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ON THE RHEONOMIC FINSLERIAN MECHANICAL SYSTEMS

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Abstract. In this paper it will be studied the dynamical system of a rheonomic Finslerian mechanical system, whose evolution curves are given, on the phase space $TM\times\mathbf{R}$, by Lagrange equations. Then one can associate to the considered mechanical system a vector field S on $TM\times\mathbf{R}$, which is called the canonical semispray. All geometric objects of the rheonomic Finslerian mechanical system one can be derived from S. So we have the fundamental notion as the nonlinear connection N, the metrical N-linear connection, etc.

1. The geometry of phases space $(TM\times\mathbf{R},\pi,M)$

Let be M a smooth C^∞ manifold of finite dimension n, called the space of configurations and (TM,π,M) be its tangent bundle.The $2n$-dimensional manifold TM is called the phases space of M.

We denote by $(x^i), i=1,2,\dots,n$, the local coordinates on M and by (x^i,y^i) the canonical local coordinates on TM.

We consider the manifold $TM\times\mathbf{R}$ and we shall use the differentiable structure on $TM\times\mathbf{R}$ as product of the manifold TM fibered over M with $\mathbf{R}$.

The manifold $E=TM\times\mathbf{R}$ is a $2n+1-$dimensional, real manifold. In a domain of a local chart $U\times(a,b)$, the point $u=(x,y,t)\in E$ have the local coordinates (x^i,y^i,t).

A change of local coordinates on E has the following form:

$$\tilde{x}^i=\tilde{x}^i(x^1,x^2,\dots,x^n);\ \tilde{y}^i=\frac{\partial\tilde{x}^i}{\partial x^j}y^j;\ \tilde{t}=\phi(t) \tag{1.1}$$

with $rank\left(\frac{\partial\tilde{x}^i}{\partial x^j}\right)=n$ and $\phi':=\frac{d\phi}{dt}\neq 0$.

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Of course, we may take on $\mathbf{R}$ only one chart, that is $\tilde{t} = t$ or we may consider the affine change of charts on $\mathbf{R}$, that is $\tilde{t} = at + b$, $a \neq 0$, $a, b \in \mathbf{R}$.

The natural basis of tangent space T_uE at the point $u \in U \times (a, b)$ is given by

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right). \tag{1.2}$$

The transformation of coordinates (1.1) determines the transformations of the natural basis as follows

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{y}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j} \\ \frac{\partial}{\partial y^i} &= \frac{\partial \tilde{y}^j}{\partial y^i}\frac{\partial}{\partial \tilde{y}^j}; \quad \frac{\partial}{\partial t} = \phi' \frac{\partial}{\partial \tilde{t}} \end{aligned} \tag{1.3}$$

where

$$\frac{\partial \tilde{y}^j}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}; \quad \frac{\partial \tilde{y}^j}{\partial x^i} = \frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^h} y^h.$$

In [4], it is introduced on the manifold E, a vertical distribution V, generated by $n+1$ local vector fields $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial t}\right)$

$$V : u \in E \to V_u \subset T_uE. \tag{1.4}$$

It follows:

$$V_u = V_{n,u} \oplus V_{0,n} \quad \forall u \in E,$$

where the linear space $V_{0,n}$ is generated by the vector field $\frac{\partial}{\partial t}\,|_u$ and it is an 1-dimensional linear subspace of the tangent space T_uE. Also, the n-dimensional linear space $V_{n,u}$ generated by the fields $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^n}\right)|_u$ is a linear subspace of T_uE.

On the manifold E there exists a tangent structure, [1],[4],[10],

$$J : \chi(E) \to \chi(E),$$

given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}; J\left(\frac{\partial}{\partial y^i}\right) = 0; J\left(\frac{\partial}{\partial t}\right) = 0; \quad i = 1, 2, \ldots, n. \tag{1.5}$$

J is an integrable structure. [1],[4],[10].

On $TM \times R$ there exists a globally defined vector field

$$C = y^i \frac{\partial}{\partial y^i}.$$

It is the *Liouville vector field.*

A semispray on E is a vector field $S \in \chi(E)$ which has the property

$$JS = C. \tag{1.6}$$

Proposition 1 ([8]). *a) Locally a semispray S has the form*

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y,t)\frac{\partial}{\partial y^i} - G^0(x,y,t)\frac{\partial}{\partial t} \tag{1.7}$$

where $G^i(x,y,t)$ and $G^0(x,y,t)$ are the coefficients of S.

b) The functions $\{G^i(x,y,t)\ ,G^0(x,y,t)\}$ transform under a change of coordinates (1.1), as follows:

$$2\tilde{G}^i = 2\frac{\partial \tilde{x}^i}{\partial y^j}G^j - \frac{\partial \tilde{y}^i}{\partial x^j}y^j; \quad \tilde{G}^0 = \phi' G^0. \tag{1.8}$$

The integrals curves of S are the solutions of the following system of differential equations

$$\begin{aligned} &\frac{dx^i}{d\tau} = y^i(\tau); \quad \frac{dy^i}{d\tau} + 2G^i(x(\tau),y(\tau),t(\tau)) = 0 \\ &\frac{dt}{d\tau} + G^0(x(\tau),y(\tau),t(\tau)) = 0. \end{aligned} \tag{1.9}$$

We shall say that S is a dynamical system on the phases manifold $TM \times R$ and the equations (1.9) are the evolution equations of dynamical system S.

When $G^0 \equiv 1$, we may take $t = \tau$ and the system (1.9) reduces to the second order differential equation (SODE):

$$\frac{d^2x^i}{dt^2} + 2G^i(x(t),y(t),t) = 0; y^i = \frac{dx^i}{dt}.$$

In the following, we put $t = y^0$ and we introduce the Greek indices $\alpha, \beta, \ldots$ ranging on the set $\{0,1,2,\ldots,n\}$.

A non-linear connection in E is a smooth distribution:

$$N\colon u \in E \to N_u \subset T_u, E \tag{1.10}$$

which is supplementary to the vertical distribution V:

$$T_uE = N_u \oplus V_u, \quad \forall u = (x,y,t) \in E. \tag{1.11}$$

The local basis adapted to the descomposition (1.11), is

$$\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^\alpha}\right) \tag{1.12}$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y,t)\frac{\partial}{\partial y^j} - N_i^0(x,y,t)\frac{\partial}{\partial t}. \tag{1.13}$$

$(N_i^0(x,y,t), N_i^j(x,y,t))$ are the local coefficients of the non-linear connection N on E.

The following transformation rule, under (1.1), hold:

$$\tilde{N}_m^j \frac{\partial \tilde{x}^m}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^m}N_i^m - \frac{\partial \tilde{y}^j}{\partial x^i}; \quad \frac{\partial \tilde{x}^j}{\partial x^i}\tilde{N}_j^0 = \phi'\, N_i^0 \tag{1.14}$$

Conversely, a set of local functions $(N_i^0(x,y,t), N_i^j(x,y,t))$ satisfying (1.14) determines $\frac{\delta}{\delta x^i}$, hence it uniquely determines a non-linear connection N.

The dual basis of (1.12) is $(\delta x^i, \delta y^i, \delta t)$ with

$$\delta x^i = dx^i; \delta y^i = dy^i + N_j^i dx^j; \delta t = dt + N_i^0 dx^i. \tag{1.15}$$

2. Rheonomic Finsler Spaces. Preliminaries

Definition 1. A rheonomic Finsler space is a pair $RF^n = (M, F(x,y,t))$, for which $F\colon TM \times R \to R$ satisfy the following axioms:

(1) F is a positive scalar function on $E = TM \times R$;
(2) F is a positive $1-$homogenous with respect to the variables y^i;
(3) F is differentiable on $\tilde{E} = E \setminus \{0\}$ and continuous in the points (x,0,t);
(4) The Hessian of F, with the entries:

$$g_{ij}(x,y,t) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{2.1}$$

is positively defined on $\tilde{TM} \times R$.

F is called the fundamental function and $g_{ij}(x,y,t)$ is the fundamental tensor of space RF^n.

Remark 1. (1) F is a scalar function with respect to (1.1).
(2) $g_{ij}(x,y,t)$ is a tensor field with respect to (1.1). It is covariant of order 2, symmetric and nesingular.
(3) The pair $(M, L = F^2(x,y,t))$ is a rheonomic Lagrange space.

The geometrical theory of rheonomic Finsler space F^n can be found in the books [8],[10].

Using Remark 1 we can use the theory of rheonomic Lagrange spaces [1], [5], [8], for developing the geometry of rheonomic Finsler spaces.

The variational problem for the rheonomic Lagrangian $L(x,y,t) = F^2(x,y,t)$ lead us to the Euler-Lagrange equations:

$$\frac{d^2x^i}{dt^2} + \gamma_{jk}^i(x,y,t)\frac{dx^j}{dt}\frac{dx^k}{dt} + g^{ih}\frac{\partial g_{hj}}{\partial t}y^j = 0; y^i = \frac{dx^i}{dt} \tag{2.2}$$

where γ_{jk}^i are the Christoffel symbols of the fundamental tensor $g_{ij}(x,y,t)$.

Theorem 2. *The Euler-Lagrange equations are equivalent with the Lorentz equations:*

$$\frac{d^2x^i}{dt^2} + \gamma_{jk}^i(x,y,t)\frac{dx^j}{dt}\frac{dx^k}{dt} = F_j^i(x, \frac{dx}{dt}, t)\frac{dx^j}{dt} \tag{2.3}$$

where

$$F_j^i(x,y,t) = -g^{ih}\frac{\partial g_{hj}}{\partial t}$$

is the electromagnetic tensor field determined by the fundamental tensor field g_{ij}.

The system of equations (2.3) locally determine a dynamical system on the phase space $TM \times R$. We consider the following functions on $\tilde{TM} \times R$

$$2G^i(x,y,t) = \gamma^i_{jk}(x,y,t)y^j y^k$$

$$N^i_0(x,y,t) = g^{ih}\frac{\partial g_{hj}}{\partial t}y^j$$

Using the theory of the rheonomic Lagrange spaces it is obtain the canonical spray S of RF^n, as follows

$$S = y^i\frac{\partial}{\partial x^i} - (N^i_0(x,y,t) + N^i_k(x,y,t)y^k)\frac{\partial}{\partial y^i} + \frac{\partial}{\partial t} \tag{2.4}$$

with

$$N^i_j(x,y,t) = \frac{1}{2}\frac{\partial}{\partial y^j}(\gamma^i_{rs}(x,y,t)y^r y^s);\ N^0_j(x,y,t) = \frac{1}{2}\frac{\partial g_{jk}}{\partial t}y^k. \tag{2.5}$$

Equations of evolution (2.3) are the equations of the integral curves of the semispray S.

The semispray S determines the Cartan non-linear connection N, [8], [10], with the coefficients $(N^i_j(x,y,t), N^0_j(x,y,t))$.

Then N is a differentiable distribution on $\tilde{TM} \times R$, supplementary to the vertical distribution V, i.e.:

$$T_u\tilde{TM} \times R = N(u) \oplus V(u), \forall u \in \tilde{TM} \times R. \tag{2.6}$$

Let $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t})_u$ be the adapted basis to decomposition (2.6), with

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x,y,t)\frac{\partial}{\partial y^j} - N^0_i(x,y,t)\frac{\partial}{\partial t}. \tag{2.7}$$

The canonical metrical (or Cartan) $N-$ connection $C\Gamma(N)$ has the coefficients $(F^i_{jk}(x,y,t), C^i_{j\alpha}(x,y,t))$ given by the generalized Christoffel symbols:

$$F^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s}\right), \tag{2.8}$$

$$C^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s}\right), \tag{2.9}$$

$$C^i_{j0} = \frac{1}{2}g^{is}\left(\frac{\partial g_{s0}}{\partial y^j} + \frac{\partial g_{sj}}{\partial t} - \frac{\partial g_{j0}}{\partial y^s}\right). \tag{2.10}$$

3. Rheonomic Finslerian Mechanical Systems

The dynamical system of a nonconservative Lagrangian mechanical system can not be correctly defined without geometrical frameworks of the phases manifold TM. The Lagrangian mechanical systems, their equations and the associated dynamical systems were studied in [1],[2],[4],[5],[7],[3],[10], and the Finslerian mechanical systems in [6],[9]. The geometric study of the sclerhonomic

Finslerian mechanical systems given by equations with the external forces a priori given was studied in [4], [9].

Definition 2. A rheonomic Finslerian mechanical system is a triple

$$\Sigma = (M, F^2(x,y,t), \sigma(x,y,t))$$

where $F(x,y,t)$ is the fundamental function of a rheonomic Finsler space $RF^n = (M, F(x,y,t))$ and $\sigma(x,y,t) = \sigma^i(x,y,t)\frac{\partial}{\partial y^i}$ is a vertical vector field called the external force of Σ.

A rheonomic Lagrange space $RL^n = (M, L(x,y,t))$ reduces to a Finsler space $RF^n = (M, F(x,y,t))$ if the Lagrangian function is second order homogeneous with respect to the velocity coordinates.

A first consequence of the homogeneity condition is the energy of a Finsler space coincides with the square of the fundamental function of the space:

$$E_{F^2}(x,y,t) = y^i \frac{\partial F^2}{\partial y^i} - F^2 = 2F^2 - F^2 = F^2 = g_{ij}(x,y,t)y^i y^j, \tag{3.1}$$

and it is verified the next equality

$$\frac{dF^2}{dt} = -\frac{dx^i}{dt}E_i(F^2) - \frac{\partial F^2}{\partial t}, \tag{3.2}$$

where

$$E_i(F^2) = \frac{\partial F^2}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial F^2}{\partial y^i}\right).$$

Taking into account the variational problem of the integral action of $L(x,y,t) = F^2(x,y,t)$ we introduce the evolution equations of Σ by:

The evolutions equations of the rheonomic Finslerian mechanical system Σ are the following Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = \sigma_i(x,y,t);\ y^i = \frac{dx^i}{dt} \tag{3.3}$$

where $\sigma_i(x,y,t) = g_{ij}(x,y,t)\sigma^j(x,y,t)$.

One can write an equivalent form of Lagrange equations (3.3) as a system of second order differential equations, given by

$$\frac{d^2x^i}{dt^2} + 2\Gamma^i(x,y,t) = \frac{1}{2}\sigma^i(x,y,t), \tag{3.4}$$

where

$$2\Gamma^i = 2G^i(x,y,t) + N^i_0(x,y,t),$$

$2G^i(x,y,t) = \gamma^i_{jk}(x,y,t)y^j y^k$ and $N^i_0(x,y,t) = \frac{1}{2}g^{ih}\frac{\partial^2 L}{\partial t \partial y^h}$.

The equations (3.4) are called equations of evolution of the mechanical system Σ. The solutions of these equations are called evolution curves of the mechanical system Σ.

With respect to (1.1), the functions $\breve{\Gamma}^i$:

$$2\breve{\Gamma}^i(x,y,t) = (2G^i(x,y,t) - \frac{1}{2}\sigma^i(x,y,t)) + N^i_0(x,y,t) \tag{3.5}$$

transform as

$$2\,\tilde{\breve{\Gamma}}^i(x,y,t) = 2\breve{\Gamma}^j(x,y,t)\frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{y}^i}{\partial x^j}y^j.$$

We can prove:

Theorem 3. *a) $\breve{S}$ given by:*

$$\breve{S} = y^i\frac{\partial}{\partial x^i} - 2\breve{\Gamma}^i(x,y,t)\frac{\partial}{\partial y^i} + \frac{\partial}{\partial t} \tag{3.6}$$

is a semispray on $\tilde{TM} \times R$.

b) $\breve{S}$ is a dynamical system on $\tilde{TM} \times R$ depending only on the rheonomic Finslerian mechanical system Σ. We call this semispray the evolution semispray of the mechanical system Σ.

c) The integral curves of $\breve{S}$ are the evolution curves of Σ given by (3.3).

We can say:

The geometry of the rheonomic Finslerian mechanical system Σ is the geometry of the pair $(RF^n, \breve{S})$, where RF^n is a rheonomic Finsler space and $\breve{S}$ is the evolution semispray.

The variation of the kinetic energy E_{F^2} along the evolution curves of the rheonomic mechanical system Σ, is given by:

$$\frac{dE_{F^2}}{dt} = y^i\sigma_i(x,y,t) - \frac{\partial F^2}{\partial t}.$$

The kinetic energy of the Finsler space RF^n is not conserved along the evolution curves of the mechanical system.

Now we can consider some geometric objects determined by the evolution semispray $\breve{S}$ and we will refer to these as the geometric objects of the mechanical system Σ.

a) The non-linear connection $\breve{N}$ of mechanical system Σ has the coefficients $(\breve{N}^i_j, \breve{N}^0_j)$:

$$\breve{N}^i_j = N^i_j - \frac{1}{4}\frac{\partial \sigma^i}{\partial y^j} = \frac{\partial \breve{G}^i}{\partial y^j};\ \breve{N}^0_j = \frac{1}{2}\frac{\partial^2 L}{\partial t \partial y^j}, \tag{3.7}$$

with $\breve{G}^i = G^i(x,y,t) - \frac{1}{2}\sigma^i(x,y,t)$.

$\breve{N}$ is the canonical non-linear connection of mechanical system Σ.

The adapted basis to the distributions $\breve{N}$ and $V = V_n \oplus V_0$ is given by

$$\{\frac{\breve{\delta}_i}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\} \tag{3.8}$$

where

$$\frac{\breve{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \breve{N}^i_j(x,y,t)\frac{\partial}{\partial y^j} - \breve{N}^0_j(x,y,t)\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial \sigma^j}{\partial y^i}\frac{\partial}{\partial y^j}. \tag{3.9}$$

The Lie brackets of the local vector fields from this basis are as follows:

$$\left[\frac{\breve{\delta}}{\delta x^j}, \frac{\breve{\delta}}{\delta x^h}\right] = \breve{R}^i_{jh}\frac{\partial}{\partial y^i} + \breve{R}^0_{jh}\frac{\partial}{\partial t};$$

$$\left[\frac{\breve{\delta}}{\delta x^j}, \frac{\partial}{\partial t}\right] = \frac{\partial \breve{N}^i_j}{\partial t}\frac{\partial}{\partial y^i} + \frac{\partial \breve{N}^0_j}{\partial t}\frac{\partial}{\partial t}; \left[\frac{\breve{\delta}}{\delta x^j}, \frac{\partial}{\partial y^h}\right] = \frac{\partial \breve{N}^i_j}{\partial y^h}\frac{\partial}{\partial y^i} + \frac{\partial \breve{N}^0_j}{\partial y^h}\frac{\partial}{\partial t}; \tag{3.10}$$

$$\left[\frac{\breve{\delta}}{\delta y^j}, \frac{\partial}{\partial y^h}\right] = \left[\frac{\partial}{\partial y^j}, \frac{\partial}{\partial t}\right] = \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right] = 0,$$

where

$$\breve{R}^i_{jh} = \frac{\breve{\delta}\breve{N}^i_j}{\delta x^h} - \frac{\breve{\delta}\breve{N}^i_h}{\delta x^j}; \breve{R}^0_{jh} = \frac{\breve{\delta}\breve{N}^0_j}{\delta x^h} - \frac{\breve{\delta}\breve{N}^0_h}{\delta x^j}. \tag{3.11}$$

The dual basis $\{dx^i, \breve{\delta}y^i, \breve{\delta}t\}$ is given by

$$\breve{\delta}y^i = dy^i + \breve{N}^i_j dx^j - \frac{1}{4}\frac{\partial \sigma^i}{\partial y^j}dx^j; \breve{\delta}t = dt + \breve{N}^0_i dx^i \tag{3.12}$$

and we have

$$\begin{aligned} d(dx^i) &= 0; \\ d(\breve{\delta}y^i) &= \frac{1}{2}\breve{R}^i_{jh}dx^h \wedge dx^j + \frac{\partial \breve{N}^i_j}{\partial y^h}\breve{\delta}y^h \wedge dx^j + \frac{\partial \breve{N}^i_j}{\partial t}\breve{\delta}t \wedge dx^j; \\ d(\breve{\delta}t) &= \frac{1}{2}\breve{R}^0_{jh}dx^h \wedge dx^j + \frac{\partial \breve{N}^0_j}{\partial y^h}\breve{\delta}y^h \wedge dx^j + \frac{\partial \breve{N}^0_j}{\partial t}\breve{\delta}t \wedge dx^j. \end{aligned} \tag{3.13}$$

We can prove the following theorem

Theorem 4. *a)The canonical non-linear connection $\breve{N}$ is integrable if and only if $\breve{R}^i_{jh} = 0$ and $\breve{R}^0_{jh} = 0$.*

b)The canonical metrical $\breve{N}-$connection of the rheonomic mechanical system Σ, $C\Gamma(\breve{N})$, has the coefficients given by the generalized Christoffel symbols:

$$\begin{aligned} \breve{L}^i_{jk} &= \frac{1}{2}g^{ih}\left(\frac{\breve{\delta}g_{hk}}{\delta x^j} + \frac{\breve{\delta}g_{jh}}{\delta x^k} - \frac{\breve{\delta}g_{jk}}{\delta x^h}\right) \\ \breve{C}^i_{jk} &= \frac{1}{2}g^{ih}\left(\frac{\partial g_{hk}}{\partial y^j} + \frac{\partial g_{jh}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^h}\right) \\ \breve{C}^i_{j0} &= \frac{1}{2}g^{ih}\frac{\partial g_{jh}}{\partial t}. \end{aligned} \tag{3.14}$$

c)The $h-$ and $v-$covariant derivation with respect $C\Gamma(\breve{N})$ of Liouville vector field $C = y^i \frac{\partial}{\partial y^i}$ lead us to introduce followings $h-$ and $v-$deflection tensors of $C\Gamma(\breve{N})$:

$$\breve{D}^i_j = y^i_{|j};\ \breve{d}^i_\alpha = y^i|_\alpha. \tag{3.15}$$

We may also introduce the $h-$ and $v-$electromagnetic tensors

$$\breve{\mathcal{F}}_{ij} = \frac{1}{2}\left(\breve{D}_{ij} - \breve{D}_{ji}\right);\ \breve{f}_{ij} = \frac{1}{2}\left(\breve{d}_{ij} - \breve{d}_{ji}\right) \tag{3.16}$$

where $\breve{D}_{ij} = g_{ir}\breve{D}^r_j$, $\breve{d}_{i\alpha} = g_{ir}\breve{d}^r_\alpha$.

Let us consider the helicoidal tensor of Σ:

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial\sigma^i}{\partial y^j} - \frac{\partial\sigma^j}{\partial y^i}\right). \tag{3.17}$$

We obtain $\breve{f}_{ij} = 0$ and the following theorem

Theorem 5. *Between the $h-$electromagnetic tensor of the rheonomic Finslerian mechanical system $\breve{\mathcal{F}}_{ij}$, the $h-$electromagnetic tensor of the rheonomic Finsler space $\mathcal{F}_{ij}$ and the helicoidal tensor σ_{ij} of Σ the following relation holds:*

$$\breve{\mathcal{F}}_{ij} = \mathcal{F}_{ij} + \frac{1}{4}\sigma_{ij}. \tag{3.18}$$

The proof is not difficult.

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