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C_0-BIČECH SPACES AND C_1-BIČECH SPACES

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Abstract. The aim of this paper is to introduce the concepts of C_0-biČech spaces and C_1-biČech spaces and study its basic properties.

1. Introduction

Čech closure spaces were introduced by Čech [2] and then studied by many authors, see e.g. [4, 5, 6, 7]. BiČech closure spaces were introduced by Chandrasekhara Rao, Gowri and Swaminathan [3]. Caldas and Jafari [1] introduced the notions of $\wedge_\delta - R_0$ and $\wedge_\delta - R_1$ topological spaces as a modification of the known notions of R_0 and R_1 topological spaces. In this paper, we introduce the concepts of C_0 - biČech spaces and C_1-biČech spaces and study its basic properties in biČech closure spaces.

2. Preliminaries

An operator $u\colon P(X) \to P(X)$ defined on the power set $P(X)$ of a set X satisfying the axioms:
(C1) $u\emptyset = \emptyset$,
(C2) $A \subseteq uA$ for every $A \subseteq X$,
(C3) $u(A \cup B) = uA \cup uB$ for all $A, B \subseteq X$.
is called a *Čech closure operator* and the pair (X, u) is a *Čech closure space*. For short, the space will be noted by X as well, and called a *closure space*. A closure operator u on a set X is called *idempotent* if $uA = uuA$ for all $A \subseteq X$.

A subset A is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too.

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Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f\colon (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$. Clearly, if $f\colon (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces and let $f\colon (X, u) \to (Y, v)$ be a map. If f is continuous, then $f^{-1}(G)$ is an open subset of (X, u) for every open subset G of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha\in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha\in I} X_\alpha$ denotes the Cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha\colon \prod\limits_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\alpha, u_\alpha)$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha\in I} u_\alpha\pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha\in I} X_\alpha$.

The following statement is evident:

Proposition 2.1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then the projection map $\pi_\beta\colon \prod\limits_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous.*

Proposition 2.2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let $\beta \in I$ and let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

Proposition 2.3. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let $\beta \in I$ and let G be an open subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(G)$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(G) = G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$, therefore $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ be an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Then $\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha = (X_\beta - G) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$, hence $(X_\beta - G) \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. By Proposition 2.2, $X_\beta - G$ is a closed subset of (X_β, u_β). Consequently, G is an open subset of (X_β, u_β). □

3. C_0-BiČech Spaces and C_1-BiČech Spaces

Definition 3.1. Two maps u_1 and u_2 from power set X to itself are called *biČech closure operator* (simply biclosure operator) for X if they satisfies the following properties:

(i) $u_1\emptyset = \emptyset$ and $u_2\emptyset = \emptyset$,
(ii) $A \subseteq u_1 A$ and $A \subseteq u_2 A$ for every $A \subseteq X$,
(iii) $u_1(A \cup B) = u_1 A \cup u_1 B$ and $u_2(A \cup B) = u_2 A \cup u_2 B$ for all $A, B \subseteq X$.

A structure (X, u_1, u_2) is called a *biČech closure space*

Definition 3.2. A biČech closure space (X, u_1, u_2) is said to be a C_0-*biČech space* if, for every open subset G of (X, u_1) such that $x \in G$, $u_2\{x\} \subseteq G$.

Example 3.3. Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = u_1\{b\} = u_1 X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, and $u_2\{a\} = u_2\{b\} = u_2 X = X$. Then (X, u_1, u_2) is a C_0-biČech space.

Proposition 3.4. *A biČech closure space (X, u_1, u_2) is a C_0-biČech space if and only if, for every closed subset F of (X, u_1) such that $x \notin F$, $u_2\{x\} \cap F = \emptyset$.*

Proof. Let F be a closed subset of (X, u_1) and let $x \notin F$. Since $x \in X - F$ and $X - F$ is an open subset of (X, u_1), $u_2\{x\} \subseteq X - F$. Consequently, $u_2\{x\} \cap F = \emptyset$.

Conversely, let G be an open subset of (X, u_1) and let $x \in G$. Since $X - G$ is a closed subset of (X, u_1) and $x \notin X - G$, $u_2\{x\} \cap (X - G) = \emptyset$. Consequently, $u_2\{x\} \subseteq G$. Hence, (X, u_1, u_2) is a C_0-biČech space. □

Definition 3.5. A biČech closure space (X, u_1, u_2) is said to be a C_1-*biČech space* if, for each $x, y \in X$ such that $u_1\{x\} \neq u_2\{y\}$, there exist a disjoint open subset G of (X, u_2) and an open subset V of (X, u_1) such that $u_1\{x\} \subseteq G$ and $u_2\{y\} \subseteq V$.

Example 3.6. Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = \{a\}$, $u_1\{b\} = \{b\}$ and $u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$, $u_2\{b\} = \{b\}$ and $u_2X = X$. Then (X, u_1, u_2) is a C_1-biČech space.

Proposition 3.7. *Every C_1-biČech space is a C_0 -biČech space.*

Proof. Let (X, u_1, u_2) be a C_1-biČech space. Let G be an open subset of (X, u_1) and let $x \in G$. If $y \notin G$, then $u_2\{x\} \neq u_1\{y\}$ because $x \notin u_1\{y\}$. Then there exists an open subset V_y of (X, u_2) such that $u_1\{y\} \subseteq V_y$ and $x \notin V_y$, which implies $y \notin u_2\{x\}$. Consequently, $u_2\{x\} \subseteq G$. Hence, (X, u_1, u_2) is a C_0-biČech space. □

The converse is not true as can be seen from the following example.

Example 3.8. Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = u_1\{b\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$, and $u_2\{b\} = u_2X = X$. Then (X, u_1, u_2) is a C_0-biČech space but it is not a C_1-biČech space.

Proposition 3.9. *A biČech closure space (X, u_1, u_2) is a C_1-biČech space if and only if every pair of points x, y of (X, u_1, u_2) such that $u_1\{x\} \neq u_2\{y\}$, there exist an open subset G of (X, u_1) and open subset V of (X, u_2) such that $x \in V$, $y \in G$ and $G \cap V = \emptyset$.*

Proof. Suppose that (X, u_1, u_2) is a C_1-biČech space. Let x, y be points of (X, u_1, u_2) such that $u_1\{x\} \neq u_2\{y\}$. There exist a disjoint open subset G of (X, u_1) and an open subset V of (X, u_2) such that $x \in u_1\{x\} \subseteq V$ and $y \in u_2\{y\} \subseteq G$.

Conversely, suppose that there exist an open subset G of (X, u_1) and an open subset V of (X, u_2) such that $x \in V$, $y \in G$ and $G \cap V = \emptyset$. Since every C_1-biČech space is a C_0-biČech space, $u_1\{x\} \subseteq V$ and $u_2\{y\} \subseteq G$. This gives the statement. □

Proposition 3.10. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biČech closure spaces. If $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_0-biČech space, then $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_0-biČech space for each $\alpha \in I$.*

Proof. Suppose that $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_0-biČech closure space. Let $\beta \in I$ and let G be an open subset of (X_β, u^1_β) such that $x_\beta \in G$. Then $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $(x_\alpha)_{\alpha\in I} \in G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_0-biČech space, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha(\{(x_\alpha)_{\alpha\in I}\}) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta\{x_\beta\} \subseteq G$. Hence, $(X_\beta, u^1_\beta, u^2_\beta)$ is a C_0-biČech space. □

Proposition 3.11. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biČech closure spaces. If $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_1-biČech space for each $\alpha \in I$, then $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_1-biČech space.*

Proof. Suppose that $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_1-biČech space for each $\alpha \in I$. Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be points of $\prod_{\alpha\in I} X_\alpha$ such that

$$\prod_{\alpha\in I} u^1_\alpha \pi_\alpha(\{(x_\alpha)_{\alpha\in I}\}) \neq \prod_{\alpha\in I} u^2_\alpha \pi_\alpha(\{(y_\alpha)_{\alpha\in I}\}).$$

There exists $\beta \in I$ such that $u^1_\beta\{x_\beta\} \neq u^2_\beta\{y_\beta\}$. Since $(X_\beta, u^1_\beta, u^2_\beta)$ is a C_1-biČech space, there exist an open subset U of (X_β, u^1_β) and V is an open subset of (X_β, u^2_β) such that $U \cap V = \emptyset$, $u^2_\beta\{y_\beta\} \subseteq U$ and $u^1_\beta\{x_\beta\} \subseteq V$. Consequently, $\prod_{\alpha\in I} u^2_\alpha \pi_\alpha(\{(y_\alpha)_{\alpha\in I}\}) \subseteq U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ and $\prod_{\alpha\in I} u^1_\alpha \pi_\alpha(\{(x_\alpha)_{\alpha\in I}\}) \subseteq V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ such that $U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$, $V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$ and $(U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) \cap (V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) = \emptyset$. Hence, $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a C_1-biČech space. □

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