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CERTAIN FAMILY OF ANALYTIC AND UNIVALENT FUNCTIONS WITH NORMALIZED CONDITIONS

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Abstract. There are many subclasses of analytic and univalent functions. The object of this paper is to introduce new classes and we have attempted to obtain coefficient estimate, distortion theorem, radius of starlikeness, convexity and closure theorem for the classes $S^*(\alpha, \beta, \xi, \gamma)$ and $K^*(\alpha, \beta, \xi, \gamma)$ on the lines of [1] and [2]. Results obtained by [1] and [2] which are particular cases of the parameters involved here, are also pointed out.

1. Introduction

Let A denote the class of functions given by,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and normalized by $f(0) = 0$; $f'(0) = 1$, and let S be the subclass of A consisting of analytic and univalent functions of the form (1.1). We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of S consisting of all functions which are, respectively starlike and convex of order α in E with $(0 \le \alpha < 1)$. Thus,

$$S^*(\alpha) = \left\{ f \in S : \mathrm{Re}\left(z\frac{f'(z)}{f(z)} \right) > \alpha; \quad 0 \le \alpha < 1, z \in E \right\}$$

and

$$K(\alpha) = \left\{ f \in S : \mathrm{Re}\left(1 + z\frac{f''(z)}{f'(z)} \right) > \alpha; \quad 0 \le \alpha < 1, z \in E \right\}.$$

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We say that a function $f(z)$ is in the class $S(\alpha, \beta, \xi, \gamma)$ if and only if,

$$\left| \frac{z\frac{f'(z)}{f(z)} - 1}{2\xi\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)} - 1\right)} \right| < \beta \text{ for } |z| < 1$$

where $0 < \beta \leq 1$, $\frac{1}{2} \leq \xi \leq 1$, $0 \leq \alpha \leq \frac{1}{2}\xi$, $\frac{1}{2} < \gamma \leq 1$.

If we replace γ by 1 in the above inequality; we obtain the result of Aghalary and Kulkarni [1] and Silverman and Silvia [2]. If we replace ξ by 1 we obtain the result of [3].

Furthermore a function $f(z)$ is said to belong to the class $K(\alpha, \beta, \xi, \gamma)$ if and only if $zf'(z) \in S(\alpha, \beta, \xi, \gamma)$. Let T denote the subclass of S consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

We denote by $S^*(\alpha, \beta, \xi, \gamma)$ and $K^*(\alpha, \beta, \xi, \gamma)$ the classes obtained by taking intersection, respectively of the classes $S(\alpha, \beta, \xi, \gamma)$ and $K(\alpha, \beta, \xi, \gamma)$ with T.

In this paper we obtain sharp result for coefficient estimates, distortion theorem, radius of starlikeness and convexity, and other related results for the classes $S^*(\alpha, \beta, \xi, \gamma)$ and $K^*(\alpha, \beta, \xi, \gamma)$.

2. Coefficient Estimates

Theorem 1. *A function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$$

is in $S^*(\alpha, \beta, \xi, \gamma)$ *if and only if;*

$$\sum_{n=2}^{\infty}[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2\xi n]a_n \leq 2\beta\xi(1-\alpha).$$

Proof. Suppose,

$$\sum_{n=2}^{\infty}[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2\xi n)]a_n \leq 2\beta\xi(1-\alpha).$$

We have

$$|zf' - f| - \beta|2\xi(zf' - \alpha f) - \gamma(zf' - f)| < 0$$

with the provision:

$$\left|\sum_{n=2}^{\infty}(n-1)a_n z^n\right| - \beta\left|2\xi(1-\alpha) + \sum_{n=2}^{\infty}(2\xi\alpha - 2\xi n + \gamma n - \gamma)a_n z^n\right| < 0 \tag{2.1}$$

for $|z| = r < 1$; then the condition (2.1) is bounded above by

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-1)a_n r^n - 2\beta\xi(1-\alpha) - \beta\sum_{n=2}^{\infty}(2\xi\alpha - 2\xi n + \gamma n - \gamma)a_n r^n \\
&= \sum_{n=2}^{\infty}\{(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)\}a_n r^n - 2\beta\xi(1-\alpha) \\
&\leq \sum_{n=2}^{\infty}\{(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)\}a_n - 2\beta\xi(1-\alpha) \leq 0.
\end{aligned}$$

Therefore $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$.

Now, we prove the converse result. Let

$$\left|\frac{z\frac{f'(z)}{f(z)} - 1}{2\xi\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \gamma\left(z\frac{f'(z)}{f(z)} - 1\right)}\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(n-1)a_n z^n}{2\xi(1-\alpha) + \sum\limits_{n=2}^{\infty}(2\xi\alpha - 2\xi n + \gamma n - \gamma)a_n z^n}\right| < \beta.$$

As $|\operatorname{Re}(z)| \leq |z|$ for all z, we have

$$\operatorname{Re}\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)a_n z^n}{2\xi(1-\alpha) + \sum\limits_{n=2}^{\infty}(2\xi\alpha - 2\xi n + \gamma n - \gamma)a_n z^n}\right| < \beta.$$

We choose values of z on real axis such that $zf'(z)/f(z)$ is real and upon clearing the denominator of above expression and allowing $z \to 1$ through real values, we obtain

$$\sum_{n=2}^{\infty}\{(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)\}a_n - 2\beta\xi(1-\alpha) \leq 0.$$

□

Remark. If $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$, then

$$a_n \leq \frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)\}} \text{ for } n = 2, 3, 4, \ldots$$

and equality holds for

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)\}}z^n.$$

Corollary 1. *If $f(z) \in S^*(\alpha, \beta, \xi, 1)$ i.e. replacing $\gamma = 1$, we get*

$$a_n \leq \frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(2\xi\alpha - 2\xi n + n - 1)\}}, \quad n = 2, 3, 4, \dots$$

and equality holds for

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(2\xi\alpha - 2\xi n + n - 1)\}} z^n$$

which is a known result of [1] *and* [2].

Corollary 2. *If $f(z) \in S^*(\alpha, \beta, 1, 1)$ we get*

$$a_n \leq \frac{2\beta(1-\alpha)}{\{(n-1) - \beta(2\alpha + 3n - 1)\}}$$

and equality holds for

$$f(z) = z - \frac{2\beta(1-\alpha)}{\{(n-1) - \beta(2\alpha + 3n - 1)\}} z^n$$

which is a known result of [3].

Corollary 3. *$f(z) \in S^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} (n + \alpha) a_n \leq (1 - \alpha).$$

Theorem 2. *A function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$$

is in $K^*(\alpha, \beta, \xi, \gamma)$ if and only if,

$$\sum_{n=2}^{\infty} n[(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)] a_n \leq 2\beta\xi(1-\alpha).$$

Proof. The proof of this theorem is analogous to that of theorem [1], because a function $f(z) \in K^*(\alpha, \beta, \xi, \gamma)$, if and only if $zf'(z) \in S^*(\alpha, \beta, \xi, \gamma)$ so it is enough that a_n in Theorem 1 is replaced with na_n. □

Remark. If $f \in K^*(\alpha, \beta, \xi, \gamma)$, then

$$a_n \leq \frac{2\beta\xi(1-\alpha)}{n\{(n-1) - \beta(2\xi\alpha - 2n\xi + \gamma n - \gamma)\}} \text{ for } n = 2, 3, 4, \dots$$

and equality holds for

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{n\{(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)\}} z^n.$$

Corollary 4. *If $f(z) \in K^*(\alpha, \beta, \xi, 1)$ i.e. replacing $\gamma = 1$, we get*

$$a_n \leq \frac{2\beta\xi(1-\alpha)}{n\{(n-1) - \beta(2\xi\alpha - 2\xi n + n - 1)\}} \quad n = 2, 3, 4, \ldots$$

and equality holds for

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{\{(n-1) - \beta(2\xi\alpha - 2\xi n + n - 1)\}} z^n.$$

This result is due to [1] and [2].

Corollary 5. *If $f(z) \in K^*(\alpha, \beta, 1, 1)$, then*

$$a_n \leq \frac{2\beta(1-\alpha)}{n\{(n-1) - \beta(2\alpha + 3n - 1)\}}, \quad n = 2, 3, 4, \ldots$$

with equality for

$$f(z) = z - \frac{2\beta(1-\alpha)}{n\{(n-1) - \beta(2\alpha + 3n - 1)\}} z^n.$$

Corollary 6. *$f(z) \in K^*(\alpha)$ if and only if,*

$$\sum_{n=2}^{\infty} n(n+\alpha) a_n \leq (1-\alpha).$$

3. Growth and Distortion Theorem

Theorem 3. *If $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$ then*

$$r - \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi) - \beta(\gamma + 2\xi\alpha)} r^2 \leq |f(z)| \leq r + \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi) - \beta(\gamma + 2\xi\alpha)} r^2. \tag{3.1}$$

Equality holds for

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi) - \beta(\gamma + 2\xi\alpha)} z^2 \ \text{ at } z = \pm r.$$

Proof. By Theorem 1, we have $f(z) \in S^*(\alpha, \beta, \xi, \gamma)$ if and only if,

$$\sum_{n=2}^{\infty} [(n-1) - \beta(2\xi\alpha - 2\xi n + \gamma n - \gamma)] a_n \leq 2\beta\xi(1-\alpha)$$

or equivalently,

$$\sum_{n=2}^{\infty} a_n \left\{ n - \left(1 - \frac{2\beta\xi(1-\alpha)}{1 + 2\beta\xi - \gamma\beta} \right) \right\} \leq \frac{2\beta\xi(1-\alpha)}{1 + 2\beta\xi - \gamma\beta}. \tag{3.2}$$

So $f \in S^*(\alpha, \beta, \xi, \gamma)$ if and only if, by (3.2)

$$\sum_{n=2}^{\infty} a_n (n-t) \leq (1-t) \text{ where } t = 1 - \frac{2\beta\xi(1-\alpha)}{1 + 2\beta\xi - \gamma\beta}. \tag{3.3}$$

But

$$(2-t)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} a_n(n-t) \leq (1-t).$$

This last inequality follows from (3.3). We obtain;

$$|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + r^2 \left(\frac{1-t}{2-t}\right).$$

Similarly,

$$|f(z)| \geq r - \sum_{n=2}^{\infty} a_n r^n \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - r^2 \left(\frac{1-t}{2-t}\right).$$

So

$$r - \left(\frac{1-t}{2-t}\right) r^2 \leq |f(z)| \leq r + \left(\frac{1-t}{2-t}\right) r^2$$

that is

$$r - \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi)-\beta(\gamma+2\xi\alpha)} r^2 \leq |f(z)| \leq r + \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi)-\beta(\gamma+2\xi\alpha)} r^2.$$

Hence the result (3.1). □

Corollary 7. *If $f \in S^*(\alpha,\beta,\xi,1)$ i.e. replacing $\gamma = 1$, then*

$$r - \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi)-\beta(1+2\xi\alpha)} r^2 \leq |f(z)| \leq r + \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi)-\beta(1+2\xi\alpha)} r^2.$$

With equality for,

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{(1+4\beta\xi)-\beta(1+2\xi\alpha)} z^n \ \text{ at } z = \pm r.$$

This result is due to [1] and [2].

Corollary 8. *If $f \in S^*(\alpha,\beta,1,1)$ i.e. replacing ξ by 1 and γ by 1, then*

$$r - \frac{2\beta(1-\alpha)}{(1+4\beta)-\beta(1+2\alpha)} r^2 \leq |f(z)| \leq r + \frac{2\beta(1-\alpha)}{(1+4\beta)-\beta(1+2\alpha)} r^2.$$

With equality for

$$f(z) = z - \frac{2\beta(1-\alpha)}{(1+4\beta)-\beta(1+2\alpha)} z^n \ \text{ at } z = \pm r.$$

Theorem 4. *If $f(z) \in K^*(\alpha,\beta,\xi,\gamma)$, then*

$$r - \frac{\beta\xi(1-\alpha)}{(1+4\beta\xi)-\beta(\gamma+2\xi\alpha)} \leq |f(z)| \leq r + \frac{\beta\xi(1-\alpha)}{(1+4\beta\xi)-\beta(\gamma+2\xi\alpha)}.$$

Proof. The proof of this theorem is analogous to that of theorem (3), because a function $f(z) \in K^*(\alpha,\beta,\xi,\gamma)$ if and only if $zf'(z) \in S^*(\alpha,\beta,\xi,\gamma)$. So it will be enough that a_n in Theorem 3 is replace with na_n. □

Corollary 9. *If $f \in K^*(\alpha, \beta, \xi, 1)$ i.e. replacing γ by 1, then*

$$r - \frac{\beta\xi(1-\alpha)}{(1+4\beta\xi) - \beta(1+2\xi\alpha)} \leq |f(z)| \leq r + \frac{\beta\xi(1-\alpha)}{(1+4\beta\xi) - \beta(1+2\xi\alpha)}$$

with equality for

$$f(z) = z - \frac{\beta\xi(1-\alpha)}{(1+4\beta\xi) - \beta(1+2\xi\alpha)} z^2 \ \text{at}\ z = \pm r.$$

This is due to [1] *and* [2].

Corollary 10. *If $f \in K^*(\alpha, \beta, 1, 1)$ i.e.; replacing $\xi = 1$ and $\gamma = 1$, then*

$$r - \frac{\beta(1-\alpha)}{(1+3\beta-2\alpha)} \leq |f(z)| \leq r + \frac{\beta(1-\alpha)}{(1+3\beta-2\alpha)}$$

with equality for

$$f(z) = z - \frac{\beta(1-\alpha)}{(1+3\beta-2\alpha)} z^2 \ \text{at}\ z = \pm r.$$

This corollary is due to [3].

Theorem 5. *If $f \in S^*(\alpha, \beta, \xi, \gamma)$, then*

$$1 - \frac{4\beta\xi(1-\alpha)}{(1-\gamma\beta) + 2\beta\xi(2-\alpha)} r \leq |f'(z)| \leq 1 + \frac{4\beta\xi(1-\alpha)}{(1-\gamma\beta) + 2\beta\xi(2-\alpha)} r. \tag{3.4}$$

Proof. Since $f \in S^*(\alpha, \beta, \xi, \gamma)$, we have

$$\sum_{n=2}^{\infty} a_n(n-t) \leq (1-t) \ \text{where}\ t = 1 - \frac{2\beta\xi(1-\alpha)}{1+2\beta\xi-\gamma\beta}. \tag{3.5}$$

Now in view of Theorem 3, we have

$$\sum_{n=2}^{\infty} na_n = \sum_{n=2}^{\infty}(n-t)a_n + t\sum_{n=2}^{\infty} a_n \leq (1-t) + t\left(\frac{1-t}{2-t}\right) = \frac{2(1-t)}{(2-t)}.$$

Therefore,

$$|f'| \leq 1 + \sum_{n=2}^{\infty} na_n|z|^{n-1} \leq 1 + r\sum_{n=2}^{\infty} na_n \leq 1 + r\frac{2(1-t)}{(2-t)}.$$

Similarly,

$$|f'| \geq 1 - \sum_{n=2}^{\infty} na_n|z|^{n-1} \geq 1 - r\sum_{n=2}^{\infty} na_n \geq 1 - r\frac{2(1-t)}{(2-t)}.$$

So

$$1 - r\left(\frac{2(1-t)}{(2-t)}\right) \leq |f'(z)| \leq 1 + r\left(\frac{2(1-t)}{(2-t)}\right) \tag{3.6}$$

by substituting t from (3.5) in (3.6), the result (3.4) is obtained. ∎

Corollary 11. *If $f \in S^*(\alpha, \beta, \xi, 1)$, then*

$$1 - \frac{4\beta\xi(1-\alpha)}{(1-\beta)+2\beta\xi(2-\alpha)}r \leq |f'(z)| \leq 1 + \frac{4\beta\xi(1-\alpha)}{(1-\beta)+2\beta\xi(2-\alpha)}r \text{ for } |z| = r.$$

This result is due to [1] and [2].

Corollary 12. *If $f \in S^*(\alpha, \beta, 1, 1)$, then*

$$1 - \frac{4\beta(1-\alpha)}{1+3\beta-2\alpha\beta}r \leq |f'(z)| \leq 1 + \frac{4\beta(1-\alpha)}{1+3\beta-2\alpha\beta}r.$$

This inequality is due to Kulkarni [3].

Theorem 6. *If $f \in K^*(\alpha, \beta, \xi, \gamma)$, then*

$$1 - \frac{2\beta\xi(1-\alpha)}{(1-\gamma\beta)+2\beta\xi(2-\alpha)}r \leq |f'(z)| \leq 1 + \frac{2\beta\xi(1-\alpha)}{(1-\gamma\beta)+2\beta\xi(2-\alpha)}r \text{ for } |z| = r.$$

Proof. The proof of this theorem is similar to that of Theorem 5 because a function $f \in K^*(\alpha, \beta, \xi, \gamma)$ if and only if $zf' \in S^*(\alpha, \beta, \xi, \gamma)$. □

Corollary 13. *If $f \in K^*(\alpha, \beta, \xi, 1)$, then*

$$1 - \frac{2\beta\xi(1-\alpha)}{(1-\beta)+2\beta\xi(2-\alpha)}r \leq |f'(z)| \leq 1 + \frac{2\beta\xi(1-\alpha)}{(1-\beta)+2\beta\xi(2-\alpha)}r \text{ for } |z| = r.$$

This result is due to [1] and [2].

Corollary 14. *If $f \in K^*(\alpha, \beta, 1, 1)$ then*

$$1 - \frac{2\beta(1-\alpha)}{1+3\beta-2\alpha\beta}r \leq |f'(z)| \leq 1 + \frac{2\beta(1-\alpha)}{1+3\beta-2\alpha\beta}r \text{ for } |z| = r.$$

This inequality is due to Kulkarni [3].

4. Radius of Convexity

Theorem 7. *If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha, \beta, \xi, \gamma)$ then f is convex in the disc,*

$$0 < |z| < r = r(\alpha, \beta, \xi, \gamma) = \inf_n \left[\frac{(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)}{(n^2(1+2\beta\xi-\gamma\beta) + n(\gamma\beta-1)(1-\alpha)} \right]^{1/n-1}.$$

This result is sharp, with the extremal function,

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)]} z^n \text{ for some } n.$$

Proof. We know that $f(z) \in K(\alpha, \beta, \xi, \gamma)$. So it is sufficient to show that,

$$\left| \frac{\frac{z(zf')'}{zf'} - 1}{2\xi\left(z\frac{(zf')'}{zf'} - \alpha\right) - \gamma\left(z\frac{(zf')'}{zf'} - 1\right)} \right| < \beta \text{ for } |z| \leq r(\alpha, \beta, \xi, \gamma) \tag{4.1}$$

We have

$$\left|\frac{z\frac{(zf')'}{zf'}-1}{\beta\left[2\xi\left(\frac{z(zf')'}{zf'}-\alpha\right)-\gamma\left(z\frac{(zf')'}{zf'}-1\right)\right]}\right|$$

$$=\left|\frac{zf''}{\beta[(2\xi-\gamma)zf''+2\xi(1-\alpha)f']}\right|$$

$$=\left|\frac{-\sum\limits_{n=2}^{\infty}n(n-1)a_nz^{n-1}}{\beta\left[2\beta\xi(1-\alpha)-\sum\limits_{n=2}^{\infty}\left[(2\xi-\gamma)n(n-1)-2n\xi(1-\alpha)\right]a_nz^{n-1}\right]}\right|$$

$$\leq\frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{2\beta\xi(1-\alpha)-\sum\limits_{n=2}^{\infty}[\beta n(n-1)(2\xi-\gamma)-2n\xi(1-\alpha)]a_n|z|^{n-1}}.$$

Thus (4.1) holds if,

$$\sum_{n=2}^{\infty}\left[n(n-1)[1+\beta(2\xi-\gamma)]+2\beta\xi n(1-\alpha)\right]a_n|z|^{n-1}\leq 2\beta\xi(1-\alpha).$$

In view of coefficient inequality in Theorem 1, we have

$$|z|^{n-1}\leq\frac{(n-1)-\beta(\gamma n-\gamma+2\xi\alpha-2n\xi)}{n^2(1+2\beta\xi-\beta\gamma)-n(\beta\gamma+2\beta\xi\alpha-1)},\quad n=2,3,\ldots \tag{4.2}$$

The desired result follows by substituting $|z|=r(\alpha,\beta,\xi,\gamma)$ in the above expression. □

Corollary 15. *If $f(z)\in S^*(\alpha,\beta,\xi,1)$ then f is convex in the disc,*

$$0<|z|<r=r(\alpha,\beta,\xi)=\inf_n\left\{\frac{(n-1)-\beta(n-1+2\xi\alpha-2n\xi)}{[n^2(1+2\beta\xi-\beta)+n(\beta+2\beta\xi\alpha-1)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp with the extremal function,

$$f(z)=z-\frac{2\beta\xi(1-\alpha)}{[(n-1)-\beta(n-1+2\xi\alpha-2n\xi)]}z^n$$

for some n.

This result is due to [1] and [2].

Corollary 16. *If $f(z)\in S^*(\alpha,\beta,1,1)$ then f is convex in the disc,*

$$0<|z|<r=r(\alpha,\beta)=\inf_n\left\{\frac{(n-1)-\beta(2\alpha-n-1)}{n^2(1+\beta)-n(\beta+2\beta\alpha-1)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp with the extremal function,

$$f(z) = z - \frac{2\beta(1-\alpha)}{[(n-1) - \beta(2\alpha - n - 1)]} z^n$$

for some n.

This result is due to Kulkarni [3].

Corollary 17. *If* $f(z) \in S^*(0,1,1,1)$ *then* f *is convex in the disc,*

$$0 < |z| < r = r(0,1,1,1) = \inf_n \left\{ \frac{1}{n} \right\}^{\frac{1}{n-1}}, \quad n = 2,3,4,\ldots.$$

5. Closure Theorem

Theorem 8. *Let* $f_1(z) = z$ *and*

$$f_n(z) = \frac{2\beta\xi(1-\alpha)}{[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)]} z^n \text{ for } n = 2,3,4,\ldots.$$

Then $f(z) \in K^*(\alpha,\beta,\xi,\gamma)$ *if and only if,* $f(z)$ *can be expressed in the forms,*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \text{ where } \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1.$$

Proof. Suppose $f(z)$ can be written in the form

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{2\beta\xi(1-\alpha)}{(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)} z^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\lambda_n 2\beta\xi(1-\alpha)}{[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)]} \\ &\times \frac{[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)]}{2\beta\xi(1-\alpha)} = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1. \end{aligned}$$

Therefore $f(z) \in K^*(\alpha,\beta,\xi,\gamma)$.

Conversely, suppose $f(z) \in K^*(\alpha,\beta,\xi,\gamma)$ then remark of Theorem 1 gives us;

$$a_n \leq \frac{2\beta\xi(1-\alpha)}{[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)]}.$$

We take,

$$\lambda_n = \frac{[(n-1) - \beta(\gamma n - \gamma + 2\xi\alpha - 2n\xi)]}{2\beta\xi(1-\alpha)} a_n, \quad n = 2,3,4,\ldots$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$. Then $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$. □

Corollary 18. *If $f_1(z) = z$ and*

$$f_n(z) = z - \frac{2\beta\xi(1-\alpha)}{[(n-1) - \beta(n-1+2\xi\alpha-2n\xi)]} z^n \text{ for } n = 2, 3, \dots.$$

Then $f(z) \in K^*(\alpha, \beta, \xi, 1)$ if and only if, $f(z)$ can be expressed in the form,

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \text{ where } \lambda_n \geq 0, n = 1, 2, \dots, \quad \sum_{n=1}^{\infty} \lambda_n = 1.$$

This result is due to [1] and [2].

Corollary 19. *If $f_1(z) = z$ and*

$$f_n(z) = z - \frac{2\beta(1-\alpha)}{[(n-1) - \beta(2\alpha-n-1)]} z^n \text{ for } n = 2, 3, \dots.$$

Then $f(z) \in K^*(\alpha, \beta)$ if and only if, $f(z)$ can be expressed in the form,

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \text{ where } \lambda_n \geq 0, n = 1, 2, \cdots, \quad \sum_{n=1}^{\infty} \lambda_n = 1.$$

This result is due to Kulkarni [3].

Corollary 20. *If $f_1(z) = z$ and*

$$f_n(z) = z - \frac{1}{n} z^n.$$

Then $f(z) \in K^*(0, 1, 1, 1)$ if and only if, $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \text{ where } \lambda_n \geq 0, \quad \sum_{n=1}^{\infty} \lambda_n = 1.$$

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