SUFFICIENT CONDITIONS FOR OSCILLATORY BEHAVIOUR OF A FIRST ORDER NEUTRAL DIFFERENCE EQUATION WITH OSCILLATING COEFFICIENTS

R. N. RATH, N. MISRA, AND S. K. RATH

ABSTRACT. In this paper, we obtain sufficient conditions so that every solution of neutral functional difference equation

$$\Delta(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n$$

oscillates or tends to zero as $n \to \infty$. Here Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, and p_n, q_n, f_n are the terms of oscillating infinite sequences; $\{\tau_n\}$ and $\{\sigma_n\}$ are non-decreasing sequences, which are less than n and approaches ∞ as n approaches ∞. This paper generalizes and improves some recent results.

1. INTRODUCTION

In this work, we find sufficient conditions, so that every solution of neutral functional difference equation

$$(1) \qquad \Delta(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n$$

oscillates or tends to zero as $n \to \infty$, where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}, \{q_n\}$ and $\{f_n\}$ are infinite sequences of real numbers (not necessarily positive) and $G \in C(R, R), \tau(n)$ and $\sigma(n)$ are non-decreasing sequences, which are less than n and approaches ∞ as n approaches ∞.

Let n_0 be a fixed nonnegative integer. Let $\rho = \min\{\tau(n_0), \sigma(n_0)\}$. By a solution of (1) we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geq \rho$ and satisfies (1) for $n \geq n_0$. Clearly if the initial condition

$$(2) \qquad y_n = a_n \quad \text{for} \quad \rho \leq n \leq n_0$$

is given then the equation (1) has a unique solution satisfying the given initial condition (2). A solution $\{y_n\}$ of (1) is said to be oscillatory if for every

positive integer $n_0 > 0$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

If we put $\tau(n) = n - m$ and $\sigma(n) = n - k$, where m, k are non-negative integers then (1) reduces to

$$(3) \qquad \Delta(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) = f_n.$$

Further, if we put $p_n = 0$ in (3), then we obtain the delay difference equation

$$(4) \qquad \Delta(y_n) + q_n G(y_{n-k}) = f_n.$$

Hence (1) is more general than (3) and (4).

Recently the oscillatory and asymptotic behavior of (3) and (4) have been investigated by many authors (see [13]–[23], [25]) when q_n is non-negative. However, the general case, when q_n is allowed to oscillate, it is difficult to study the oscillation of (3) or (4), since the difference $\Delta(y_n - p_n y_{n-m} - \sum_{i=n_0}^{n-1} f_i)$ of any non-oscillatory solution of (3) is always oscillatory. Therefore, the results on oscillation of (4), (3), with oscillating q_n, are relatively scarce; see [18, 24, 22]. The motivation of this work is because of the interesting open problem for the above case in [9].

The open problem 7.11.3 of [9, pp197] reads as:

Problem 1.1. Extend the following result to difference equation with oscillating coefficients q_n.

Theorem 1.2 ([9, Theorem 7.5.1]). *Suppose that $\{q_n\}$ is a non negative sequence of real numbers and k be a positive integer. Then*

$$(5) \qquad \liminf_{n \to \infty} \left[\sum_{i=n-k}^{n-1} q_i \right] > \left(\frac{k}{k+1} \right)^{k+1}$$

is a sufficient condition for every solution of

$$(6) \qquad y_{n+1} - y_n + q_n y_{n-k} = 0$$

to be oscillatory.

Note that if $k = 2$ and $q_n = \left(\frac{1}{e^2} - \frac{1}{e^3} \right)$ then the difference equation (6) admits a positive solution $y_n = e^{-n}$, which approaches zero as $n \to \infty$. In this case (5) does not hold. However,

$$(7) \qquad \sum_{n=n_0}^{\infty} q_n = \infty$$

holds and (5) implies (7).

In view of this, we deal in this paper, with the problem which is slightly different from the Problem 1.1. In fact, our main result would be the following theorem, where sufficient conditions are obtained so that every solution of (6) oscillates or tends to zero as $n \to \infty$.

Theorem 1.3. *Suppose that q_n changes sign and satisfies the condition*

$$\sum_{n=0}^{\infty} q_n^- < \infty \quad where \quad q_n^- = \max(-q_n, 0).$$

Then the condition

$$(8) \qquad \sum_{n=n_0}^{\infty} |q_n| = \infty$$

is sufficient for every solution of (6) *to oscillate or to tend to zero as $n \to \infty$.*

2. MAIN RESULTS

We need the following hypothesis to be used in this article.

(H1) There exists integers $n_0 > 0, r \le 0$ and $R \ge 0$ such that $R - r < 1$ and $r \le p_n \le R$ for $n \ge n_0$.

(H2) G is bounded.

(H3) $xG(x) > 0$ for $x \ne 0$.

(H4) $\sum_{n=0}^{\infty} q_n^+ = \infty$, where $q_n^+ = \max(q_n, 0)$.

(H5) $\sum_{n=0}^{\infty} q_n^- < \infty$, where $q_n^- = \max(-q_n, 0)$.

(H6) $\sum_{n=n_0}^{\infty} |f_n| < \infty$.

As a proto type of an infinite sequence satisfying (H4) and (H5), we have

$$(9) \qquad q_n = \frac{n|\sin(n\pi/2)| - |\sin((n-1)\pi/2)|}{n^2}$$

Hence, for $n = 1, 2, 3, \ldots$ we obtain

$$q_{2n-1}^+ = 1/(2n-1), q_{2n}^+ = 0, q_{2n-1}^- = 0 \text{ and } q_{2n}^- = 1/(2n)^2.$$

A proto type of a function satisfying (H2)-(H3) is $G(u) = ue^{-u^2}$ or $G(u) = \frac{u}{u^2+1}$, which are monotonic non-increasing.

¿From the definitions of q_n^+ and q_n^-, it follows that $q_n^+ \ge 0$, $q_n^- \ge 0$ and $q_n = q_n^+ - q_n^-$. Then using this decomposition, (1) can be rewritten as

$$(10) \qquad \Delta(y_n - p_n y_{\tau(n)}) + q_n^+ G(y_{\sigma(n)}) - q_n^- G(y_{\sigma(n)}) = f_n$$

Now, we present our first result.

Theorem 2.1. *Suppose (H1)–(H6) hold. Then every solution of* (1) *oscillates or tends to zero as n tends to ∞.*

Proof. Let $\{y_n\}$ be any solution of (1). If it oscillates, then we have nothing to prove. If it does not oscillate then assume that $\{y_n\}$ be an eventually positive solution of (1) for $n \ge n_0$. Our intention is to prove that y_n approaches zero as $n \to \infty$. If necessary increment n_0 so that

$$(11) \qquad y_{\tau(n)} > 0, y_{\sigma(n)} > 0 \quad \text{for } n > n_0.$$

For simplicity of notation, define

$$(12) \qquad z_n = y_n - p_n y_{\tau(n)}.$$

Using (H2) and (H5), define

$$(13) \qquad T_n = \sum_{i=n}^{\infty} q_i^- G(y_{\sigma(i)}),$$

and

$$(14) \qquad w_n = y_n - p_n y_{\tau(n)} + T_n - \sum_{i=n_0}^{n-1} f_i.$$

It may be noted that $T_n > 0$, when $y_n > 0$ and $T_n < 0$, when $y_n < 0$. From (10), using Eqs(12)– Eq(14) for $n > N_1 > N_0$ we obtain

$$(15) \qquad \Delta w_n = -q_n^+ G(y_{\sigma(n)}) \leq 0.$$

Hence, $\{w_n\}$ is non-increasing, implying $w_n > 0$ or $w_n < 0$ for large n. By (H6), the function $\sum_{i=n_0}^{n} f_i$ being bounded, we have

$$(16) \qquad w_{n_0} + \sup_{n \geq n_0} \sum_{i=n_0}^{n} f_i \geq y_n - p_n y_{\tau(n)} \geq y_n - R y_{\tau(n)}.$$

Next, we claim that $\{y_n\}$ is bounded. Otherwise, there exists a sequence $\{y_{n_k}\}$ such that $y_{n_k} \to \infty$ as $k \to \infty$ and $y_{n_k} = \max(y_n : n_0 \leq n \leq n_k)$. Note that $y_{\tau(n_k)} \to \infty$ as $k \to \infty$. Since $\tau(n) \leq n$, from (16) it follows that for each n_k

$$w_{n_0} + \sup_{k} \sum_{i=n_0}^{n_k} f_i \geq (1 - R) y_{n_k}.$$

Since $(1 - R) > 0$, the right hand side approaches ∞, as $k \to \infty$. This is a contradiction, that proves, $\{y_n\}$ is bounded. Using (H5) and (H6),and noting that, $\{p_n\}$ being bounded, we see that $\{z_n\}$ and $\{w_n\}$ are bounded sequences. Then $\{w_n\}$ must converge, as it is monotonic. By (H5) the sequence $\{T_n\}$ is convergent and tends to zero as $n \to \infty$. This along with(H6), implies $\{z_n\}$ is also convergent. Let

$$(17) \qquad l := \lim_{n \to \infty} z_n = \lim_{n \to \infty} w_n.$$

Next, we claim $\liminf_{n \to \infty} y_n = 0$. Otherwise, for $n \geq N_2 > N_1$ we have $y_n > 0$. Then $y_{\sigma(n)} > 0$. From the definition of $\liminf$, there exists constants α and $N_3 > N_2$ such that $y_{\sigma(n)} \geq \alpha$ for all $n \geq N_3$. Since $\{y_n\}$ is bounded, then we can find a upper bound β for $\{y_n\}$. The continuity of G and (H3) imply the existence of a positive lower bound m for G on $[\alpha, \beta]$; i.e., $0 < m < G(y_{\sigma(n)})$ for all $n \geq N_4 \geq N_3$. Then summing (15) from $i = N_4$ to $n-1$, we obtain that

$$w_{N_4} - w_n = \sum_{i=N_4}^{n} q_i^+ G(y_{\sigma(i)}) \geq m \sum_{i=N_4}^{n} q_i^+.$$

Since the left hand side is a member of a bounded sequence, while the right hand side approaches $+\infty$, we have a contradiction. Therefore, $\liminf y_n = 0$. Next, we prove that $\limsup y_n = 0$.

Since $y_n \geq 0$, from assumption (H1), it follows that $y_n - p_n y_{\tau(n)} \geq y_n - R y_{\tau(n)}$. As we know that, for bounded functions,

$$\limsup\{f_n + g_n\} \geq \limsup\{f_n\} + \liminf\{g_n\}.$$

Therefore, by taking $\limsup$ in (12), we obtain that

(18)
$$\begin{aligned}
l &= \limsup_{n\to\infty}\{y_n - p_n y_{\tau(n)}\} \\
&\geq \limsup_{n\to\infty}\{y_n\} + \liminf_{n\to\infty}\{-R y_{\tau(n)}\} \\
&\geq \limsup_{n\to\infty}\{y_n\} - R\limsup_{n\to\infty}\{y_{\tau(n)}\} \\
&\geq (1-R)\limsup_{n\to\infty}\{y_n\}.
\end{aligned}$$

From (H1), it follows that $y_n - p_n y_{\tau(n)} \leq y_n - r y_{\tau(n)}$. Since

$$\liminf\{f_n + g_n\} \leq \liminf\{f_n\} + \limsup\{g_n\},$$

taking $\liminf$ in (12), we obtain

$$\begin{aligned}
l &= \liminf_{n\to\infty}\{y_n - r y_{\tau(n)}\} \\
&\leq \liminf_{n\to\infty}\{y_n\} + \limsup_{n\to\infty}\{-r y_{\tau(n)}\} \\
&= 0 - r\limsup_{n\to\infty}\{y_n\}.
\end{aligned}$$

From (18) and above inequality, we have

$$(1 - R + r)\limsup_{n\to\infty}\{y_n\} \leq 0.$$

Since $y_n \geq 0$, by (H1), it follows that $\limsup_{n\to\infty} y_n = 0$. Hence, $\lim_{n\to\infty} y_n = 0$. The proof for the case $y_n < 0$ is similar. □

Remark 1. It is not difficult to see that if p_n satisfies the condition $0 \leq p_n \leq p < 1$ or $-1 < -p \leq p_n \leq 0$, instead of (H1) then also the above theorem holds. Note that if $q_n \geq 0$, then $q_n^- = 0$ and $q_n^+ = q_n$. Hence, the above theorem improves and generalizes [13, Theorems 2.1 and 2.3], [17, Theorems 2.3 and 2.4], and [20, Corollary 2.5].

Note that, in the above theorem, we assumed that G is bounded. However, equation (6) that is considered in Theorem 1.3, does not satisfy this condition. To address this problem, we introduce the following hypothesis, and state another theorem.

(H7) There exists non-negative constants a, b such that $|G(u)| \leq a|u| + b$, for all u.

Theorem 2.2. *Assume (H1),(H3)-(H7) hold. Then every solution of (1) oscillates or tends to zero as $n \to \infty$.*

Proof. As in theorem 2.1, we prove that every non-oscillatory solution converges to zero as $n \to \infty$. Suppose $\{y_n\}$ be an eventually positive solution of (1) for $n \geq n_o$. If necessary increment n_0 such that (11) is satisfied, and by (H5),

$$(19) \qquad \alpha := (a+b) \sum_{i=n_0}^{\infty} q_i^- < 1 - R.$$

Using that $\tau(n)$ and $\sigma(n)$ are non-decreasing and both tend to ∞ as $n \to \infty$, we define $\tau_0 = \tau_{n_0}$ and $\sigma_0 = \sigma_{n_0}$. Select a constant M large enough so that

$$
\begin{aligned}
& 1 \leq M, \\
(20) \qquad & |y_n| \leq M \quad \text{for } \min\{\tau_0, \sigma_0\} \leq n \leq n_0, \\
& \alpha + R \leq \frac{M}{M + y_{n_0} + |p_{n_0} y_{\tau(n_0)}| + \sum_{i=n_0}^{\infty} |f_i|}.
\end{aligned}
$$

By (19), $0 \leq \alpha + R < 1$. Then for $n \leq n_0$

$$(21) \qquad 0 \leq y_n \leq M + y_{n_0} + |p_{n_0} y_{\tau(n_0)}| + \sum_{i=n_0}^{\infty} |f_i| = M_1.$$

Next, we prove $y_n < M_1$ for all $n \geq n_0$, by induction. As per induction hypothesis, assume that (21) holds for all $n < k$. Then summing (10) from $n = n_0$ to $k-1$, we obtain that

$$
\begin{aligned}
(22) \qquad y_k = {}& p_k y_{\tau(k)} + y_{n_0} - p_{n_0} y_{\sigma(n_0)} - \sum_{n=n_0}^{k-1} q_n^+ G(y_{\sigma(n)}) \\
& + \sum_{n=n_0}^{k-1} q_n^- G(y_{\sigma(n)}) + \sum_{n=n_0}^{k-1} f_n.
\end{aligned}
$$

Because $\tau(k) < k$ and $\sigma(k) < k$, we can use (21) to estimate each term in the above expression. Using $p_n \leq R$, we obtain

$$p_k y_{\sigma(k)} \leq RM_1.$$

Applying induction hypothesis,(H7) and the fact that $M_1 > 1$,we obtain for $n \leq k-1$,

$$G(y_{\sigma(n)}) \leq a|y_{\sigma(n)}| + b \leq aM_1 + b \leq (a+b)M_1.$$

Hence,

$$\sum_{n=n_0}^{k-1} q_n^- G(y_{\sigma(n)}) \leq (a+b)M_1 \sum_{n=n_0}^{k-1} q_n^- = M_1 \alpha.$$

Since (20) implies $M_1(R + \alpha) \leq M$, then it follows that

$$y_k \leq M_1(R + \alpha) + y_{n_0} + |p_{n_0} y_{\tau(n_0)}| + \sum_{i=n_0}^{\infty} |f_i| \leq M_1.$$

Thus, by the mathematical principle of induction, $y_n \leq M_1$ for $n \geq n_0$. Hence $\{y_n\}$ is bounded. Next, we define z_n and w_n as in Theorem 2.1 and prove $\lim_{n\to\infty} y_n = 0$ by the same method as in the proof of Theorem 2.1. The proof for the case when y_n is eventually negative is similar. $\square$

Since the results in Theorems 2.1 and 2.2 hold for bounded solutions, then we have the following result.

Theorem 2.3. *Under assumptions (H1) and (H3)-(H6), every bounded solution of (1) oscillates or tends to zero as $n \to \infty$.*

We have the following result in order to answer Theorem 1.3.

Theorem 2.4. *Assume that q_n changes sign and (H5) holds. If (H4) holds then every solution of (6) oscillates or tends to zero as $n \to \infty$.*

Proof. The delay equation (6) is a particular case of (1), when $p_n \equiv 0, \sigma(n) = n - k, f_n \equiv 0$. Condition (H2) is not satisfied, but, (H7) is satisfied with $a = 1$ and $b = 0$. Since (H1), (H3)-(H7) are satisfied, we apply Theorem 2.2 and obtain the desired result. $\square$

Remark 2. If q_n changes sign then $|q_n| = q_n^+ + q_n^-$. Thus, if (H5) holds then (H4) implies and implied by (8). Hence, Theorem 2.4 proves Theorem 1.3. To emphasize the need of (H5) for Theorem 1.3, we present the following example.

Example 1. Consider the delay equation

$$\Delta y_n + q_n y_{n-2} = 0,$$

where

$$q_n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ \frac{1}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Then, q_n is oscillatory but, does not satisfy (H5). Moreover, $G(u) = u$, does not satisfy (H2). Note that (H1), (H3), (H4), (H6)–(H7) hold, but we cannot apply Theorem 2.1 or Theorem 2.2. In fact,

$$y_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even,} \end{cases}$$

is a solution of the above delay equation which neither oscillates nor tends to zero as $n \to \infty$.

Before we close our article, we present an example to illustrate, one of our main results.

Example 2. Consider the NDDE

$$(23) \qquad \Delta(y_n) + q_n y_{n-2} = q_n e^{2-n} + e^{-n}(e^{-1} - 1),$$

where q_n is as given by Eq(9). It is easy to verify that Eq(23) satisfies all the conditions of Theorem 2.2. Hence, every solution oscillates or tends to zero as $n \to \infty$. Thus, $y_n = e^{-n}$ is such a solution which $\to 0$ as $n \to \infty$.

References

[1] G. E. Chatzarakis and I. P. Stavroulakis. Oscillations of first order linear delay difference equations. *Aust. J. Math. Anal. Appl.*, 3(1):Art. 14, 11 pp. (electronic), 2006.

[2] S. S. Cheng and W. T. Patula. An existence theorem for a nonlinear difference equation. *Nonlinear Anal.*, 20(3):193–203, 1993.

[3] S. S. Cheng and G. Zhang. "Virus" in several discrete oscillation theorems. *Appl. Math. Lett.*, 13(3):9–13, 2000.

[4] R. L. Devaney. *An introduction to chaotic dynamical systems.* The Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA, 1986.

[5] Y. Domshlak. What should be a discrete version of the Chanturia-Koplatadze lemma? *Funct. Differ. Equ.*, 6(3-4):299–304, 1999.

[6] L. H. Erbe, Q. Kong, and B. G. Zhang. *Oscillation theory for functional-differential equations*, volume 190 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker Inc., New York, 1995.

[7] S. R. Grace and B. S. Lalli. Oscillation theorems for forced neutral difference equations. *J. Math. Anal. Appl.*, 187(1):91–106, 1994.

[8] J. R. Graef and P. W. Spikes. Asymptotic decay of oscillatory solutions of forced nonlinear difference equations. *Dynam. Systems Appl.*, 3(1):95–102, 1994.

[9] I. Győri and G. Ladas. *Oscillation theory of delay differential equations.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. With applications, Oxford Science Publications.

[10] G. Ladas. Recent developments in the oscillation of delay difference equations. In *Differential equations (Colorado Springs, CO, 1989)*, volume 127 of *Lecture Notes in Pure and Appl. Math.*, pages 321–332. Dekker, New York, 1991.

[11] G. Ladas and C. Qian. Oscillation in differential equations with positive and negative coefficients. *Canad. Math. Bull.*, 33(4):442–451, 1990.

[12] R. E. Mickens. *Difference equations.* Van Nostrand Reinhold Co., New York, 1987.

[13] N. Parhi and A. K. Tripathy. Oscillation criteria for forced nonlinear neutral delay difference equations of first order. *Differential Equations Dynam. Systems*, 8(1):81–97, 2000.

[14] N. Parhi and A. K. Tripathy. On asymptotic behaviour and oscillation of forced first order nonlinear neutral difference equations. *Fasc. Math.*, (32):83–95, 2001.

[15] N. Parhi and A. K. Tripathy. Oscillation of a class of neutral difference equations of first order. *J. Difference Equ. Appl.*, 9(10):933–946, 2003.

[16] N. Parhi and A. K. Tripathy. Oscillation of a class of nonlinear neutral difference equations of higher order. *J. Math. Anal. Appl.*, 284(2):756–774, 2003.

[17] N. Parhi and A. K. Tripathy. Oscillation of forced nonlinear neutral delay difference equations of first order. *Czechoslovak Math. J.*, 53(128)(1):83–101, 2003.

[18] I. P. Stavroulakis. Oscillations of delay difference equations. *Comput. Math. Appl.*, 29(7):83–88, 1995.

[19] I. P. Stavroulakis. Oscillation criteria for first order delay difference equations. *Mediterr. J. Math.*, 1(2):231–240, 2004.

[20] E. Thandapani, R. Arul, and P. S. Raja. Oscillatory and asymptotic behavior of solutions of non-homogeneous neutral difference equations. *Stud. Univ. Žilina Math. Ser.*, 15(1):67–82, 2002.

[21] E. Thandapani, R. Arul, and P. S. Raja. Oscillation of first order neutral delay difference equations. *Appl. Math. E-Notes*, 3:88–94 (electronic), 2003.

[22] J. S. Yu and X. H. Tang. Sufficient conditions for the oscillation of linear delay difference equations with oscillating coefficients. *J. Math. Anal. Appl.*, 250(2):735–742, 2000.

[23] J. S. Yu and Z. C. Wang. Asymptotic behavior and oscillation in neutral delay difference equations. *Funkcial. Ekvac.*, 37(2):241–248, 1994.

[24] J. S. Yu, B. G. Zhang, and X. Z. Qian. Oscillations of delay difference equations with oscillating coefficients. *J. Math. Anal. Appl.*, 177(2):432–444, 1993.

[25] B. G. Zhang and H. Wang. The existence of oscillatory and nonoscillatory solutions of neutral difference equations. *Chinese J. Math.*, 24(4):377–393, 1996.

Radhanath Rath,
Department of Mathematics,
Khallikote Autonomous College,
Berhampur 760001 Orissa,
India.
E-mail address: radhanathmath@yahoo.co.in

Niyati Misra,
Department of Mathematics,
Berhampur University,
760007, Orissa,
India.
E-mail address: niyatimath@indiatimes.com

Subhendu Kumar Rath,
Department of MCA,
Gandhi Engineering College,
Bhubaneswar, Orissa,
India.
E-mail address: rath.subhendu@gmail.com