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CONTINUED FRACTIONS, FIBONACCI NUMBERS, AND SOME CLASSES OF IRRATIONAL NUMBERS

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Abstract. In this paper we define an equivalence relation on the set of positive irrational numbers less than 1. The relation is defined by means of continued fractions. Equivalence classes under this relation are determined by the places of some elements equal to 1 (called *essential* 1*'s*) in the continued fraction expansion of numbers. Analysis of suprema of all equivalence classes leads to a solution which involves Fibonacci numbers and constitutes the main result of this paper. The problem has its origin in the author's research on the construction of digital lines and upper and lower mechanical and characteristic words according to the hierarchy of runs.

1. Introduction

Sequences generated by an irrational rotation have been intensively studied by mathematicians, astronomers, crystallographers, and computer scientists; see Venkov (1970) [18, pp. 65–68] and Bruckstein (1991) [4, section *Some consequences and historical remarks*]. These sequences, or related objects, can be found back in the mathematical literature under many different names: rotation sequences, cutting sequences, Beatty sequences, characteristic words, upper and lower mechanical words, balanced words, Sturmian words, Christoffel words, Freeman codes (chain codes) of digital straight lines, and so on; see Pytheas Fogg (2002) [5, p. 143]. There exist some recursive descriptions by continued fractions (CF) of these sequences. The most well known is probably the one formulated by the astronomer J. Bernoulli in 1772, proven by A. Markov in 1882 and described by Venkov (1970) [18, p. 67]. Also well known is the description by Shallit (1991) [11], which can be found in Lothaire (2002) [8, p. 75, 76, 104, 105] as the method by *standard sequences.*

In H. U-W (2008) [15] the author presented a new CF based description of such sequences. The new description reflects the hierarchy of runs, by analogy

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to digital straight lines as defined by Azriel Rosenfeld in 1974 [10]. This new description appeared to be a good basis for two partitions of upper mechanical words (digital lines) with irrational slopes into equivalence classes according to the length of runs (one of the relations) and the construction of runs (the second one) on all levels in the hierarchy. This has been presented in H. U-W (2009) [16]. Partitions of upper mechanical words with irrational slopes (which are Sturmian words) can give a better understanding of their construction and, as a consequence of that, can be useful in research in combinatorics on words. In H. U-W (2009) [16] the author studied the equivalence classes obtained by both partitions. While examining suprema of the equivalence classes under the relation based on the construction of runs on all the levels in the hierarchy, the author found a solution involving Fibonacci numbers. Now we formulate the essence of this problem, independently from digital geometry and word theory.

The problem we discuss in this paper concerns least and greatest elements in some sets of irrational numbers from the interval $]0,1[$. We define (by means of CFs) an equivalence relation on $]0,1[\setminus \mathbf{Q}$; see Definition 4. This partitions the set of positive irrational numbers less than 1 into equivalence classes. Numbers with the same *sequences of essential places* (Definition 2) in their CF expansions are gathered in the same class. As we will explain in Section 3 (where we present the circumstances in which the presented problem appeared), the upper mechanical words (digital lines) with slopes belonging to the same equivalence class, have the same construction in terms of long and short runs in the hierarchy of runs, because this is fully determined by *essential places* of the slopes (Definition 1), as shown in Proposition 2. The *essential* 1*'s* make that the most frequently appearing run on the level they decide about is long (instead of short, as in case of CF elements different from 1; *non-essential* 1*'s* do not decide about the construction at all, they only determine run length). Sturmian words with slopes belonging to the same equivalence class thus share some construction-related properties, which can give rise to a new tool to the research in combinatorics on words.

The main theorem of the presented paper (Theorem 1) is a description of infima and suprema of all equivalence classes under the relation. The only class which has a greatest element is the one which contains $(\sqrt{5}-1)/2 = [0;\ \overline{1}\ ]$, the Golden Section, and the greatest element is the Golden Section itself. Suprema of all the other equivalence classes are expressed by the odd-numbered convergents of $[0;\ \overline{1}\ ]$. They are thus fractions with numerators and denominators being consecutive Fibonacci numbers.

2. An equivalence relation on the set of positive irrational numbers less than 1

In this paper we assume that the simple continued fraction (CF) expansion of each $a \in\ ]0,1[\setminus \mathbf{Q}$ is given, expressed as $a = [0; a_1, a_2, a_3, \dots]$, and we know the positive integers a_k for all $k \in \mathbf{N}^+$. These are called the *elements* of the CF. By *index* of a CF element a_k we mean the positive integer k which describes the place of the element a_k in the CF expansion of a. We recall that

$$[a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}. \tag{1}$$

In our case, when $a \in]0,1[\setminus \mathbf{Q}$, we have $a_0 = \lfloor a \rfloor = 0$ and the sequence of the CF elements $(a_1, a_2, \dots)$ is infinite. We call $[a_0; a_1, a_2, \dots, a_n]$, for each $n \in \mathbf{N}$, the n^{th} *convergent* of the CF $[a_0; a_1, a_2, \dots]$. If we define

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \quad \text{and} \quad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } \; n \geq 2 \tag{2}$$

and

$$q_0 = 1, \quad q_1 = a_1, \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } \; n \geq 2, \tag{3}$$

then

$$[a_0; a_1, a_2, \dots, a_n] = \frac{p_n}{q_n} \quad \text{for } \; n \in \mathbf{N}, \tag{4}$$

see for example Vajda (2008) [17, pp. 158–159]. For more information about CFs see Khinchin (1997) [6].

Some elements equal to 1 in the CF expansion of $a \in]0,1[\setminus \mathbf{Q}$ will receive special attention. The reason for this has its roots in the theory of digital lines or, equivalently, upper mechanical words with slope a and intercept 0. This will be explained in Section 3.

Definition 1. Let $a \in]0,1[$ be an irrational number and let $[0; a_1, a_2, a_3, \dots]$ be its CF expansion. Let $k \in \mathbf{N}^+$ be such that $a_k = 1$. The integer k is called an *essential place* for a if the following assertions hold:

- $k \geq 2$
- $\exists \; j \in \mathbf{N}, \; [0; a_1, a_2, \dots] = [0; a_1, a_2, \dots, a_{k-2j-1}, \underbrace{1, 1, \dots, 1, 1}_{2j}, a_k, \dots]$

  and, if $k - 2j - 1 \geq 2$, then $a_{k-2j-1} \geq 2$.

In other words, a natural number $k \geq 2$ is an essential place of $a = [0; a_1, a_2, \dots]$ iff $a_k = 1$ and a_k is directly preceded by an even number (i.e., by $0, 2, 4, \dots$) of consecutive 1's (i.e., elements $a_m = 1$) with an index m greater than 1. Such elements a_k (where k is an essential place) we called *essential* 1*'s* in H. U-W (2009) [16, Definition 6]. The CF elements $a_k = 1$ which are not in essential places in the CF expansion of a (i.e., if $k = 1$, or if $k \geq 3$ and a_k is directly preceded by an odd number of consecutive 1's with an index greater than 1), are called *non-essential* 1*'s*.

Definition 2. Let $a = [0; a_1, a_2, \dots]$ be irrational. We denote by A the set of all essential places for a, i.e., $A = \{k \in \mathbf{N}^+; \; k \text{ is an essential place for } a\}$, and by $|A|$ the cardinality of A. Let the set J be as follows:

- $A = \emptyset \;\; \Rightarrow \;\; J = \emptyset,$

- $|A| = \aleph_0 \;\Rightarrow\; J = \mathbf{N}^+$,
- $[\; \exists\, M \in \mathbf{N}^+,\ |A| = M \;] \;\Rightarrow\; J = [1, M]_{\mathbf{Z}}$.

We define $(s_j)_{j\in J}$, the *sequence of essential places* of the CF expansion of a as follows:

- $J = \emptyset \;\Rightarrow\; (s_j)_{j\in\emptyset} = \emptyset$,
- $J \neq \emptyset \;\Rightarrow\; (s_j)_{j\in J}$ is such that $s_1 = \min\{k \in \mathbf{N}^+;\ k \in A\}$ and, if $n \in J \setminus \{1\}$, then $s_n = \min\{k > s_{n-1};\ k \in A\}$.

In words, $J = \emptyset$ if there are no 1's in the CF expansion of a (except maybe for a_1), $J = \mathbf{N}^+$ is there are infinitely many 1's in the CF expansion of a, and $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$ if there are exactly M essential places (essential 1's) in the CF expansion of a. The sequence of essential places for a is indexed by J and we put the smallest essential place first, the next one on the second place, and so on. The sequence of essential places defined above was called the *sequence of the places of essential* 1*'s* in H. U-W (2009) [16, Definition 7].

The following lemma shows how to find essential places in an easy way.

Lemma 1. *Let $a = [0; a_1, a_2, \dots]$ be irrational and let the set J for this slope be as described in Definition 2. Then the sequence $(s_j)_{j\in J}$ of essential places of the CF expansion of a is $(s_j)_{j\in\emptyset} = \emptyset$ if $J = \emptyset$ and, if $J \neq \emptyset$, then $(s_j)_{j\in J}$ is as follows: $s_1 = \min\{k \geq 2;\ a_k = 1\}$ and*

$$n \in J \setminus \{1\} \;\Rightarrow\; s_n = \min\{k \geq s_{n-1} + 2;\ a_k = 1\}. \tag{5}$$

Proof. Let us consider any irrational $a = [0; a_1, a_2, \dots]$ and the corresponding $(s_j)_{j\in J}$ as in Definition 2. If $J = \emptyset$, then $(s_j)_{j\in\emptyset} = \emptyset$. If $J \neq \emptyset$, then we can prove the statement by induction (if $|J| = M$ for some $M \in \mathbf{N}^+$, the proof has only a finite number of steps). It follows from Definition 1 that $s_1 \geq 2$. Let us take any $m \in J \setminus \{1\}$ such that s_{m-1} is an essential place. We will show that the next essential place is $s_m = \min\{k \geq s_{m-1} + 2;\ a_k = 1\}$. First we show that $s_m - s_{m-1} \geq 2$. Suppose not, i.e., $s_m = s_{m-1} + 1$. Then both $a_{s_{m-1}} = 1$ and $a_{s_{m-1}+1} = 1$, which are consecutive CF elements of a, are essential 1's. This is not possible, however, because, as consecutive CF elements equal to 1, they cannot both be directly preceded by an even number of CF elements equal to 1 and with an index greater than 1. We get a contradiction, so it must be $s_m - s_{m-1} \geq 2$.

If the difference between s_m (as defined by (5)) and s_{m-1} is greater than 2, then $a_{s_m} = 1$ is the next essential 1 following after $a_{s_{m-1}}$, because, according to (5), there are no other 1's between $a_{s_{m-1}}$ and a_{s_m} (maybe $a_{s_{m-1}+1} = 1$, but then it is a non-essential 1, as it is directly preceded by an odd number of $a_j = 1$ with $j > 1$, and $s_{m-1} + 1 < s_m - 1$), so $a_{s_m} = 1$ is directly preceded by zero (i.e., an even number) 1's.

If $s_m - s_{m-1} = 2$ (where s_m is defined by (5)), then $a_{s_m} = 1$ is the next essential 1 following after $a_{s_{m-1}}$, because it is directly preceded by an even number of CF elements equal to 1 and with index greater than 1. This number is equal to zero if $a_{s_{m-1}+1} \geq 2$ and to the even number of such 1's corresponding to s_{m-1}, increased by 2, in case $a_{s_{m-1}+1} = 1$. □

Example 1. Let $a = [0; a_1, a_2, a_3, a_4, \dots]$, with $\forall\ k < 16,\ \ a_k = 1$ if and only if $k \in \{1, 3, 4, 6, 7, 10, 13, 14, 15\}$. We find the sequences of essential places in the CF expansion of such a in the following way:

$$\begin{array}{rcl} a &=& [0;\ 1,\ a_2,\ \underline{1},\ 1,\ a_5,\ \underline{1},\ 1,\ a_8,\ a_9,\ \underline{1},\ a_{11},\ a_{12},\ \underline{1},\ 1,\ \underline{1},\ \dots] \\ && \qquad\quad\ \downarrow \qquad\quad \downarrow \qquad\qquad\quad \downarrow \qquad\qquad \downarrow \quad\ \ \downarrow \ \ \dots \\ (s_j)_{j\in J} &=& (\qquad\ \ 3, \qquad\ 6, \qquad\qquad 10, \qquad\quad 13 \quad 15\ \ \dots). \end{array}$$

All essential 1's with index less than 16 are underlined. We have $(s_j)_{j\in J} = (3, 6, 10, 13, 15, \dots)$. The first four non-essential 1's are a_1, a_4, a_7, a_{14}.

The following proposition describes all the possible sequences of essential places for CF-expansions of positive irrational a less than 1. First, we will introduce the following definition.

Definition 3. A sequence $(t_j)_{j\in J}$ of positive integer numbers will be called an *essential sequence* iff:

- the set J is as follows: $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$,
- the sequence $(t_j)_{j\in J}$ (if not empty) is a sequence of positive integers such that $t_1 \geq 2$ and, for $k \in J \setminus \{1\}$, $t_k - t_{k-1} \geq 2$.

Proposition 1. *A sequence of positive integer numbers is an essential sequence iff it is the sequence of essential places for some irrational* $a = [0; a_1, a_2, \dots]$.

Proof. Let $(t_j)_{j\in J}$ be an essential sequence (Definition 3). We define $a = [0; a_1, a_2, \dots]$ in the following way: if $J = \emptyset$, then we take any $a_1 \in \mathbf{N}^+$ and, for each $n \geq 2$, we choose any $a_n \geq 2$. If J is not empty, we define $a_{t_i} = 1$ for all $i \in J$ and a_k for $k \in \mathbf{N}^+ \setminus \{t_j\}_{j\in J}$ can be any integer greater than or equal to 2. It follows trivially from Definitions 1 and 2 that $(t_j)_{j\in J}$ is the sequence of essential places for such a. The second implication in the statement follows from Lemma 1. □

All sequences of essential places have elements greater than or equal to 2, are increasing and the difference between each two consecutive elements is greater than or equal to 2. Each sequence, finite or infinite, with those properties (i.e., an essential sequence), is the sequence of essential places for some $a \in\,]0, 1[\setminus \mathbf{Q}$.

We can identify with each other all irrational numbers from the interval $]0, 1[$ which have the same sequences of essential places.

Definition 4. We define the following relation $\sim_{\text{ess}} \subset (]0, 1[\setminus \mathbf{Q})^2$. If a and a' are positive irrational numbers less than 1, then

$$a \sim_{\text{ess}} a' \quad \Leftrightarrow \quad \left(s_j^{(a)}\right)_{j\in J} = \left(s_k^{(a')}\right)_{k\in J'},$$

where $\left(s_j^{(a)}\right)_{j\in J}$ and $\left(s_k^{(a')}\right)_{k\in J'}$ are the corresponding sequences of essential places in the CF expansion of a and a' respectively.

The relation $\sim_{\text{ess}}$ partitions the set $]0,1[\setminus\mathbf{Q}$ into equivalence classes defined by essential sequences (Definition 3, Proposition 1). Let us consider the following examples.

Example 2. Let $(t_j)_{j\in\emptyset}=\emptyset$. The class under $\sim_{\text{ess}}$ generated by this sequence is the set of all $a\in\,]0,1[\,\setminus\mathbf{Q}$ such that $a=[0;a_1,a_2,\dots]$, where $a_1\in\mathbf{N}^+$ and $a_n\geq 2$ for all $n\geq 2$.

Example 3. Let $(t_j)_{j\in\mathbf{N}^+}=(2j)_{j\in\mathbf{N}^+}$. The class under $\sim_{\text{ess}}$ generated by this sequence is the set of all positive irrational numbers with the CF expansion $a=[0;a_1,1,a_3,1,a_5,1,\dots]$, where $a_{2n+1}\in\mathbf{N}^+$ for all $n\in\mathbf{N}$. The Golden Section $(\sqrt{5}-1)/2$ belongs to this class.

The problem we want to solve in this paper is the question about supremum and infimum of each class under $\sim_{\text{ess}}$. The following lemma, which shows how to compare two CFs with each other, will help us to find the solution.

Lemma 2. *Let $a_0,b_0\in\mathbf{Z}$ and $a_i,b_i\in\mathbf{N}^+$ for all $i\in\mathbf{N}^+$. Then*

$$[a_0;a_1,a_2,\dots]<[b_0;b_1,b_2,\dots]\quad\Leftrightarrow$$

$$(a_0,-a_1,a_2,-a_3,a_4,-a_5,\dots)\overset{\text{lexic.}}{<}(b_0,-b_1,b_2,-b_3,b_4,-b_5,\dots),$$

where the first inequality is according to the order $<$ on the real numbers, and the second inequality is according to the lexicographical order on sequences.

Theorem 1 (Main Theorem)**.** *There exists no equivalence class under $\sim_{\text{ess}}$ with a least element according to the order $\leq$ on the real numbers. The infimum is equal to 0 for all the classes.*

There exists exactly one class under $\sim_{\text{ess}}$ which has a greatest element according to the order $\leq$ on the real numbers. This class is defined by the sequence $(t_j)_{j\in\mathbf{N}^+}=(2j)_{j\in\mathbf{N}^+}$ and the maximum is the Golden Section $(\sqrt{5}-1)/2$. Moreover, the following statement describes suprema of all the classes under $\sim_{\text{ess}}$ different from $[(2j)_{j\in\mathbf{N}^+}]_{\sim_{\text{ess}}}$. For all $n\in\mathbf{N}^+$

$$[(\forall\ k\in[1,n-1]_{\mathbf{Z}},\ \ t_k=2k)\ \wedge\ (t_n>2n\ \ \vee\ \ |J|=n-1)]$$

$$\Rightarrow\ \sup\{a\in\,]0,1[\,\setminus\mathbf{Q};\ \ a\in[(t_j)_{j\in J}]_{\sim_{\text{ess}}}\}=\frac{F_{2n-1}}{F_{2n}},$$

where $(F_n)_{n\in\mathbf{N}^+}$ is the Fibonacci sequence, i.e.,

$$F_1=1,\ F_2=1\ \text{ and, for }\ k\geq 3,\ \ F_k=F_{k-1}+F_{k-2}, \tag{6}$$

$|J|$ denotes the cardinality of J, and $(t_j)_{j\in J}$ is any essential sequence different from $(2j)_{j\in\mathbf{N}^+}$.

Proof. To prove the statement about infimum we observe that in each equivalence class under $\sim_{\text{ess}}$ there exist numbers $a=[0;a_1,a_2,\dots]$ with $a_1=1$, numbers with $a_1=2$, etc. When a_1 tends to infinity, then $a=[0;a_1,a_2,\dots]$ tends to zero, so zero is infimum and we have no least element in the class (which is a subset of $]0,1[\,\setminus\mathbf{Q}$).

To prove the statement about supremum, we take $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$ and we consider all the classes generated by all possible essential sequences, i.e., by sequences $(t_j)_{j \in J}$ of integers such that $t_1 \geq 2$ and $t_i - t_{i-1} \geq 2$ for all $i \in J \setminus \{1\}$.

The flowchart on p. 24 analyzes all such possible classes with respect to greatest elements and suprema. We use Lemma 2 in each step of the flowchart. To make a CF as large as possible, the even-numbered CF elements must be as large as possible (it is represented by the left-hand side of the flowchart) and the odd-numbered CF elements must be as small as possible, thus equal to 1 (see the right-hand side of the flowchart).

If $J = \emptyset$, then there is clearly no greatest element in the equivalence class $[\emptyset]_{\sim_{\text{ess}}}$, because all the numbers $[0; 1, a_2, \dots]$ with $a_n \geq 2$ for $n \geq 2$ (which are the only candidates for the position of maximum) belong to it and, when a_2 tends to infinity, then $[0; 1, a_2, \dots]$ tends to 1, which does not belong to $]0, 1[\setminus \mathbf{Q}$, so there is no greatest element. The supremum is equal to 1. The same reasoning holds for $J \neq \emptyset$ in case when $t_1 > 2$ (thus $a_2 \geq 2$).

If $J \neq \emptyset$ and $t_1 = 2$ (thus $a_2 = 1$), we consider the only (according to Lemma 2) candidate for a greatest element and it is $[0; 1, 1, 1, a_4, \dots]$. If $t_2 > 4$ (thus $a_4 \geq 2$), we repeat the same reasoning again: when a_4 tends to infinity, then $[0; 1, 1, 1, a_4, \dots]$ tends to $\frac{2}{3}$, so it is the supremum. There is no greatest element, because the supremum is rational. We go on like this, using Lemma 2 about comparison of CFs.

The rightmost way of the flowchart from p. 24 leads to $[0; \overline{1}\ ]$, which is irrational, equal to $(\sqrt{5} - 1)/2$. This means that the only class which has a greatest element is the class as described in Example 3, generated by the essential sequence $(t_j)_{j \in \mathbf{N}^+} = (2j)_{j \in \mathbf{N}^+}$.

The statement about suprema of all the classes can be derived from the flowchart. It follows from (2), (3), (4) and (6), that the odd-numbered convergents of the Golden Section $[0; \overline{1}\ ]$ are $\frac{p_{2n-1}}{q_{2n-1}} = \frac{F_{2n-1}}{F_{2n}}$ for $n \in \mathbf{N}^+$; see for example Vajda (2008) [17, pp. 101–105] or Benjamin and Quinn (2003) [3, p. 52]. The odd-numbered convergents are thus $\frac{F_1}{F_2} = 1$, $\frac{F_3}{F_4} = \frac{2}{3}$, $\frac{F_5}{F_6} = \frac{5}{8}$, $\frac{F_7}{F_8} = \frac{13}{21}$, $\frac{F_9}{F_{10}} = \frac{34}{55}, \dots$. Moreover, when analyzing the flowchart one can ensure oneself that it covers all the possible classes under $\sim_{\text{ess}}$. $\square$

In Theorem 1 we answered questions about least and greatest elements in classes generated by the relation $\sim_{\text{ess}}$. No equivalence class under $\sim_{\text{ess}}$ has a least element. The infimum in each class is equal to zero. The answer related to largest elements is much more interesting. The partition of all the irrational numbers from the interval $]0, 1[$ into equivalence classes under $\sim_{\text{ess}}$ gives the sets with suprema equal to the odd-numbered convergents of the Golden Section, thus with no largest element belonging to the class (which is a set of irrational numbers). The only exception is the class generated by $(t_j)_{j \in \mathbf{N}^+} = (2j)_{j \in \mathbf{N}^+}$, which has a greatest element and it is equal to the Golden Section.

$\boxed{\text{the candidates for max for } \textbf{any } J:\ [0;1,a_2,\dots]}$

$t_1 > 2$ or $J = \emptyset$ (i.e., $a_2 \geq 2$)

$\boxed{[0;1,a_2^{(n)},\dots] \overset{a_2^{(n)}\to\infty}{\longrightarrow} \mathbf{1}}$

no max

$t_1 = 2$
(i.e., $a_2 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,a_4,\dots]}$

$t_2 > 4$ or $|J| = 1$ (i.e., $a_4 \geq 2$)

$\boxed{[0;1,1,1,a_4^{(n)},\dots] \overset{a_4^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{2}{3}}}$

no max

$t_2 = 4$
(i.e., $a_4 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,a_6,\dots]}$

$t_3 > 6$ or $|J| = 2$ (i.e., $a_6 \geq 2$)

$\boxed{[0;1,1,1,1,1,a_6^{(n)},\dots] \overset{a_6^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{5}{8}}}$

no max

$t_3 = 6$
(i.e., $a_6 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,1,1,a_8,\dots]}$

$t_4 > 8$ or $|J| = 3$ (i.e., $a_8 \geq 2$)

$\boxed{[0;1,1,1,1,1,1,1,a_8^{(n)},\dots] \overset{a_8^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{13}{21}}}$

no max

$t_4 = 8$
(i.e., $a_8 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,1,1,1,1,a_{10},\dots]}$

$t_5 > 10$ or $|J| = 4$ (i.e., $a_{10} \geq 2$)

$\boxed{[0;1,1,1,1,1,1,1,1,1,a_{10}^{(n)},\dots] \overset{a_{10}^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{34}{55}}}$

no max

$t_5 = 10$
(i.e., $a_{10} = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,1,1,1,1,1,1,a_{12},\dots]}$

and so on. . .

3. The origin of the problem. Digital lines and Sturmian words

In this section we will give some information about the circumstances in which the presented problem arose. In H. U-W (2009) [16] we analyzed two equivalence relations defined on the set of all slopes $a \in]0,1[\setminus \mathbf{Q}$ of digital straight lines $y = ax$ (or, equivalently, of upper mechanical words $u(a)\colon \mathbf{N} \to \{0,1\}$, $u_n(a) = \lceil a(n+1) \rceil - \lceil an \rceil$ for each $n \in \mathbf{N}$). One of those relations is the one discussed in the presented paper. This relation identifies with each other all slopes which have the same sequences of essential places (Definitions 1 and 2) in their CF expansions. We know from H. U-W (2008) [14] that essential 1's determine the construction of digital lines. How exactly, will be shown in Proposition 2.

General information about digital straightness can be found in the review by R. Klette and A. Rosenfeld from 2004 [7]. Very good sources of information are also Reveillès (1991) [9] and Stephenson (1998) [12]. The digitization $D_{R'}$ of $y = ax$ for some $a \in]0,1[\setminus \mathbf{Q}$ as defined in H. U-W (2007) [13] is the following:

$$D_{R'}(y = ax) = \{(k, \lceil ak \rceil);\ \ k \in \mathbf{Z}\}. \tag{7}$$

We illustrate it with an example in Figure 1.

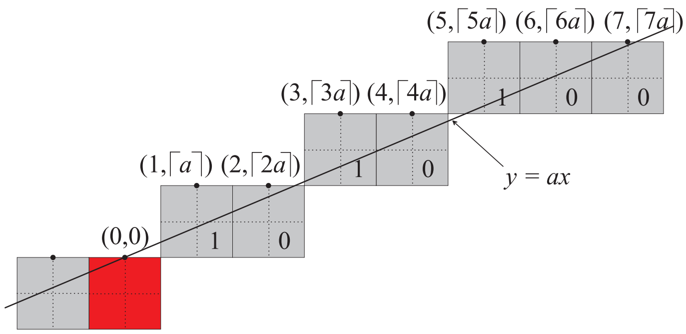

Figure 1. Digitization of $y = ax$ for some $a \in]0,1[\setminus \mathbf{Q}$; $u(a) = 1010100\cdots$.

The 0's and 1's on the squares in the picture show the relationship between digital lines and upper and lower mechanical and characteristic words. Let us recall the definitions of those words; see Lothaire (2002) [8, p. 53].

Definition 5. For each $a \in]0,1[\setminus \mathbf{Q}$ we define two binary words in the following way: $l(a)\colon \mathbf{N} \to \{0,1\}$, $\quad u(a)\colon \mathbf{N} \to \{0,1\}$ are such that for each $n \in \mathbf{N}$

$$l_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor, \quad u_n(a) = \lceil a(n+1) \rceil - \lceil an \rceil.$$

The word $l(a)$ is the *lower mechanical word* and $u(a)$ is the *upper mechanical word* with *slope* a and *intercept* 0.

We have $l_0(a) = \lfloor a \rfloor = 0$ and $u_0(a) = \lceil a \rceil = 1$ and, because $\lceil x \rceil - \lfloor x \rfloor = 1$ for irrational x, we have

$$l(a) = 0c(a), \quad u(a) = 1c(a) \tag{8}$$

(meaning 0, resp. 1 concatenated to $c(a)$). The word $c(a)$ is called the *characteristic word* of a. For each $a \in \,]0,1[\setminus \mathbf{Q}$, the characteristic word associated with a is thus the following infinite word $c(a)\colon \mathbf{N}^+ \to \{0,1\}$:

$$c_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor = \lceil a(n+1) \rceil - \lceil an \rceil, \quad n \in \mathbf{N}^+. \tag{9}$$

Formulae (7), (8) and (9), together with the 0's and 1's in Figure 1, illustrate and explain the relationship between digital lines and lower and upper mechanical and characteristic words. The developed theory is thus also valid for upper and lower mechanical words and characteristic words; see for example Lothaire (2002) [8, p. 53, 2.1.2 *Mechanical words, rotations*], H. U-W (2008) [15]. According to Theorem 2.1.13 in Lothaire (2002) [8, p. 57], irrational (lower or upper) mechanical words are Sturmian words.

Our description of digital lines in the author's papers [13, 14, 16] reflected the hierarchy of runs on all digitization levels. The concept of *runs* was already introduced and explored by Azriel Rosenfeld (1974) [10, p. 1265]. We call $\mathrm{run}_k(j)$ for $k, j \in \mathbf{N}^+$ a *run of digitization level k*. Each $\mathrm{run}_1(j)$ can be identified with a subset of $\mathbf{Z}^2$: $\{(i_0+1, j), (i_0+2, j), \dots, (i_0+m, j)\}$, where m is the length $\|\mathrm{run}_1(j)\|$ of the run. For upper mechanical words, the corresponding run is 10^{m-1}, where $m-1$ is the number of all the letters 0 between the letter 1 in the beginning of the run and the next occurring letter 1 in the word. For each $a \in \,]0,1[\setminus \mathbf{Q}$ we have only two possible run_1 lengths: $\lfloor \frac{1}{a} \rfloor$ and $\lfloor \frac{1}{a} \rfloor + 1$. All runs with one of those lengths always occur alone, i.e., do not have any neighbors of the same length in the sequence $(\mathrm{run}_1(j))_{j \in \mathbf{N}^+}$, while the runs of the other length can appear in sequences. The same holds for the sequences $(\mathrm{run}_k(j))_{j \in \mathbf{N}^+}$ on each level $k \geq 2$. We use the notation $S_k^m L_k$, $L_k S_k^m$, $L_k^m S_k$ and $S_k L_k^m$, when describing the form of digitization runs_{k+1}. For example, $S_k^m L_k$ means that the run_{k+1} consists of m short runs_k (S_k) and one long run_k (L_k) in this order. In Figure 2 we can see an example of the run hierarchical structure for the line $y = ax$ with $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \dots]$, where $a_8, a_9, \dots \in \mathbf{N}^+$.

The basis for the author's CF description, from H. U-W (2008) [14], of digital lines $y = ax$ for $a \in \,]0,1[\setminus \mathbf{Q}$ according to the definition (7) constitutes the following *index jump function*.

Definition 6. Let $a = [0; a_1, a_2, a_3, \dots]$ be a positive irrational number less than 1. We define the *index jump function* $i_a\colon \mathbf{N}^+ \to \mathbf{N}^+$ for a as follows: $i_a(1) = 1$, $i_a(2) = 2$, and, for $k \geq 2$, $i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)})$, where $\delta_1(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases}$ and a_n for $n \in \mathbf{N}^+$ are the CF elements of a.

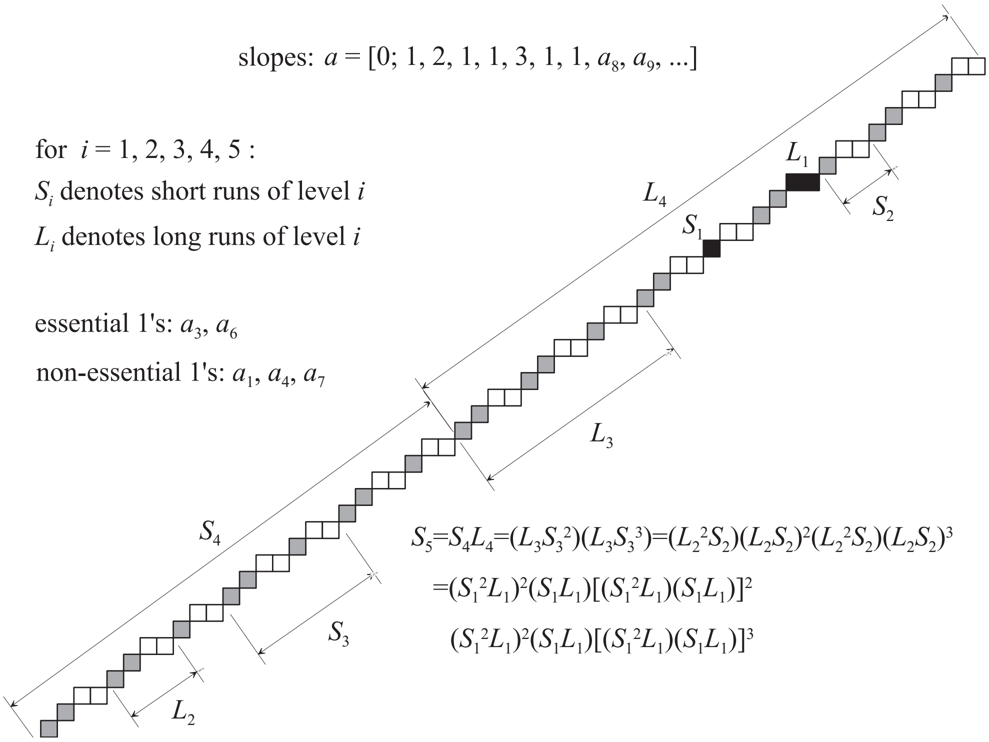

FIGURE 2. Hierarchy of long and short runs on the first four digitization levels; to translate this hierarchy for the case of upper mechanical words, put $S_1 = 1$ and $L_1 = 10$.

In H. U-W (2009) [16, Definition 6], essential 1's for $[0; a_1, a_2, \dots]$ were defined as such $a_k = 1$ that $k = i_a(m)$ for some $m \geq 2$ (compare Definition 6 with Lemma 1). The index jump function registers the essential places from the CF expansion of a. We have $(i_a(k))_{k\in\mathbf{N}^+} = \mathbf{N}^+ \setminus (s_j+1)_{j\in J}$ for all $a \in]0,1[\setminus\mathbf{Q}$; for more details see H. U-W (2009) [16].

The following proposition, which is an immediate consequence of Theorem 4 from H. U-W (2008) [14], *explains the role of essential 1's in the construction of digital lines, and, equivalently, in the run hierarchical structure of upper mechanical words.*

Proposition 2. *If a is irrational and $a = [0; a_1, a_2, \dots]$, then for the digitization of $y = ax$ (the run hierarchical structure of $u(a)$) we have the following. The CF elements $a_2, a_3, \dots$ determine the run hierarchical construction of $y = ax$ (of $u(a)$) in the following way. For each $k \in \mathbf{N}^+$*

- $a_{i_a(k+1)} \geq 2 \quad \Rightarrow \quad$ *S_k is the most frequent run on level k,*
- $a_{i_a(k+1)} = 1 \quad \Rightarrow \quad$ *L_k is the most frequent run on level k,*

where i_a is the corresponding index jump function as defined in Definition 6.

The only 1's in the CF expansion of a which influence the run-hierarchical construction of digital line $y = ax$ (upper mechanical word $u(a)$) are thus those which are indexed by the values of the index jump function, equivalently, those which are directly preceded by an even number of consecutive 1's with an index greater than 1. Briefly, only essential 1's cause the change of the most frequent run on the level they correspond to from short (S_k) to long (L_k). Which level they correspond to, is determined by the index jump function generated by a, as shown in Proposition 2. We will illustrate this proposition with the following example.

Example 4. We consider the lines as in Figure 2, thus lines $y = ax$ with slopes $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \ldots]$, where $a_8, a_9, \cdots \in \mathbf{N}^+$. Each CF element of a is responsible for some digitization level. According to Proposition 2, if $a_{i_a(k+1)} \geq 2$, then the most frequent run on level k is the short one, S_k. Otherwise, i.e., if $a_{i_a(k+1)} = 1$, the dominating run$_k$ is L_k. For the lines as in Figure 2, we have thus the following, which can be compared with the picture:

level k	$a_{i_a(k+1)}$	the most frequent run$_k$
1	$a_{i_a(2)} = a_2 = 2 \geq 2$	S_1
2	$a_{i_a(3)} = a_3 = 1$	L_2
3	$a_{i_a(4)} = a_5 = 3 \geq 2$	S_3
4	$a_{i_a(5)} = a_6 = 1$	L_4

Indeed, we have $L_2 = S_1^2 L_1,\ L_3 = L_2^2 S_2,\ L_4 = L_3 S_3^3,\ L_5 = S_4 L_4^2$.

4. Conclusion

We have presented a partition of the set $]0, 1[\setminus \mathbf{Q}$ into equivalence classes under a CF-defined equivalence relation. The relation groups together all positive irrational numbers less than 1 which have the same sequences of essential places in their CF expansions. All digital lines (upper mechanical words) with slopes belonging to the same equivalence class have the same construction in terms of long and short runs on all the levels in the hierarchy of runs. We have proven that the only class which has a greatest element is the class represented by the Golden Section. All the other classes have suprema defined by Fibonacci numbers.

The problem comes originally from digital geometry and word theory, but it can be formulated independently from these domains, as a problem concerning irrational numbers.

Because of the strong relationship between our description of digitization and the Gauss map (see the concept of digitization parameters from [13]), it would be interesting to compare our results to those of Bates et al. (2005) [2] and examine the relationship between the symmetry partners described there and our equivalence relation.

Another possible continuation of the research on our equivalence relation could be analysis of properties of the CFs with sequences of essential places determined by well-known sequences such like the Fibonacci numbers or the Pell numbers. One could try, for example, formulate the rules for transcendentality of CFs depending on the sequences of essential places. Examples of analysis of transcendentality of CFs can be found in Adamczewski et al. (2006) [1].

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