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LINKAGES BETWEEN THE GAUSS MAP AND THE STERN-BROCOT TREE

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Abstract. We discover a bijective map between the Gauss Map and the left-half of the Stern-Brocot Tree. The domain of the Gauss Map is then extended to cover all reals, and the coverage of the Stern-Brocot Tree is extended to include all positive and negative rationals in a manner that preserves the map between the two constructions.

1. Introduction

That the Gauss Map and the Stern-Brocot Tree have corresponding features seems, at first glance, unintuitive. The Stern-Brocot Tree is a number-theoretic construction built on a strange algebra (child's addition) that seems far removed from a locally differentiable function based on the function $\frac{1}{x}$. Yet nonetheless, there are interesting areas of correspondence. We begin with the Gauss Map, which we define based on the notation in Corless [2].

Definition 1 (Gauss Map)**.** The *Gauss Map*, $G(x)$ is defined as

$$G(x) = \begin{cases} \frac{1}{x} \bmod 1 = \operatorname{frac} \frac{1}{x}, & \text{for } x \in (0,1] \\ 0, & \text{for } x = 0. \end{cases}$$

The Gauss Map and its iterates are made up of disjoint continuous parts. The following definition of these parts is a variation of the notation found in Bates et al [1].

Definition 2 (Parts in the Gauss Map)**.** $[0; j_1, \ldots, j_k]$ is that part of G^k whose domain is

$$[\{0; j_1, \ldots, j_k\}, \{0; j_1, \ldots, j_k, 1\})$$

if k is even, and

$$(\{0; j_1, \ldots, j_k, 1\}, \{0; j_1, \ldots, j_k\}]$$

if k is odd.

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The points of discontinuity in the k^{th} iterate, G^k, of the Gauss Map are $\{x : G^k(x) = 0\}$. Accordingly, the set of points of discontinuity in the domain of G is

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}. \tag{1.1}$$

In continued fraction notation, these same points of discontinuity are values of x that increment by one in the last entry of their continued fraction expansion. Thus within G, the set (1.1) in continued fraction notation becomes:

$$\{\{0; 1\}, \{0; 2\}, \{0; 3\}, \{0; 4\}, \ldots\}.$$

In general, $[0; j_1, \ldots, j_k]$ is discontinuous with the rest of G^kat $\{0; j_1, \ldots, j_k\}$ and $\{0; j_1, \ldots, j_k, 1\}$. Note that there is no part defined at the origin.

From hereon, where there is no ambiguity, we refer to the Gauss Map and its iterates as simply the Gauss Map.

2. Clusters within the Gauss Map

Definition 3 (Clusters). A *cluster* within G^k is an infinite set of consecutive parts of G^k whose slopes become progressively more vertical and approach infinity in their limit. The cluster is bounded by a line with vertical slope called the *cluster line.* This bound is not part of the cluster.

Example 1. G is made up of the elements of a single cluster whose cluster line is $x = 0$.

Theorem 1. *The set of consecutive parts* $[0; j_1, \ldots, j_k, 1], [0; j_1, \ldots, j_k, 2], \ldots$ *forms a cluster in* G^{k+1} *with cluster line* $x = \{0; j_1, \ldots, j_k\}$.

Proof. For k odd, the union of the set of consecutive parts

$$[0; j_1, \ldots, j_k, 1], [0; j_1, \ldots, j_k, 2], [0; j_1, \ldots, j_k, 3], \ldots$$

has least upper and greatest lower bounds $\{0; j_1, \ldots, j_k\} = \frac{p_k}{q_k}$ and

$$\{0; j_1, \ldots, j_k, 1\} = \frac{p_k + p_{k-1}}{q_k + q_{k-1}}$$

respectively. For k even the bounds are reversed.

We have shown in [1] that, if $x \in (\{0; j_1, \ldots, j_{k+1}\}, \{0; j_1, \ldots, j_{k+1} + 1\})$,

$$G^{k+1}(x) = \frac{q_{k+1}x - p_{k+1}}{p_k - q_k x}$$

where $\frac{p_k}{q_k} = \{0; j_1, \ldots, j_k\}$ and $\frac{p_{k+1}}{q_{k+1}} = \{0; j_1, \ldots, j_{k+1}\}$.

Since

$$\begin{aligned} \frac{d}{dx}G^{k+1}(x) &= \frac{p_k q_{k+1} - p_{k+1} q_k}{(p_k - q_k x)^2} \\ &= \frac{(-1)^{k+1}}{(p_k - q_k x)^2} \end{aligned}$$

$$\to \pm\infty \text{ as } x \to \frac{p_k}{q_k}$$

it follows that each consecutive part in the set

$$[0; j_1, \ldots, j_k, 1]\,, [0; j_1, \ldots, j_k, 2]\,, \ldots$$

becomes progressively more vertical. The vertical line $x = \frac{p_k}{q_k} = \{0; j_1, \ldots, j_k\}$ lies outside the set but is the limit of the set. Thus by Definition 3 the infinite set of consecutive parts $[0; j_1, \ldots, j_k, 1]$, $[0; j_1, \ldots, j_k, 2]\,, \ldots$ forms a cluster in G^{k+1} with cluster line $x = \{0; j_1, \ldots, j_k\}$. □

We denote the cluster

$$[0; j_1, \ldots, j_k, 1]\,,\ [0; j_1, \ldots, j_k, 2]\,, \ldots$$

in G^{k+1} by $\langle\langle 0, j_1, \ldots, j_k, \rangle\rangle$. The cluster in G is $\langle\langle 0\rangle\rangle$ and represents the entire first iterate.

Definition 4 (Domains and Open Domains of Sets)**.** Let the domain of each function in a set of functions be known. Then the union of the domains of each function in the set is called the *domain of the set*. The domain of a function minus its endpoints (if any) is called the *open domain* of the function. Similarly, the domain of a set minus its endpoints (if any) is called the *open domain of the set*.

For simplicity, let the interval (a, b), where $a < b$, be equivalently represented by (b, a). It follows from Definition 4 that:

i) The domain of a cluster is the union of the domains of each part in the cluster.

ii) The open domain of $[0; j_1, \ldots, j_k, t]$ is the interval

$$(\{0; j_1, \ldots, j_k, t\}\,, \{0; j_1, \ldots, j_k, t+1\})\,.$$

Theorem 2. *The open domain of the part* $[0; j_1, \ldots, j_k]$ *in* G^k *is the open domain of the cluster* $\langle\langle 0, j_1, \ldots, j_k\rangle\rangle$ *in* G^{k+1}.

Proof. The open domain of $[0; j_1, \ldots, j_k]$ in G^k is

$$(\{0; j_1, \ldots, j_k\}\,, \{0; j_1, \ldots, j_k, 1\})\,.$$

Consider any two consecutive parts $[0; j_1, \ldots, j_k, t]$ and $[0; j_1, \ldots, j_k, t+1]$ in the cluster $\langle\langle 0, j_1, \ldots, j_k\rangle\rangle$ in G^{k+1}. By Definition 2, $[0; j_1, \ldots, j_k, t]$ has domain

$$[\{0; j_1, \ldots, j_k, t\}\,, \{0; j_1, \ldots, j_k, t+1\})$$

for k even and

$$(\{0; j_1, \ldots, j_k, t+1\}\,, \{0; j_1, \ldots, j_k, t\}]$$

for k odd.

Similarly, $[0; j_1, \ldots, j_k, t+1]$ has domain

$$[\{0; j_1, \ldots, j_k, t+1\}\,, \{0; j_1, \ldots, j_k, t+2\})$$

for k even and

$$(\{0;j_1,\ldots,j_k,t+2\},\{0;j_1,\ldots,j_k,t+1\}]$$

for k odd. It follows that the domain of the set of two consecutive parts in $\langle\langle 0,j_1,\ldots,j_k\rangle\rangle$ is continuous and non-overlapping. Therefore the domain of the cluster $\langle\langle 0,j_1,\ldots,j_k\rangle\rangle$ is continuous and non-overlapping. Accordingly, the only endpoints of $\langle\langle 0,j_1,\ldots,j_k\rangle\rangle$ are $\{0;j_1,\ldots,j_k\}$ and $\{0;j_1,\ldots,j_k,1\}$, and its open domain is

$$(\{0;j_1,\ldots,j_k\},\{0;j_1,\ldots,j_k,1\}),$$

establishing our theorem. □

Corollary 1. *The only clusters within G^{k+1} are those of the form*

$$\langle\langle 0,j_1,\ldots,j_k\rangle\rangle$$

where the open domain of $\langle\langle 0,j_1,\ldots,j_k\rangle\rangle$ is the open domain of $[0;j_1,\ldots,j_k]$ in G^k.

Proof. $G(x)=\langle\langle 0\rangle\rangle$ and represents a single cluster with open domain $(0,1)$. By Theorem 2, $G^2(x)$ consists entirely of clusters of the form $\langle\langle 0,j_1\rangle\rangle$ where the open domain of $\langle\langle 0,j_1\rangle\rangle$ is the open domain of $[0;j_1]$ in G. Repeating Theorem 2 for G^3, $G^4,\ldots$ establishes our corollary. □

Since from the proof of Theorem 1, $\frac{d}{dx}G^k(x)=\frac{(-1)^k}{(p_{k-1}-q_{k-1}x)^2}$, it follows that within G^k, parts have negative slope for k odd and positive slope for k even.

Summary 1. The following is a summary of important characteristics of the Gauss Map:

1. Every part of the Gauss Map belongs to a cluster.
2. Each cluster has an infinite number of parts.
3. All parts within odd iterates have negative slope; all parts within even iterates have positive slope.
4. The slopes of successive parts within a cluster become progressively steeper.
5. The cluster line is vertical and is not part of the cluster.
6. Clusters within G^{k+1} are all of the form $\langle\langle 0,j_1,\ldots,j_k\rangle\rangle$ where the open domain of $\langle\langle 0,j_1,\ldots,j_k\rangle\rangle$ is the open domain of the part $[0;j_1,\ldots,j_k]$ in G^k.

3. Terms in the Stern-Brocot Tree

Let the level of the Stern-Brocot Tree consisting of the terms $\frac{0}{1}$ and $\frac{1}{0}$ be called Level 0. Except for terms in level 0, each term of the Stern-Brocot Tree is the *mediant* of two terms, called *parents*, found in lower ordinal levels of the tree. The following definitions formalise our understanding of entries in each level of the tree

Definition 5 (Mediant)**.** If m, n, s, t are integers then the mediant of $\frac{m}{n}$ and $\frac{s}{t}$, written as $\frac{m}{n} + \frac{s}{t}$ is $\frac{m+s}{n+t.}$ The operation $+$ is called child's addition.

Definition 6 (Interleave Operator)**.** We denote by $\#$, the *interleave operator* acting on two ordered sets $A = \langle a_1, a_2, \dots, a_{k+1}\rangle$ and $B = \langle b_1, b_2, \dots, b_k\rangle$, such that $A\#B = C$ where $C = \langle a_1, b_1, a_2, b_2, \dots, b_k, a_{k+1}\rangle$.

Definition 7 (Stern-Brocot Sequence)**.** Let $H_0 = \left\langle \frac{0}{1}, \frac{1}{0} \right\rangle$ and for $k \geq 1$,

$$H_k = H_{k-1}\#\,\mathrm{med}\, H_{k-1}$$

where $\mathrm{med}\, H_{k-1}$ denotes the increasing sequence of mediants that are generated from consecutive terms in H_{k-1}.

That is, if

$$H_{k-1} = \left\langle h_{k-1,1}, h_{k-1,2}, \dots, h_{k-1,2^k+1} \right\rangle$$

then

$$\mathrm{med}\, H_{k-1} = \left\langle (h_{k-1,1} + h_{k-1,2}), (h_{k-1,2} + h_{k-1,3}), \dots, \left(h_{k-1,2^k} + h_{k-1,2^k+1}\right) \right\rangle .$$

H_k represents the increasing sequence containing both the first k generations of mediants based on H_0, and the terms of H_0 itself. It is styled the Stern-Brocot sequence.

Definition 8 (Stern-Brocot Tree)**.** The Stern-Brocot Tree is a series of levels given by:

$$\begin{array}{ll} \textit{Level } 0 & \frac{0}{1}\ \frac{1}{0} \\ \textit{Level } 1 & \mathrm{med}\, H_0 \\ \textit{Level } 2 & \mathrm{med}\, H_1 \\ \textit{Level } 3 & \mathrm{med}\, H_2 \\ \vdots & \vdots \end{array}$$

The right half of Figure 1 (with the inclusion of the term $\frac{0}{1}$) represents the first five levels of the Stern-Brocot Tree.

We can uniquely locate any term in the tree by introducing *Right-Left* notation. Consider the term $\frac{8}{11}$. We locate $\frac{8}{11}$ by moving from level 0, one level at a time, through successive mediants. Thus beginning at $\frac{0}{1}$, we move to the right and down to $\frac{1}{1}$, to the left and down to $\frac{1}{2}$, to the right and down to $\frac{2}{3}$, to the right and down to $\frac{3}{4}$, to the left and down to $\frac{5}{7}$and finally to the right and down to $\frac{8}{11}$. It is obvious that this is the *only* route by which we can move from $\frac{0}{1}$ to $\frac{8}{11}$, traversing one level at a time and one successive mediant at a time. Generalising for the tree, every term can be uniquely located in the tree by a succession of these right or left and down moves. Thus our Right-Left notation is a handy way of describing any term in the tree. In our example we adopt the notation RLR^2LR to represent $\frac{8}{11}$. This denotes that we move from $\frac{0}{1}$ to the right and down once, then to the left and down once, then to the right and down twice, then to the left and down once and finally to the right and down once to arrive at $\frac{8}{11}$.

Based on Graham et al [3], it can be easily shown that, for $a_0 \geq 0, a_1 \geq 1, a_2 \geq 1, \ldots, a_k \geq 1$,

$$\{a_0; a_1, a_2, \ldots, a_k + 1\} = \begin{cases} R^{a_0+1}L^{a_1}R^{a_2}\ldots L^{a_k} \text{ for } k \text{ odd} \\ R^{a_0+1}L^{a_1}R^{a_2}\ldots R^{a_k} \text{ for } k \text{ even} \end{cases} \tag{3.1}$$

where the left hand side of (3.1) is the continued fraction expansion of the term represented by the right hand side of (3.1). Thus $\frac{8}{11} = \{0; 1, 2, 1, 2\} = R^1LR^2LR$. Note that we have chosen a variation on the notation found in [3]. Graham et al do not commence the right and left movements from $\frac{0}{1}$ because the first movement to $\frac{1}{1}$ is common to all entries in the tree and can be ignored by defining that all movements commence from $\frac{1}{1}$. However we later extend the Stern-Brocot Tree to include negative fractions. In order to locate entries in this extended tree, and thereby generalise the process for all fractions, the scheme of right and left movements needs to begin at $\frac{0}{1}$.

4. Branches in the Stern-Brocot Tree

We now introduce branches within the Stern-Brocot Tree as a prelude to discovering their linkage to clusters in the Gauss Map. Firstly, we categorise mediants according to the following definition:

Definition 9 (Left and Right Mediants)**.** A *left (right) mediant* is the mediant formed in level $k + 1$ that is smaller (greater) than its parent found in level k.

Definition 10 (Left and Right Branches)**.** The set of all mediants possessing a common parent μ and whose elements are smaller than μ is called the *left branch* of μ; the set of all mediants possessing a common parent μ and whose elements are greater than μ is called the *right branch* of μ.

Definition 10 can be alternatively stated as follows: Let μ be a term in the Stern-Brocot Tree and μ_l its left mediant. The left branch of μ is the set consisting of μ_l and all successive right mediants of μ_l. Similarly, let μ_r be the right mediant of μ. The right branch of μ is the set consisting of μ_r and all successive left mediants of μ_r. It follows from Definition 10 that each term ordinally above level 0 in the tree belongs to two branches - the left branch of one parent and the right branch of the other parent.

We can represent each rational number by a terminating continued fraction that has a short and a long form. Thus for $a_0 \geq 0, a_1 \geq 1, a_2 \geq 1, \ldots, a_k > 1$, we have $\{a_0; a_1, a_2, \ldots, a_k\} = \{a_0; a_1, a_2, \ldots, a_k - 1, 1\}$ where the continued fraction on the left is the short form and the continued fraction on the right is the long form. The following theorem links branches and the short and long form of terms in the Stern-Brocot Tree. It shows that the left and right branch of a term is built upon the short and long form of the term.

Theorem 3. *Let $\{a_0; a_1, a_2, \ldots, a_k\}$ be the short form of μ. Then* *i) for k odd, the right branch of μ is the set*

$$\{\{a_0; a_1, a_2, \ldots, a_k - 1, 1, t\} \mid t \geq 1\}$$

and the left branch of μ is the set

$$\{\{a_0; a_1, a_2, \ldots, a_k, t\} \mid t \geq 1\};$$

ii) for k even, the right branch of μ is the set

$$\{\{a_0; a_1, a_2, \ldots, a_k, t\} \mid t \geq 1\}$$

and the left branch of μ is the set

$$\{\{a_0; a_1, a_2, \ldots, a_k - 1, 1, t\} \mid t \geq 1\}.$$

Proof. From (3.1), let $\mu = R^{a_0+1}L^{a_1}R^{a_2}\ldots L^{a_k-1} = \{a_0; a_1, a_2, \ldots, a_k\}$ where k is odd. Then

$$\mu_l = R^{a_0+1}L^{a_1}R^{a_2}\ldots L^{a_k} = \{a_0; a_1, a_2, \ldots, a_k + 1\} = \{a_0; a_1, a_2, \ldots, a_k, 1\}$$

and the left branch of μ is the set

$$\left\{R^{a_0+1}L^{a_1}R^{a_2}\ldots L^{a_k}R^t \mid t \geq 0\right\} = \{\{a_0; a_1, a_2, \ldots, a_k, t\} \mid t \geq 1\}.$$

Similarly, from (3.1),

$$\begin{aligned}\mu_r &= R^{a_0+1}L^{a_1}R^{a_2}\ldots L^{a_k-1}R = \{a_0; a_1, a_2, \ldots, a_k - 1, 2\} \\ &= \{a_0; a_1, a_2, \ldots, a_k - 1, 1, 1\}\end{aligned}$$

and the right branch of μ is the set

$$\left\{R^{a_0+1}L^{a_1}R^{a_2}\ldots L^{a_k}RL^t \mid t \geq 0\right\} = \{\{a_0; a_1, a_2, \ldots, a_k - 1, 1, t\} \mid t \geq 1\}.$$

The case for k even, follows the same reasoning as that for k odd. □

Corollary 2. *Left (right) mediants have an even (odd) number of terms in their continued fraction expansions.*

Proof. From Theorem 3, let $\{a_0; a_1, a_2, \ldots, a_k\}$ be the short form of μ. Then
i) for k odd, the right mediant of μ is

$$\{a_0; a_1, a_2, \ldots, a_k - 1, 1, 1\} = \{a_0; a_1, a_2, \ldots, a_k - 1, 2\}$$

and the left mediant of μ is

$$\{a_0; a_1, a_2, \ldots, a_k, 1\} = \{a_0; a_1, a_2, \ldots, a_k + 1\};$$

ii) for k even, the right mediant of μ is

$$\{a_0; a_1, a_2, \ldots, a_k, 1\} = \{a_0; a_1, a_2, \ldots, a_k + 1\}$$

and the left mediant of μ is

$$\{a_0; a_1, a_2, \ldots, a_k - 1, 1, 1\} = \{a_0; a_1, a_2, \ldots, a_k - 1, 2\}.$$

□

We call μ in Theorem 3 the *pivot* of the two branches that are described in the theorem. Clearly every term in the tree except $\frac{0}{1}$ and $\frac{1}{0}$ is a pivot for a left branch and a right branch. ($\frac{0}{1}$ is a pivot only for a right branch but no left branch; and $\frac{1}{0}$ is defined as the pivot for the left branch $\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\right\}$. $\frac{1}{0}$ is not a pivot for any right branch). Theorem 3 shows that every branch of the Stern-Brocot Tree possesses members whose continued fraction expansions are identical except in their last terms. Their last terms increment as we move down the branch. Thus if a branch has k terms in the continued fraction expansion of its members, the first $k-1$ terms are common for every member of the branch and correspond to the short or long form of the continued fraction expansion of the pivot.

Definition 11 (Continued Fraction Notation for a Branch)**.** The branch in the Stern-Brocot Tree consisting of all continued fractions of the form

$$\{0; j_1, j_2, \ldots, j_k, t\}$$

where $t = 1, 2, 3, \ldots,$ is denoted by $\overrightarrow{\{0, j_1, j_2, \ldots, j_k\}}$ and its *extended branch* consists of all continued fractions of the form $\{0; j_1, j_2, \ldots, j_k, t\}$ where $t = 0, 1, 2, \ldots,$ is denoted by $\widetilde{\{0, j_1, j_2, \ldots, j_k\}}$. The *domain of a branch* consists of all reals that lie between the first term of the branch and its pivot.

We note that an extended branch is formed when the parent of the first term of a branch that is not the pivot of the branch, is added to the branch.

Example 2. The right branch $\overrightarrow{\{0\}} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ has pivot $\frac{0}{1}$. Its members are of the form $\{0; t\}$ for $t = 1, 2, 3, \ldots,$ whilst the pivot is $\{0\}$. Its extended branch is $\widetilde{\{0\}} = \{\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$. The domain of $\overrightarrow{\{0\}}$ consists of all reals between 0 and 1.

Summary 2. The following is a summary of important characteristics of the Stern-Brocot Tree:

1. Every term found ordinally above level 0 of the Stern-Brocot Tree belongs to both a left and a right branch.
2. Each branch contains an infinite number of terms.
3. The domain of a branch consists of all reals that lie between the first term of the branch and its pivot.

5. Mapping the Gauss Map to the Stern-Brocot Tree

We are now able to formally state the linkage that exists between the Gauss Map and the Stern-Brocot Tree.

Definition 12 (Map between Gauss Map and Left-Half of the Stern-Brocot Tree)**.** Let P be the class of all parts of all iterates of the Gauss Map. Let $\mathbb{K} : P \to \mathbb{Q}$ denote the map between the Gauss Map and the left-half of the

Stern-Brocot Tree given by:

$$[0; j_1, \ldots, j_k] \xrightarrow{\mathbb{K}} \{0; j_1, j_2, \ldots, j_k\} .$$

That is, under $\mathbb{K}$, the part $[0; j_1, \ldots, j_k]$ in G^k is transformed into the term $\{0; j_1, j_2, \ldots, j_k\}$ in the Stern-Brocot Tree.

We note that $\mathbb{K}$ is one-to-one and onto between the Gauss Map and the left-half of the Stern-Brocot Tree.

Under $\mathbb{K}$:

1. Clusters in the Gauss Map become branches in the left-half of the Stern-Brocot Tree. That is, sequential parts within a given cluster in the Gauss Map are mapped to sequential members of the corresponding branch in the left-half of the Stern-Brocot Tree.
2. Right branches are mapped to clusters found within even iterates; left branches are mapped to clusters found within odd iterates.
3. The domain of a cluster in the Gauss Map and the domain of its corresponding branch in the Stern-Brocot Tree are identical.
4. The cluster line $x = \{0; j_1, j_2, \ldots, j_k\}$ is associated with a cluster which is mapped to a branch possessing the pivot $\{0; j_1, j_2, \ldots, j_k\}$.

An interesting consequence of the mapping defined in Definition 12 is that every positive rational number less than 1 has a corresponding part in the Gauss Map. This follows from the fact that all positive rationals less than 1 are represented in the left-half of the Stern-Brocot Tree and $\mathbb{K}$ is bijective.

We now show further correspondences between the Gauss Map and the Stern-Brocot Tree based on $\mathbb{K}$.

6. Symmetry Clusters and Symmetry Branches

We have shown in [1] that $[0; j_1, \ldots, j_k]$ in G^k is symmetric with

$$[0; 1, j_1 - 1, j_2, \ldots, j_k]$$

in G^{k+1} where symmetry implies that one part is the mirror reverse of the other around $x = \frac{1}{2}$. Note that a reverse symmetry around $x = \frac{1}{2}$ occurs for $j_1 = 1$. That is, for $j_1 = 1$,

$$\{0; 1, j_1 - 1, j_2, \ldots, j_k\} = \{0; 1 + j_2, j_3, \ldots, j_k\}$$

which is of the form $\{0; h_1, h_2, \ldots, h_{k-1}\}$ where $h_1 > 1, h_i \in \mathbb{Z}^+_, \ i > 1$. Therefore for $j_1 = 1$, $[0; 1, j_1 - 1, j_2, \ldots, j_k]$ is found in the left-half of G^{k-1}. Its symmetry partner is $[0; j_1, j_2, j_3, \ldots, j_k]$ found in the right-half of G^k. Accordingly,

i) For $j_1 = 1$, $[0; j_1, j_2, \ldots, j_k]$, located in the right-half of G^k, is symmetric around $x = \frac{1}{2}$ with $[0; 1 + j_2, j_3, \ldots, j_k]$, located in the left-half of G^{k-1}.

ii) For $j_1 > 1$, $[0; j_1, j_2, \ldots, j_k]$, located in the left-half of G^k, is symmetric around $x = \frac{1}{2}$ with $[0; 1, j_1 - 1, j_2, \ldots, j_k]$, located in the right-half of G^{k+1}.

In either case it follows for $k > 0$, that the cluster $\langle\langle 0, j_1, j_2, \ldots, j_k\rangle\rangle$ is symmetric with the cluster $\langle\langle 0, 1, j_1 - 1, j_2, \ldots, j_k\rangle\rangle$ around $x = \frac{1}{2}$ (in G the first part of the cluster has no symmetry with any other part in the map). We call $\langle\langle 0, j_1, j_2, \ldots, j_k\rangle\rangle$ and $\langle\langle 0, 1, j_1 - 1, j_2, \ldots, j_k\rangle\rangle$ *symmetry clusters.* We now define *symmetry branches* in the Stern-Brocot Tree.

Definition 13 (Symmetry Branches)**.** Two branches in the Stern-Brocot Tree are called *symmetry branches* if terms found on the same level from each branch sum to 1 and the width of the domain of each branch is identical.

Theorem 4. *The only symmetry branches are those of the form*

$$\overrightarrow{\{0, j_1, j_2, \ldots, j_k\}} \text{ and } \overrightarrow{\{0, 1, j_1 - 1, j_2, \ldots, j_k\}}.$$

Proof. From (3.1), we have

$$\{0; j_1, j_2, \ldots, j_k, t\} = \begin{cases} R^1 L^{j_1} R^{j_2} \ldots L^{t-1} \text{ for } k \text{ even} \\ R^1 L^{j_1} R^{j_2} \ldots R^{t-1} \text{ for } k \text{ odd.} \end{cases}$$

It must therefore be found in level $(j_1 + j_2 + \ldots + j_k + t - 1)$ of the tree corresponding to the sum of the right and left movements used to place it in the tree. Similarly, the term

$$\{0; 1, j_1 - 1, j_2, \ldots, j_k, t\} = \begin{cases} R^1 L^1 R^{j_1 - 1} \ldots L^{t-1} \text{ for } k \text{ odd} \\ R^1 L^1 R^{j_1 - 1} \ldots R^{t-1} \text{ for } k \text{ even} \end{cases}$$

exists (since it is the continued fraction of a rational number between 0 and 1); and is found in level

$$(1 + j_1 - 1 + j_2 + \ldots + j_k + t - 1) = (j_1 + j_2 + \ldots + j_k + t - 1)$$

of the tree. Thus the terms $\{0; j_1, j_2, \ldots, j_k, t\}$ and $\{0; 1, j_1 - 1, j_2, \ldots, j_k, t\}$ both exist and are found on the same level of the tree.

We have shown in [1] that if $x = \{0; a_1, a_2, \ldots\}$, then

$$1 - x = \{0; 1, a_1 - 1, a_2, \ldots\} \tag{6.1}$$

Therefore $\{0; j_1, j_2, \ldots, j_k, t\}$ and $\{0; 1, j_1 - 1, j_2, \ldots, j_k, t\}$ sum to 1.

Now $\overrightarrow{\{0, j_1, j_2, \ldots, j_k\}}$ has pivot $\frac{p_k}{q_k} = \{0; j_1, j_2, \ldots, j_k\}$ and is bounded by

$$\frac{p_k}{q_k} = \{0; j_1, j_2, \ldots, j_k\}$$

and $\frac{p_k + p_{k-1}}{q_k + q_{k-1}} = \{0; j_1, j_2, \ldots, j_k, 1\}$. Similarly

$$\overrightarrow{\{0, 1, j_1 - 1, j_2, \ldots, j_k\}}$$

has pivot $\{0; 1, j_1 - 1, j_2, \ldots, j_k\}$ and is bounded by

$$1 - \frac{p_k}{q_k} = \{0; 1, j_1 - 1, j_2, \ldots, j_k\}$$

and $1-\frac{p_k+p_{k-1}}{q_k+q_{k-1}}=\{0;1,j_1-1,j_2,\ldots,j_k,1\}$. It follows that the widths of the domains of the two branches are identical.

This is also true in the particular case, $j_1=1$, since

$$\overrightarrow{\{0,1,j_1-1,j_2,\ldots,j_k\}}=\overrightarrow{\{0,1+j_2,\ldots,j_k\}}=\overrightarrow{\{0,h_1,h_2,\ldots,h_{k-1}\}}$$

where we have $h_1=1+j_1$, $h_i=j_{i+1}$, $i=2,\ldots,k-1$, and

$$\overrightarrow{\{0,1,1-h_1,h_2,\ldots,h_{k-1}\}}=\overrightarrow{\{0,1,j_2,\ldots,j_k\}}=\overrightarrow{\{0,j_1,j_2,\ldots,j_k\}}.$$

Since every branch is of the form $\overrightarrow{\{0,j_1,j_2,\ldots,j_k\}}$, every symmetry branch must therefore be of the form $\overrightarrow{\{0,1,j_1-1,j_2,\ldots,j_k\}}$. □

Corollary 3. *For every pair of symmetry clusters*

$$\langle\langle 0,j_1,j_2,\ldots,j_k\rangle\rangle \text{ and } \langle\langle 0,1,j_1-1,j_2,\ldots,j_k\rangle\rangle$$

in the Gauss Map there exists a corresponding pair of symmetry branches

$$\overrightarrow{\{0,j_1,j_2,\ldots,j_k\}} \text{ and } \overrightarrow{\{0,1,j_1-1,j_2,\ldots,j_k\}}$$

in the Stern-Brocot Tree.

Proof. This follows from Definition 12 □

For $m\in\overrightarrow{\{0,j_1,j_2,\ldots,j_k\}}$, we have $1-m\in\overrightarrow{\{0,1,j_1-1,j_2,\ldots,j_k\}}$, that is, each member of $\overrightarrow{\{0,j_1,j_2,\ldots,j_k\}}$ has a corresponding member in

$$\overrightarrow{\{0,1,j_1-1,j_2,\ldots,j_k\}}$$

that is equidistant from $\frac{1}{2}$. Accordingly, we say that both symmetry branches and symmetry clusters are symmetric around $x=\frac{1}{2}$, as are their respective pivots and cluster lines.

7. The Enlarged Gauss Map

Consider the following extended definition of the Gauss Map, styled the Enlarged Gauss Map, $\mathcal{G}(x)$.

Definition 14 (Enlarged Gauss Map)**.** The *Enlarged Gauss Map*, $\mathcal{G}(x)$, is defined as

$$\mathcal{G}(x)=\begin{cases}\frac{1}{x}\bmod 1=\operatorname{frac}\frac{1}{x}, & \text{for } x>0.\\ 0, & \text{for } x=0.\end{cases}$$

Note that the Enlarged Gauss Map is identical to the Gauss Map for the domain $0<x\le 1$.

Theorem 5. *Let $x=\{a_0;a_1,a_2,\ldots\}$ be any non-negative real number. Then*

$$\mathcal{G}^k(x)=\begin{cases}\{0;a_{k+1},a_{k+2},\ldots\} & \textit{if } a_0=0\\ \{0;a_{k-1},a_k,\ldots\} & \textit{if } a_0\ge 1.\end{cases}$$

Proof. i) If $a_0 = 0, \mathcal{G}(x) = G(x)$. So the result is that of Theorem 3 of [1].

ii) If $a_0 > 0, \mathcal{G}(x) = \{0; a_0, a_1, \ldots\}$. So

$$\mathcal{G}^k(x) = G^{k-1}(\{0; a_0, a_1, \ldots\}) = \{0; a_{k-1}, a_k, \ldots\}$$

by i). □

Theorem 5 can be restated in terms of $G(x)$.

i) For $k = 1$,

$$\mathcal{G}(x) = \begin{cases} G(x) & \text{for } 0 < x \le 1 \\ \frac{1}{x} & \text{for } x > 1. \end{cases}$$

ii) For $k > 1$,

$$\mathcal{G}^k(x) = \begin{cases} G^k(x) & \text{for } 0 < x \le 1 \\ G^{k-2}(\text{frac } x) & \text{for } x > 1. \end{cases}$$

That is, the graph of the k^{th} iterate in the domain $0 < x \le 1$ is identical in appearance to the graph of the $(k+2)^{th}$ iterate in each of the domains $a < x \le a + 1$ where $a = 1, 2, 3, \ldots$. Figure 2 shows a portion of the third iterate of the Enlarged Gauss Map.

We now extend the definition of parts in the Gauss Map so that it covers all parts of the Enlarged Gauss Map.

Definition 15 (Parts of the Enlarged Gauss Map)**.** $[j_0; j_1, \ldots, j_k]$ is that part of $\mathcal{G}^k$ whose domain is $[\{j_0; j_1, \ldots, j_k, 1\}, \{j_0; j_1, \ldots, j_k\})$.

8. Mapping the Enlarged Gauss Map to the Stern-Brocot Tree

In earlier sections we explored the mapping between the Gauss Map and the *left-half* of the Stern-Brocot Tree. Through the Enlarged Gauss Map we are now able to extend this mapping to include the *entire* Stern-Brocot Tree. To do this we need to extend Definition 12.

Definition 16 (Map between the Enlarged Gauss Map and the Stern-Brocot Tree)**.** Let $\mathcal{P}$ be the class of all parts of all iterates of the Enlarged Gauss Map. Let $\mathbb{K} : \mathcal{P} \to \mathbb{Q}$ denote the map between the Enlarged Gauss Map and the Stern-Brocot Tree given by:

$$[j_0; j_1, \ldots, j_k] \xrightarrow{\mathbb{K}} \{j_0; j_1, \ldots, j_k\}.$$

That is, under $\mathbb{K}$, the $(j_0, j_1, \ldots, j_k)^{th}$ part in $\mathcal{G}^k$ is transformed into the term $\{j_0; j_1, \ldots, j_k\}$ in the Stern-Brocot Tree.

Theorem 5 informs us that if t is any positive integer and $k > 1$, the graph of $\mathcal{G}^k$ over $(0, 1]$ is the same as that of $\mathcal{G}^{k+2}$ over $(t, t+1]$. Hence $\mathcal{G}^{k+2}$, for $k > 1$, has axes of symmetry at the points $x = \frac{2m+1}{2}$ for $m = 0, 1, 2, \ldots$.

Example 3. Within the Enlarged Gauss Map, $[0; j_1]$, where $j_1 > 1$, has a symmetry partner in the second iterate. It is also repeated infinitely in the third iterate as the parts $[t; j_1]$ where $t = 1, 2, 3, \ldots$ and its symmetry partner is repeated infinitely in the fourth iterate around each of the axes $x = \frac{2m+1}{2}$. Similar comments hold for every other part of the cluster in which $[0; j_1]$ is found.

Under $\mathbb{K}$, the cluster $\langle\langle j_0, j_1, \ldots, j_k \rangle\rangle$ with cluster line $x = \{j_0; j_1, \ldots, j_k\}$ is mapped to the branch $\overrightarrow{\{j_0; j_1, \ldots, j_k\}}$ that has pivot $\{j_0; j_1, \ldots, j_k\}$, and vice versa. Moreover, $\{j_0; j_1, \ldots, j_k, 1\}$ and $\{j_0; j_1, \ldots, j_k\}$ represent the endpoints of the interval that is the domain of the cluster $\langle\langle j_0, j_1, \ldots, j_k \rangle\rangle$. The endpoints of the interval that is the domain of the branch $\overrightarrow{\{j_0; j_1, \ldots, j_k\}}$ are also $\{j_0; j_1, \ldots, j_k, 1\}$ and $\{j_0; j_1, \ldots, j_k\}$.

Example 4. Consider the following clusters within $\mathcal{G}$:

$$\begin{gathered}
\langle\langle 1 \rangle\rangle = \langle\langle [1;1], [1;2], [1;3], \ldots \rangle\rangle \\
\langle\langle 2 \rangle\rangle = \langle\langle [2;1], [2;2], [2;3], \ldots \rangle\rangle \\
\langle\langle 3 \rangle\rangle = \langle\langle [3;1], [3;2], [3;3], \ldots \rangle\rangle \\
\vdots
\end{gathered}$$

The width of the domain of each of these clusters is equal to 1.

Under $\mathbb{K}$ these clusters map respectively to the following branches in the Stern-Brocot Tree:

$$\begin{gathered}
\overrightarrow{\{1\}} = \left\{ \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots \right\} \\
\overrightarrow{\{2\}} = \left\{ \frac{3}{1}, \frac{5}{2}, \frac{7}{3}, \ldots \right\} \\
\overrightarrow{\{3\}} = \left\{ \frac{4}{1}, \frac{7}{2}, \frac{10}{3} \right\} \\
\vdots
\end{gathered}$$

The width of the domain of each of these branches is equal to 1.

We conclude this section with some comments on $\mathcal{G}$ and $\mathcal{G}^2$.

We have already seen that every part in G is mapped under $\mathbb{K}$ to form the branch $\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ in the left-half of the Stern-Brocot Tree. For $x \leq 1$, $\mathcal{G}$ consists of the parts $[0;1], [0;2], [0;3], \ldots$ which meet the $x-$axis at $x = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ respectively. For $x > 1$, $\mathcal{G}$ consists of only one part which is asymptotic to the $x-$axis. We designate this part as $[\infty]$, the infinitieth part, because for x sufficiently large, $\mathcal{G}(x)$ can be made arbitrarily close to zero. Under $\mathbb{K}$, $[\infty]$ maps to $\{\infty\}$ which is the term $\frac{1}{0}$.

Consider $\mathcal{G}^2$. By Theorem 5, for $x > 1$, $\mathcal{G}^2(x) = \text{frac } x$. Hence for $x > 1$, $\mathcal{G}^2$ is made up of disjoint truncated parts of the line $y = x + 1$ displaced vertically downwards by its integer parts. Though for $x > 1$, parts of $\mathcal{G}^2$ do not form a cluster (parts do not become progressively more vertical), it is mapped under $\mathbb{K}$ to the branch $\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\right\}$.

We summarise the correspondences between the Enlarged Gauss Map and the Stern-Brocot Tree:

Summary 3. For the Enlarged Gauss Map and the Stern-Brocot Tree, under $\mathbb{K}$:

1. Clusters in the Enlarged Gauss Map become branches in the Stern-Brocot Tree. That is, sequential parts within a given cluster in the Enlarged Gauss Map are mapped to sequential members of the corresponding branch in the Stern-Brocot Tree.
2. Right branches are mapped to clusters found within even iterates; left branches are mapped to clusters found within odd iterates.
3. The domain of a cluster in the Enlarged Gauss Map and the domain of its corresponding branch in the Stern-Brocot Tree are identical.
4. The cluster line $x = \{j_0; j_1, j_2, \ldots, j_k\}$ is associated with a cluster which is mapped to a branch possessing the pivot $\{j_0; j_1, j_2, \ldots, j_k\}$.

9. The Generalised Gauss Map

In this section we extend the domain of the Gauss Map to encompass the entire number line. We call this map the *Generalised Gauss Map*,$\mathbb{G}(x)$.

Definition 17 (Generalised Gauss Map)**.** The *Generalised Gauss Map*, $\mathbb{G}(x)$, is defined as

$$\mathbb{G}(x) = \begin{cases} \frac{1}{x} \bmod 1 = \operatorname{frac} \frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$

This definition implies that, for $x \neq 0$,

$$\mathbb{G}(-x) = 1 - \mathbb{G}(x)\,. \tag{9.1}$$

This is due to the fact that for negative arguments of any modulus n, we find the difference between the next lower multiple of n and the argument itself. Since the Enlarged Gauss Map, $\mathcal{G}(x)$, is the right-half of $\mathbb{G}(x)$ we can restate $\mathbb{G}(x)$ in terms of $\mathcal{G}(x)$. That is,

$$\mathbb{G}(x) = \begin{cases} \mathcal{G}(x) & \text{for } x > 0 \\ 1 - \mathcal{G}(-x) & \text{for } x < 0 \end{cases}.$$

A portion of the first iterate of $\mathbb{G}(x)$ is shown at Figure 3.

But what transformation of $\mathcal{G}$ yields $\mathbb{G}$? That is, for $x > 0$, $0 \leq y \leq 1$, what transformation converts (x, y) into $(-x, 1 - y)$ in order to create the left-half of $\mathbb{G}$? This can be achieved through two reflections: Reflect $\mathcal{G}$ around the $y-$ axis and then again around $y = \frac{1}{2}$. We show that these reflections are equivalent to (9.1).

Let (x, y) be a point in $\mathbb{R}^2$. The transformation that reflects (x, y) around the $y-$axis to give the point $(-x, y)$ is

$$\begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The transformation that reflects (x, y) around the line $y = \frac{1}{2}$ to give the point $(x, 1-y)$ is

$$\begin{bmatrix} x \\ 1-y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We now combine these two transformations. Let (x, y) be a point in the Enlarged Gauss Map. It is transformed into the point $(-x, 1-y)$ in the left-half of the Generalised Gauss Map according to the following transformation:

$$\begin{aligned} \begin{bmatrix} -x \\ 1-y \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

We now show that an extension of the Stern-Brocot Tree, styled the *Generalised Stern-Brocot Tree*, can be made so that it covers all rational numbers, not just the non-negative rational numbers, and that the Generalised Gauss Map possesses a correspondence with the Generalised Stern-Brocot Tree.

10. The Generalised Stern-Brocot Tree

The Generalised Stern-Brocot Tree is formed from the Stern-Brocot Tree by altering Level 0 so that it becomes:

$$\text{Level } 0 \qquad \tfrac{-1}{0} \qquad \tfrac{0}{1} \qquad \tfrac{1}{0}$$

That is, we define the Generalised Stern-Brocot Sequence by redefining H_0 as $H_0 = \left\langle \frac{-1}{0}, \frac{0}{1}, \frac{1}{0} \right\rangle$ in Definition 7; and we define the Generalised Stern-Brocot Tree by using $H_0 = \left\langle \frac{-1}{0}, \frac{0}{1}, \frac{1}{0} \right\rangle$ in Definition 8. Figure 1 gives the first five levels of the Generalised Stern-Brocot Tree.

Recall from (3.1) that for the Stern-Brocot Tree:

$$\{a_0; a_1, a_2, \ldots, a_k + 1\} = \begin{cases} R^{a_0+1} L^{a_1} R^{a_2} \ldots L^{a_k} & \text{for } k \text{ odd} \\ R^{a_0+1} L^{a_1} R^{a_2} \ldots R^{a_k} & \text{for } k \text{ even.} \end{cases}$$

Since the left-half of the Generalised Stern-Brocot Tree is a reflection of the Stern Brocot Tree around $\frac{0}{1}$, we have reverse movements occurring from $\frac{0}{1}$ when locating negative entries. Thus, for $a_0 \geq 0, a_1 \geq 1, a_2 \geq 1, \ldots, a_k \geq 1$,

$$-\{a_0; a_1, a_2, \ldots, a_k + 1\} = \begin{cases} L^{a_0+1} R^{a_1} L^{a_2} \ldots R^{a_k} & \text{for } k \text{ odd} \\ L^{a_0+1} R^{a_1} L^{a_2} \ldots L^{a_k} & \text{for } k \text{ even.} \end{cases}$$

11. Mapping the Generalised Gauss Map to the Generalised Stern-Brocot Tree

Since the right halves of both generalised systems are simply the Enlarged Gauss Map and the Stern-Brocot Tree, for which we have previously identified linkages these linkages are retained in the Generalised Gauss Map and the Generalised Stern-Brocot Tree. However to perform mappings between the left halves of each construction we need an expression for negative continued fractions.

Theorem 6. *For* $a_0 \geq 0, a_1 \geq 1, \ldots, a_k \geq 1$,

$$-\{a_0; a_1, a_2, \ldots, a_k\} = \{-a_0 - 1; 1, a_1 - 1, a_2, \ldots, a_k\}.$$

Proof.

$$\begin{aligned}
-\{a_0; a_1, a_2, \ldots, a_k\} &= (-a_0 - 1) + (1 - \{0; a_1, a_2, \ldots, a_k\}) \\
&= (-a_0 - 1) + (\{0; 1, a_1 - 1, a_2, \ldots, a_k\}) \text{ by (6.1)} \\
&= \{-a_0 - 1; 1, a_1 - 1, a_2, \ldots, a_k\}
\end{aligned}$$

□

Note that where $a_1 = 1$, we have

$$\{a_0; a_1, a_2, \ldots, a_k\} = \{-a_0 - 1; 1 + a_2, a_3, \ldots, a_k\}.$$

We can now extend Definition 16 to encompass the Generalised Gauss Map and the Generalised Stern-Brocot Tree. But firstly we define parts in $\mathbb{G}^k$.

Definition 18 (Parts in the Generalised Gauss Map)**.** $[j_0; j_1, \ldots, j_k]$ is that part of $\mathbb{G}^k$ whose domain is $[\{j_0; j_1, \ldots, j_k, 1\}, \{j_0; j_1, \ldots, j_k\})$ where $j_0 \in \mathbb{Z}$, $j_i \in \mathbb{Z}^+$, $i > 0$.

Definition 19 (Map between the Generalised Gauss Map and the Generalised Stern-Brocot Tree)**.** Let $\mathbb{P}$ be the class of all parts of all iterates of the Generalised Gauss Map. Let $\mathbb{K} : \mathbb{P} \to \mathbb{Q}$ denote the map between the Generalised Gauss Map and the Generalised Stern-Brocot Tree given by:

$$[j_0; j_1, \ldots, j_k] \xrightarrow{\mathbb{K}} \{j_0; j_1, \ldots, j_k\}.$$

That is, under $\mathbb{K}$, the $(j_0, j_1, \ldots, j_k)^{th}$ part in $\mathbb{G}^k$ is transformed into the term $\{j_0; j_1, \ldots, j_k\}$ in the Generalised Stern-Brocot Tree.

Under $\mathbb{K}$:

1. Clusters in the Generalised Gauss Map become branches in the Generalised Stern-Brocot Tree. That is, sequential parts within a given cluster are mapped to sequential members of the corresponding branch.
2. The domain of a cluster and the domain of its corresponding branch are identical.
3. The cluster line $x = \{j_0; j_1, j_2, \ldots, j_k\}$ is associated with a cluster which is mapped to a branch possessing the pivot $\{j_0; j_1, j_2, \ldots, j_k\}$.

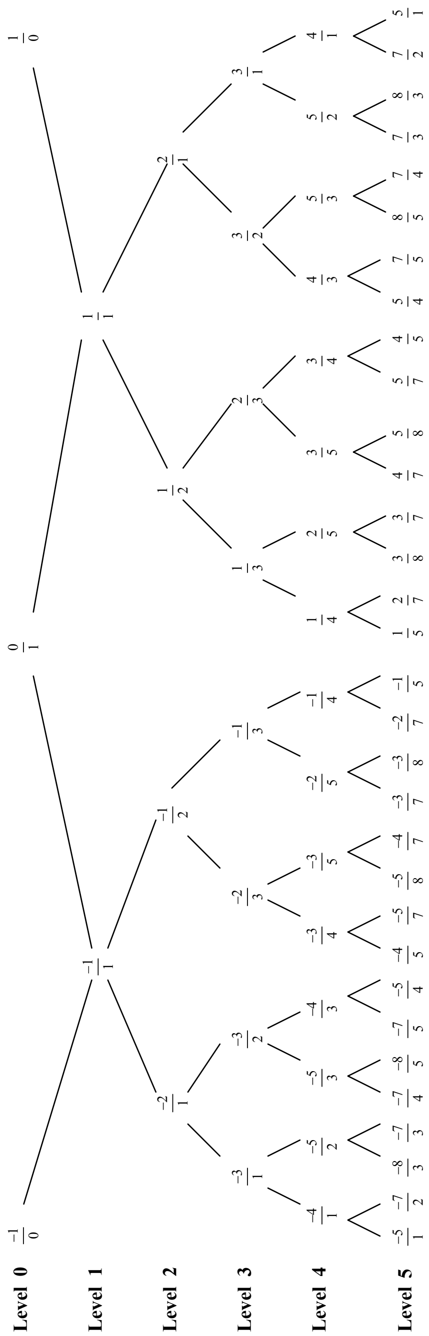


Figure 1 : Levels 0 to 5 of the Generalised Stern-Brocot Tree

Figure 2: The Third Iterate of the Enlarged Gauss Map

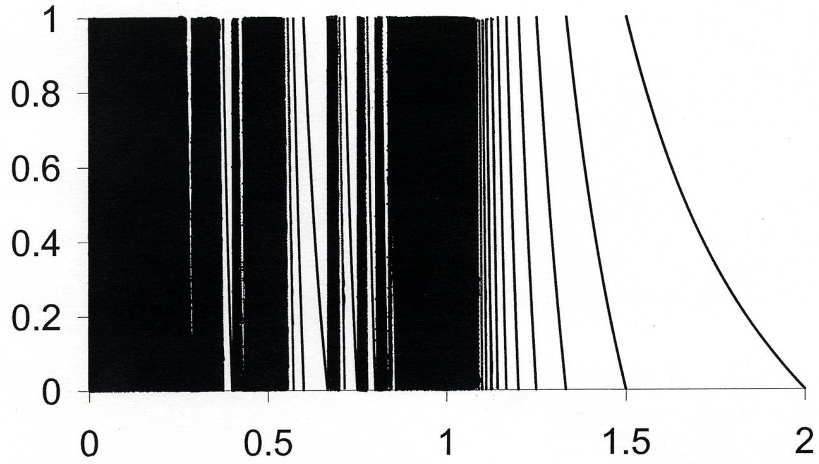
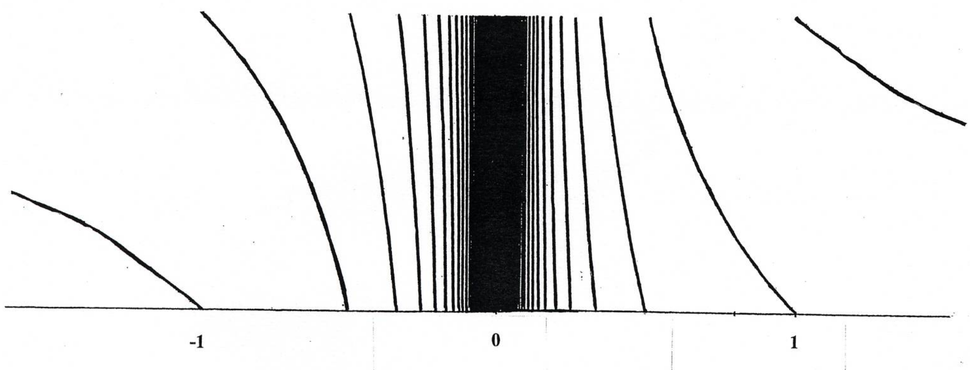


Figure 3: The Generalised Gauss Map



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