*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**21** (2005), 101–106
www.emis.de/journals
ISSN 1786-0091

# TWO DIMENSION LEGENDRE WAVELETS AND OPERATIONAL MATRICES OF INTEGRATION

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Abstract. The one dimension Legendre wavelets is a numerical method for solving one dimension variational problems and integral equations. In this paper we introduce two dimensions Legendre wavelets. These wavelets are defined over the interval $[0,1] \times [0,1]$ and an orthonormal set over this interval. The integration of the product of two dimensions Legendre wavelets over $[0,1] \times [0,1]$ is equal one. In the paper section we compute operational matrices of integration for two dimensions Legendre wavelets. These operational matrices are suitable tools for two dimensions problems. Two dimensions Legendre wavelets are a numerical method for solving two dimensions variational problems.

## 1. Introduction

The wavelet basis is constructed from a single function, called the mother wavelet. These basis functions are called wavelets and they are an orthonormal set. One of the most important wavelets are Legendre wavelets. The Legendre wavelets is constructed from Legendre polynomials and form a basis wavelet for $L^2(R)$ over [0,1]. The Legendre polynomials satisfy the Legendre differential equation and where its property is covered in many mathematical textbooks[1]. In the past ten years special attention has been given to applications of wavelets. For example,[8],[6],[7] are the direct method for solving one dimensional variational problems, [4], [3] and [5] are applications of wavelets in Scattering calculation, mathematical physics and definite integrals respectively. The main characteristic of Legendre wavelets in variational problems is that it reduces the variational problems to a system of algebraic equation.

In this paper, we introduce two dimensions Legendre wavelets. Two dimensions Legendre wavelets forms a wavelet basis for $L^3(R)$ over interval $[0,1] \times [0,1]$. They are suitable tools for solving two dimensional variational problems and other two dimension problems.

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2000 *Mathematics Subject Classification.* 65T60, 42C40.
*Key words and phrases.* Legendre wavelets, numerical integration, orthogonal set.
This work has been supported by the Bu-Ali Sina University research council.

## 2. One dimension Legendre wavelets

The function $\psi(x) \in L^2(R)$ is a mother wavelet and the $\psi_{u,v}(x) = |u|^{-\frac{1}{2}}\psi(\frac{x-v}{u})$, in which $u, v \in R$ and $u \neq 0$, is a family continuous wavelets. If we choose the dilation parameter $u = a^{-n}$ and the translation parameter $v = ma^{-n}b$, where $a > 1, b > 0$ and $n$ and $m$ positive integer, we have the discrete orthogonal wavelets set: $\{\psi_{n,m}(x) = |a|^{\frac{n}{2}}\psi(a^n x - mb) : m, n \in Z\}$ [2].

The Legendre wavelets is constructed from Legendre function. The Legendre functions satisfy the Legendre differential equation [1]. One dimension Legendre wavelets over the interval $[0, 1]$ defined as:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{(m+\frac{1}{2})}2^{\frac{k}{2}}P_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

in which $n = 1, 2, \ldots, 2^{k-1}, m = 0, 1, 2, \ldots, M-1$. In (1) $P_m$'s are ordinary Legendre functions of order $m$ defined over the interval [-1,1]. Legendre wavelets is an orthonormal set as

$$\int_0^1 \psi_{n,m}(x)\psi_{n',m'}(x)dx = \delta_{n,n'}\delta_{m,m'} \tag{2}$$

The function $f(x) \in L^2(R)$ defined over [0,1] may be expanded as

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \tag{3}$$

Razzaghi and Yousefi in [8] calculate the operational matrix of integration for Legendre wavelets and they has applied Legendre wavelets for solving variational problems [6], [7].

## 3. Two dimension Legendre wavelets

We defined two dimensions Legendre wavelets in $L^3(R)$ over the $[0, 1] \times [0, 1]$ as the form

$$\psi_{n,m,n',m'}(x, y) = \begin{cases} A_{m,m'}P_m(2^k x - 2n + 1)P_{m'}(2^{k'}y - 2n' + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ & \frac{n'-1}{2^{k'-1}} \leq y \leq \frac{n'}{2^{k'-1}}; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

in which

$$A_{m,m'} = \sqrt{(m+\frac{1}{2})(m'+\frac{1}{2})}2^{\frac{k+k'}{2}}$$

and

$$\begin{aligned} n =& 1, 2, \ldots, 2^{k-1}, n' = 1, 2, \ldots, 2^{k'-1}, \\ m =& 0, 1, 2, \ldots, M-1, m' = 0, 1, 2, \ldots, M'-1. \end{aligned}$$

Two dimensions Legendre wavelets are an orthonormal set over $[0, 1] \times [0, 1]$

$$\int_0^1 \int_0^1 \psi_{n,m,n',m'}(x, y)\psi_{n_1,m_1,n'_1,m'_1}(x, y)dxdy = \delta_{n,n_1}\delta_{m,m_1}\delta_{n',n'_1}\delta_{m',m'_1} \tag{5}$$

The function $u(x,y) \in L^3(R)$ defined over $[0,1]\times[0,1]$ may be expanded as

$$u(x,y) = X(x)Y(y) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=0}^{M'-1} c_{n,m,n',m'} \psi_{n,m,n',m'}(x,y) \tag{6}$$

where $c_{n,m,n',m'} = \int_0^1 \int_0^1 X(x)Y(y)\psi_{n,m,n',m'}(x,y)dxdy$. The truncated version of equation (6) can be expressed as the form,

$$u(x,y) = C^T \cdot \Psi(x,y). \tag{7}$$

where $C$, and $\Psi(x,y)$, are coefficients matrix and wavelets vector matrix respectively. The dimensions of those are $2^{k-1}2^{k'-1}MM' \times 1$ and given as the form

$$\begin{aligned} C =&[c_{1010}, \dots, c_{101M'-1}, c_{1020}, \dots, c_{102M'-1}, c_{1030}, \dots, c_{103M'-1}, \dots, c_{102^{k'-1}0}, \dots, \\ & c_{102^{k'-1}M'-1}, c_{1110}, \dots, c_{111M'-1}, c_{1120}, \dots, c_{112M'-1}, \dots, \\ & c_{112^{k'-1}0}, \dots, c_{112^{k'-1}M'-1}, \dots, c_{1M-12^{k'-1}0}, \dots, c_{1M-12^{k'-1}M'-1}, c_{2010}, \dots, \\ & c_{201M'-1}, c_{2020}, \dots, c_{202M'-1}, c_{2030}, \dots, c_{203M'-1}, \dots, c_{202^{k'-1}0}, c_{202^{k'-1}1}, \dots, \\ & c_{202^{k'-1}M'-1}, c_{2110}, \dots, c_{211M'-1}, c_{2120}, \dots, c_{212M'-1}, c_{2130}, \dots, \\ & c_{213M'-1}, \dots, c_{212^{k'-1}0}, c_{212^{k'-1}1}, \dots, c_{212^{k'-1}M'-1}, \dots, c_{2^{k-1}010}, \dots, \\ & c_{2^{k-1}11M'-1}, c_{2^{k-1}020}, \dots, c_{2^{k-1}02M'-1}, \dots, c_{2^{k-1}02^{k'-1}0}, \dots, c_{2^{k-1}02^{k'-1}M'-1}, \dots, \\ & c_{2^{k-1}M-12^{k'-1}0}, c_{2^{k-1}M-12^{k'-1}1}, c_{2^{k-1}M-12^{k'-1}2}, \dots, c_{2^{k-1}M-12^{k'-1}M'-1}]^T \end{aligned}$$

and in the same way for $\Psi(x,y)$

$$\begin{aligned} \Psi(x,y) =&[\psi_{1010}, \dots, \psi_{101M'-1}, \psi_{1020}, \dots, \psi_{102M'-1}, \psi_{1030}, \dots, \\ & \psi_{103M'-1}, \dots, \psi_{102^{k'-1}0}, \dots, \psi_{102^{k'-1}M'-1}, \psi_{1110}, \dots, \\ & \psi_{111M'-1}, \psi_{1120}, \dots, \psi_{112M'-1}, \dots, \psi_{112^{k'-1}0}, \dots, \psi_{112^{k'-1}M'-1}, \dots, \\ & \psi_{1M-12^{k'-1}0}, \dots, \psi_{1M-12^{k'-1}M'-1}, \psi_{2010}, \dots, \\ & \psi_{201M'-1}, \psi_{2020}, \dots, \psi_{202M'-1}, \psi_{2030}, \dots, \psi_{203M'-1}, \dots, \psi_{202^{k'-1}0}, \\ & \psi_{202^{k'-1}1}, \dots, \psi_{202^{k'-1}M'-1}, \psi_{2110}, \dots, \psi_{211M'-1}, \psi_{2120}, \dots, \psi_{212M'-1}, \\ & \psi_{2130}, \dots, \psi_{213M'-1}, \dots, \psi_{212^{k'-1}0}, \psi_{212^{k'-1}1}, \dots, \\ & \psi_{212^{k'-1}M'-1}, \dots, \psi_{2^{k-1}010}, \dots, \psi_{2^{k-1}11M'-1}, \psi_{2^{k-1}020}, \dots, \\ & \psi_{2^{k-1}02M'-1}, \dots, \psi_{2^{k-1}02^{k'-1}0}, \dots, \psi_{2^{k-1}02^{k'-1}M'-1}, \dots, \psi_{2^{k-1}M-12^{k'-1}0}, \\ & \psi_{2^{k-1}M-12^{k'-1}1}, \psi_{2^{k-1}M-12^{k'-1}2}, \dots, \psi_{2^{k-1}M-12^{k'-1}M'-1}]^T \end{aligned}$$

The integration of the product of two Legendre wavelet function vectors is obtained as

$$\int_0^1 \int_0^1 \Psi(x,y)\Psi^T(x,y)dxdy = I \tag{8}$$

where $I$ is diagonal unit matrix.

3.1. **Operational matrix of integration for $y$ variable.** The integration matrix for $y$ variable define

$$\int_0^y \Psi(x,y')dy' = P_y.\Psi(x,y). \tag{9}$$

in which

$$P_y = \frac{1}{M2^{k-1}} \begin{pmatrix} P & P & P & P & \cdots & P \\ P & P & P & P & \cdots & P \\ P & P & P & P & \cdots & P \\ P & P & P & P & \cdots & P \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P & P & P & P & \cdots & P \end{pmatrix}$$

$P$ is a $2^{k'-1}M' \times 2^{k'-1}M'$ matrix and calculated in [8]. The matrix $P$ in [8] is defined as

$$P = \frac{1}{2^{k'}} \begin{pmatrix} L & F & F & F & \cdots & F \\ O & L & F & F & \cdots & F \\ O & O & L & F & \cdots & F \\ O & O & O & L & \cdots & F \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & \cdots & L \end{pmatrix}$$

in which $O$, $F$ and $L$ are $M' \times M'$ matrices. The $O$ is null matrix and $F$ and $L$ defined as

$$F = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$L = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & \cdots & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

3.2. **Operational matrix of integration for $x$ variable.** The integration matrix for $x$ variable define

$$\int_0^x \Psi(x', y)dx' = P_x.\Psi(x, y). \tag{10}$$

in which

$$P_x = \frac{1}{M'2^{k'+k-1}} \begin{pmatrix} L & F & F & F & \cdots & F \\ O & L & F & F & \cdots & F \\ O & O & L & F & \cdots & F \\ O & O & O & L & \cdots & F \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & O & \cdots & L \end{pmatrix}$$

$P_x$ is a $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ and $L$, $F$ and $O$ are $2^{k-1}MM' \times 2^{k-1}MM'$ matrices that defined as below

$$F = 2\begin{pmatrix} D & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & O' \end{pmatrix}$$

and

$$L = \begin{pmatrix} D & \frac{1}{\sqrt{3}}D & O' & \cdots & O' \\ -\frac{\sqrt{3}}{3}D & O' & \frac{\sqrt{3}}{3\sqrt{5}}D & \cdots & O' \\ O' & -\frac{\sqrt{5}}{5\sqrt{3}}D & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & O' \end{pmatrix}$$

and

$$O = \begin{pmatrix} O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O' & O' & O' & \cdots & O' \end{pmatrix}$$

in which $D$ is $2^{k'-1}M' \times 2^{k'-1}M'$ matrix that define as below

$$D = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

and $O'$ is $2^{k'-1}M' \times 2^{k'-1}M'$ null matrix.

## 4. Conclusion

In this paper we introduce two dimensions Legendre wavelets. They are an orthonormal set over $[0,1] \times [0,1]$. The integration of the product of two Legendre wavelet functions vectors is a diagonal matrix as in the case of the one dimension Legendre wavelet. The operational matrix of integration for the Legendre wavelets is defined in this paper. Two dimensions Legendre wavelets is a suitable tools for numerical treatment of two dimensions variational problems.

## References

[1] G. B. Arfken and H. J. Weber. *Mathematical methods for physicists.* Academic Press Inc., San Diego, CA, fourth edition, 1995.

[2] J.-S. Gu and W.-S. Jiang. The Haar wavelets operational matrix of integration. *Int. J. Syst. Sci.*, 27(7):623–628, 1996.

[3] P. E. T. Jorgersen and A. Paolucci. Wavelets in mathematical physics: q oscillators. arXiv:math.FA/0212096 v3 22 Apr 2003.

[4] B. M. Kessler, G. L. Payne, and W. N. Polyou. Scattering calculation with wavelets. arXiv:nucl-th/0211016 v1 6 Nov 2002.

[5] H. Parsian. Legendre wavelets rule for definite integrals. Proceedings of the $20^{th}$ GAMM-Seminar Leipzig 2004.

[6] M. Razzaghi and S. Yousefi. Legendre wavelets direct method for variational problems. *Math. Comput. Simulation*, 53(3):185–192, 2000.
[7] M. Razzaghi and S. Yousefi. Legendre wavelets method for the solution of nonlinear problems in the calculus of variations. *Math. Comput. Modelling*, 34(1-2):45–54, 2001.
[8] M. Razzaghi and S. Yousefi. The Legendre wavelets operational matrix of integration. *Internat. J. Systems Sci.*, 32(4):495–502, 2001.

*Received May 23, 2004; revised August 28, 2004.*

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