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ON REAL POLYNOMIALS WITHOUT NONNEGATIVE ROOTS

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ABSTRACT. An elementary proof of a result of D. HANDELMAN on real Laurent polynomials without nonnegative roots is presented.

Among other things D. HANDELMAN established the following result as part of his profound theory.

Theorem 1 ([2, Theorem A]). *Let $F, P \in \mathbb{R}[X, X^{-1}]$ be non-constant Laurent polynomials such that P has only positive coefficients and $F(r) > 0$ for all $r \in \mathbb{R}_{>0}$. Then there exists a positive integer m such that $P^m F$ has only positive coefficients.*

In this short note we provide an elementary proof of this statement. Moreover, we exhibit an effectively computable bound for m. Elementary proofs for a special case of Theorem 1 were given in [3] and independently in [1].

We start with a series of Lemmas. Apart from Lemma 4 they in fact treat special cases of our assertion. The essential contents of Lemmas 1 and 2 are well known for the choice $r = 1$ (see [3, 1]).

Lemma 1. *Let $b, c, r \in \mathbb{R}$, $r > 0$ and assume $b^2 < 4c$. Then there exists a nonnegative integer n such that $(X + r)^n \cdot (X^2 + bX + c)$ has only positive coefficients. One can choose*

$$n \leq \max \left\{ \lceil \frac{1-\beta}{\alpha} \rceil,\ \lceil 1 - \gamma \rceil,\ 0 \right\}$$

where

$$\alpha = 4\delta c - (2c - br)^2, \qquad \beta = 12\delta c - 2\sigma(2c - br), \qquad \gamma = 8\delta c - \sigma^2$$

and

$$\delta = r^2 - br + c, \qquad \sigma = 3c - 2br + r^2.$$

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Proof. Pick an integer $n > \max\left\{\frac{1-\beta}{\alpha}, 1-\gamma, 0\right\}$ and write

$$(X+r)^n \cdot (X^2+bX+c) = r^n c + r^{n-1}(cn+br)X + \sum_{k=2}^{n} p_k X^k + (nr+b)X^{n+1} + X^{n+2}$$

where

$$p_k = \binom{n}{k-2} r^{n-k+2} + b\binom{n}{k-1} r^{n-k+1} + c\binom{n}{k} r^{n-k} = \binom{n}{k-1} r^{n-k} f(k) \qquad (2 \le k \le n)$$

with

$$f(k) = \frac{n+1-k}{k} c + br + \frac{k-1}{n+2-k} r^2.$$

Now we have

$$k(n-k+2)f(k) = g(k)$$

with

$$g(x) = \delta x^2 - ((2c-br)n + \sigma)x + c(n^2+3n+2).$$

Clearly, $\delta > 0$ and $\alpha = r^2(4c - b^2) > 0$. In view of

$$4\delta c(n^2+3n+2) - ((2c-br)n+\sigma)^2 = n(n\alpha+\beta) + \gamma > 0$$

this implies $g(x) > 0$ for all $x \in \mathbb{R}$, and by our choice of n the claim is completely proved. □

Lemma 2. *Let $f \in \mathbb{R}[X]$ be a monic polynomial having no nonnegative roots and $r \in \mathbb{R}_{>0}$. Then there exists some*[1] *$m \in \mathbb{N}$ bounded by an effectively computable constant such that $(X+r)^m f$ has only positive coefficients.*

Proof. We observe that f can be written as a product of monic quadratic polynomials with negative discriminants and of linear polynomials with positive coefficients. Then the statement is clear by a straightforward induction on the degree of f and Lemma 1. □

Lemma 3. *Let $f \in \mathbb{R}[X]$ be a monic polynomial having no nonnegative roots and $p \in \mathbb{R}[X]$ be a nonconstant polynomial with only positive coefficients. Then there exists some $m \in \mathbb{N}$ bounded by an effectively computable constant such that $p^m f$ has only positive coefficients.*

The proof of Lemma 3 is based on the following trivial, but useful statement which was also applied in the proof of [2, Theorem V1 A].

Lemma 4. *Let R be a commutative unital ring such that 2 is a unit in R. Then we have*

$$\sum_{i=0}^{n} a_i X^i = X^{n-1}\big(a_n X + \frac{a_{n-1}}{2}\big) + \sum_{i=1}^{n-2} X^i \big(\frac{a_{i+1}}{2} X + \frac{a_i}{2}\big) + \frac{a_1}{2} X + a_0$$

for all $a_0, \ldots, a_n \in R$ $(n \ge 1)$.

[1] We denote by $\mathbb{N}$ the set of nonnegative integers.

Proof of Lemma 3. By Lemma 4 there exist linear polynomials $p_0, \dots, p_n \in \mathbb{R}_{>0}[X]$ such that

$$p = \sum_{i=0}^{n} X^i p_i.$$

Using Lemma 2 we find nonnegative integers $m_0, \dots, m_n$ bounded by an effectively computable constant such that

$$p_i^{m_i} f \in \mathbb{R}_{>0}[X] \qquad (i = 0, \dots, n). \tag{1}$$

Let $m = m_0 + \cdots + m_n$ and

$$K = \left\{(k_0, \dots, k_n) \in \mathbb{N}^{n+1} \ : \ k_0 + \cdots + k_n = m\right\}.$$

Using multinomial coefficients $c_{(k_0,\dots,k_n)} \in \mathbb{N}_{>0}$ we can write

$$p^m f = \Big(\sum_{i=0}^{n} p_i X^i\Big)^m f = \sum_{(k_0,\dots,k_n)\in K} c_{(k_0,\dots,k_n)} \Big(\prod_{i=0}^{n} \big(p_i X^i\big)^{k_i}\Big) f. \tag{2}$$

For every $\mathbf{k} = (k_0, \dots, k_n) \in K$ the polynomial

$$g_{\mathbf{k}} = \Big(\prod_{i=0}^{n} p_i^{k_i}\Big) f$$

has degree

$$\sum_{i=0}^{n} k_i + \deg(f) = m + \deg(f),$$

and by (1) we find $g_{\mathbf{k}} \in \mathbb{R}_{>0}[X]$ because for some $i \in \{0, \dots, n\}$ we must have $k_i \geq m_i$. Thus, all polynomials

$$h_{\mathbf{k}} = c_{\mathbf{k}} g_{\mathbf{k}} \in \mathbb{R}_{>0}[X] \qquad (\mathbf{k} \in K)$$

have equal degree $m + \deg(f)$, and by (2)

$$p^m f = \sum_{(k_0,\dots,k_n)\in K} h_{(k_0,\dots,k_n)} \, X^{\sum_{i=1}^{n} i k_i} \in \mathbb{R}_{>0}[X],$$

because it can easily be verified that for each integer $s \in [0, n^2]$ there are some integers $k_1, \dots, k_n \in [0, n]$ such that $s = \sum_{i=1}^{n} i k_i$. □

Proof of Theorem 1. It is easy to check that the leading coefficient c of F is positive. Following Handelman we multiply F with a suitable power of X such that $f := c^{-1} X^n F$ is monic without negative exponents and $f(0) > 0$. Similarly we find $k \in \mathbb{N}$ with

$$p := X^k P \in \mathbb{R}_{>0}[X].$$

By Lemma 3 there exists a positive m bounded by an effectively computable constant such that $p^m f$ has only positive coefficients. But then also

$$P^m F = c X^{-(mk+n)} p^m f \in \mathbb{R}_{>0}[X, X^{-1}].$$

□

Remark 1. (i) Theorem 1 can be seen as a dynamical problem in the set W of univariate Laurent polynomials with positive leading coefficients having no nonnegative roots with respect to the (continuous) transformation

$$T_P(F) = P \cdot F$$

with some fixed non-constant $P \in \mathbb{R}_{>0}[X, X^{-1}]$. It asserts that the orbit of $T_P(F)$ eventually meets the open set $\mathbb{R}_{>0}[X, X^{-1}]$. Obviously, $\mathbb{R}_{>0}[X, X^{-1}]$ is contained in the cone $W \cup \{0\}$.

(ii) The upper bound for the constant m as delivered by the proof of Lemma 3 cannot expected to be sharp. However, this does not affect the obvious algorithm for the determination of the least exponent m turning $P^m F$ into a Laurent polynomial with only positive coefficients (see (i)).

(iii) The interested reader is referred to [2, 3] for historical notes on questions related to Lemma 3.

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