f can be written in the form $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where a_j are atoms and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$.

$$\sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ \frac{1}{\beta} \le \frac{n}{m} \le \beta}}^{\infty} \frac{|\hat{f}(n,m)|}{nm} = \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ \frac{1}{\beta} \le \frac{n}{m} \le \beta}}^{\infty} \frac{|\sum_{j=1}^{\infty} \lambda_j \hat{a}_j(n,m)|}{nm} \le \sum_{j=1}^{\infty} |\lambda_j| \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ \frac{1}{\beta} \le \frac{n}{m} \le \beta}}^{\infty} \frac{|\hat{a}_j(n,m)|}{nm}.$$

Because of this the only thing we need to prove is that for an arbitrary atom a one has

$$\sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ \frac{1}{\beta} \le \frac{n}{m} \le \beta}}^{\infty} \frac{|\hat{a}(n,m)|}{nm} \le C.$$

Let $I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2)$ be an interval for which (i), (ii) and (iii) hold. Thus

$$\hat{a}(n,m) = \int_{I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2)} a(x,y) \overline{\psi_{n,m}(x,y)}.$$

If $0 \le n < M_k$ and $0 \le m < \tilde{M}_{\tilde{k}}$ then $\psi_{n,m}(x,y) = \psi_n(x) \tilde{\psi}_m(y)$ is constant on the set $I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2)$. Consequently, $\hat{a}(n,m) = 0$ and

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ \frac{1}{\beta} \le \frac{n}{m} \le \beta}}^{\infty} \frac{|\hat{a}(n,m)|}{nm} &\le \sum_{n=1}^{M_k-1} \sum_{m=1}^{\tilde{M}_{\tilde{k}}-1} \frac{|\hat{a}(n,m)|}{nm} + \sum_{n=M_k}^{\infty} \sum_{m=\tilde{M}_{\tilde{k}}}^{\infty} \frac{|\hat{a}(n,m)|}{nm} \\ &+ \sum_{n=1}^{M_k-1} \sum_{\substack{m=\tilde{M}_{\tilde{k}} \\ \frac{1}{\beta} \le \frac{n}{m} \le \beta}}^{\infty} \frac{|\hat{a}(n,m)|}{nm} + \sum_{n=M_k}^{\infty} \sum_{\substack{m=1 \\ \frac{1}{\beta} \le \frac{n}{m} \le \beta}}^{\tilde{M}_{\tilde{k}}-1} \frac{|\hat{a}(n,m)|}{nm} \\ &=: 0 + \sum_1 + \sum_2 + \sum_3. \end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality and Bessel's inequality,

$$\sum_1 \le \sqrt{\sum_{n=M_k}^{\infty} \sum_{m=\tilde{M}_{\tilde{k}}}^{\infty} |\hat{a}(n,m)|^2} \sqrt{\sum_{n=M_k}^{\infty} \sum_{m=\tilde{M}_{\tilde{k}}}^{\infty} \frac{1}{(nm)^2}} \le \|a\|_2 \sqrt{\sum_{n=M_k}^{\infty} \sum_{m=\tilde{M}_{\tilde{k}}}^{\infty} \frac{1}{(nm)^2}}.$$

Using the properties of the atoms we have

$$\|a\|_2 = \sqrt{\int_{I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2)} |a|^2} \le \sqrt{M_k^2 \tilde{M}_{\tilde{k}}^2 \mu(I_k(x^1) \times \tilde{I}_{\tilde{k}}(x^2))} = \sqrt{M_k \tilde{M}_{\tilde{k}}}.$$

Notice that $\sum_{k=n}^{m}\frac{1}{k^2}\le\frac{2}{n}-\frac{2}{m}$. From this

$$\sum_{1}\le\sqrt{M_k\tilde{M}_{\tilde{k}}}\sqrt{\frac{2}{M_k\tilde{M}_{\tilde{k}}}}\le C.$$

Discuss $\sum\limits_{2}$.

If $M_k<\frac{\tilde{M}_{\tilde{k}}}{\beta}$ then $\sum\limits_{2}=0$. Consequently we have $M_k\ge\frac{\tilde{M}_{\tilde{k}}}{\beta}$. From Cauchy-Buniakovski-Schwarz inequality and Bessel's inequality we have

$$\sum_{2}\le\sqrt{M_k\tilde{M}_{\tilde{k}}}\sqrt{\sum_{n=1}^{M_k-1}\sum_{\substack{m=\tilde{M}_{\tilde{k}}\\ \frac{1}{\beta}\le\frac{n}{m}\le\beta}}^{\infty}\frac{1}{(nm)^2}}.$$

$$\sum_{n=1}^{M_k-1}\sum_{\substack{m=\tilde{M}_{\tilde{k}}\\ \frac{1}{\beta}\le\frac{n}{m}\le\beta}}^{\infty}\frac{1}{(nm)^2}\le\sum_{[\frac{\tilde{M}_{\tilde{k}}}{\beta}]}^{M_k-1}\sum_{m=\tilde{M}_{\tilde{k}}}^{[\beta M_k]}\frac{1}{(nm)^2}\le C\sum_{l=[\frac{\tilde{M}_{\tilde{k}}}{\beta}]\tilde{M}_{\tilde{k}}}^{[\beta M_k]M_k}\frac{1}{l^2}\le\frac{C}{\tilde{M}_{\tilde{k}}^2}.$$

The definition of $\tilde{k}$ implies $\sum\limits_{2}\le C$.

$\sum\limits_{3}\le C$ can be proved in the similar way as we have done in case $\sum\limits_{2}$.

This completes the proof. □

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