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A NOTE ON VARIATIONAL AND METRIZABLE CONNECTIONS

ALENA VANŽUROVÁ

ABSTRACT. We treat the problem of metrizability of a linear connection in the context of the inverse problem of the calculus of variations. Particularly, we show that in a 2-dimensional manifold endowed with a nowhere flat connection, Helmholtz conditions can be used to answer the metrizability problem in terms of components of the connection. In $\dim M \geq 3$, the question is open.

1. SODEs in normal form, the formalism

Let M be a (smooth) n-dimensional manifold with an atlas $(U_\alpha, (x^i)_\alpha)$, let TM denote its tangent bundle with bundle projection $p = p_M : TM \to M$ and the adapted atlas $(p^{-1}(U_\alpha), (x^i, y^i)_\alpha)$. Consider the system of SODEs for functions $t \mapsto x^k(t)$ solved to second derivatives:

$$\ddot{x}^i = f^i(t, x, \dot{x}), \quad i = 1, \dots, n. \tag{1}$$

Solutions to (1) are smooth curves $c\colon \mathbb{R} \to M$, $c(t) = (x^i(t))$ defined on an open neighborhood of $0 \in \mathbb{R}$, such that

$$\ddot{c}(t) = \frac{d^2 x^i}{dt^2} = f^i\left(t, x(t), \frac{dx}{dt}\right). \tag{2}$$

If such curves exist they are in fact paths (geodesics) of the so-called semispray connection Γ on $\mathbb{R} \times TM$ (non-linear connection in general, with components $\Gamma^i = f^i$). Particularly, when the functions f^i in (1) are quadratic forms in derivatives, (1) represent geodesics of a linear connection (defined on M or on

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TM). To give the geometrical setting for modelling time-dependent SODEs (1), one possibility is to accept the following mathematical formalism. We take an extended $(n+1)$-dimensional "configuration" graph space $\mathbb{R}\times M$ regarded as the fibred manifold $(\mathbb{R}\times M, \pi_0, \mathbb{R})$ with projection $\pi_0 = \mathrm{pr}_1 : \mathbb{R}\times M \to \mathbb{R}$. On $\mathbb{R}\times M$, SODEs can be expressed locally as (1) where solutions to (1) are curves on $\mathbb{R}\times M$ with the standard graph property that they never go vertical over the fibration. Further, we consider the $(2n+1)$-dimensional "evolution" graph space $E = \mathbb{R}\times TM$, regarded again as fibred manifold over $\mathbb{R}$, with projection π_1 onto the first component, eventually, which is more precise, the jet prolongations of π_0 up the oder two. Instead of curves in M, we use curve graphs $\gamma(t) = (t, c(t))$, curves in $\mathbb{R}\times M$, i.e. sections $\gamma\colon \mathbb{R}\to\mathbb{R}\times M$ of the projection $\pi_0 = \mathbb{R}\times M\to\mathbb{R}$. Note that the first and second (first, respectively) derivatives appearing in (1) may be viewed as local fiber coordinates in the second jet prolongation $J^2(\mathbb{R}\times M)$ (in the first jet prolongation $J^1(\mathbb{R}\times M)$, respectively) of the fibred manifold π_0. Moreover, the first jet prolongation $J^1\pi_0\colon J^1(\mathbb{R}\times M)\to\mathbb{R}$ is canonically identified with $\pi_1\colon \mathbb{R}\times TM\to\mathbb{R}$. Similarly, the second jet prolongation $J^2\pi_0\colon J^2(\mathbb{R}\times M)\to\mathbb{R}$ identifies with $\pi_2\colon \mathbb{R}\times T^2M\to\mathbb{R}$. For any (local) section $\gamma\colon\mathbb{R}\to\mathbb{R}\times M$ of the fibred manifold π_0 we can take its first jet prolongation $j^1\gamma\colon\mathbb{R}\to J^1(\mathbb{R}\times M)$, $J^1\pi_0\circ j^1\gamma = \mathrm{id}$, which can be regarded (after identification) as a section of $\mathbb{R}\times TM\to\mathbb{R}$, and the second jet prolongation $j^2\gamma\colon\mathbb{R}\to J^2(\mathbb{R}\times M)$, $J^2\pi_0\circ j^2\gamma = \mathrm{id}$, regarded as a section of $\mathbb{R}\times T^2M\to\mathbb{R}$. We also make use of the canonical jet projections $\pi_{2,1}\colon J^2(\mathbb{R}\times M)\to J^1(\mathbb{R}\times M)$, $\pi_{1,0}\colon J^1(\mathbb{R}\times M)\to J^0(\mathbb{R}\times M) = \mathbb{R}\times M$; besides we have $\pi\colon J^1(\mathbb{R}\times M)\to M$, $j^1_t\gamma\mapsto\gamma(t)$.

On $\mathbb{R}\times M$, it is natural to consider local coordinates (t, x^i), $i = 1,\dots n$, adapted to the product structure, where t is the (global) coordinate on $\mathbb{R}$, and x^i are local coordinates on M (t and the family of coordinates x^i transform independently). On $\mathbb{R}\times TM$, fiber coordinates $(t, x^i, y^i = \dot{x}^i)$ adapted to the projection are induced, with coordinate transformation $\bar{y}^i = \frac{\partial\bar{y}^i}{\partial y^j}y^j$. On TTM, adapted local coordinates (x^i, y^i, dx^i, dy^i) are used, [7].

Recall that the vector bundle of 2-velocities $T^2M\subset TTM$ (i.e. of two-jets of curves into M) is distinguished as a submanifold of the second tangent bundle consisting of all $\xi\in TTM$ such that $p_{TM}(\xi) = Tp_M(\xi)$ (a "common kernel"), or alternatively, consisting of vectors on which the canonical involution[1] $s\colon TTM\to TTM$ acts as the identity map, [7]. Elements of T^2M are distinguished by $y^i = dx^i$, and the last coordinate will be written here[2] as z^i; (t, x^i, y^i, z^i) are local fiber coordinates on $\mathbb{R}\times T^2M$. For jet prolongation, locally, when $\gamma(t) = (t, c(t))$

[1]Locally, $s\colon (x^i, y^i, dx^i, dy^i)\mapsto(x^i, dx^i, y^i, dy^i)$.
[2]instead of more common $\ddot{x}^i$.

then $j^1_t\gamma = (t, c(t), \frac{dc(t)}{dt})$ and $j^2_t\gamma = (t, c(t), \frac{dc(t)}{dt}, \frac{d^2c(t)}{dt^2})$. On E, the SODEs are

$$\frac{dx^i}{dt} = y^i, \; \frac{dy^i}{dt} = f^i(t, x^k, y^k).$$

The tangent bundle TM is equipped with the Liouville's vector field $\Delta \in \mathcal{X}(TM)$ that generates the one-parameter group $\{h_t\}$ of homotheties on TM (locally, $h_t\colon (x^i, y^i) \mapsto (x^i, e^t y^i)$, hence coordinate expression of Δ is $y_i \frac{\partial}{\partial y^i})$ and with the canonical type $(1,1)$ tensor field $S \in \mathcal{T}^1_1(TM)$ that arises as a vertical lift of type $(1,1)$ identity tensor field I on M, [6], and is called *vertical endomorphism* on TM. In adapted local coordinates, $S = \frac{\partial}{\partial y^i} \otimes dx^i$; regarded as a vector-valued 1-form, it is characterized by $S\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$, $S\left(\frac{\partial}{\partial y^i}\right) = 0$. For any $w \in TM$, $\operatorname{Ker} S_w = \operatorname{Im} S_w = \operatorname{Ver}_w TM$, and $S^2_w = 0$. That is why it is also called the *canonical almost tangent structure* on TM.

A vector field $W \in \mathcal{X}(TM)$ satisfying $SW = \Delta$ is often called a *semispray*, or *second order differential equation*; W is called a *spray* if moreover, it is homogeneous of degree 1, i.e. satisfies[3] $[\Delta, W] = W$.

The so-called (homogeneous) *Grifone connection* on TM is a $(1,1)$-tensor field $\hat{\Gamma}$ on TM (or a vector-valued one-form) satisfying [8]

$$S\hat{\Gamma} = S, \; \hat{\Gamma}S = -S.$$

If $\tilde{\Gamma}^i_k(x^j, y^j)$ are components of $\tilde{\Gamma}$ in local coordinates, the equations for geodesics read $\ddot{x}^i + \tilde{\Gamma}^i_k \dot{x}^k = 0$.

2. Tensor fields and connections related to SODEs in normal form

Similar concepts as above can be considered on the (extended) evolution space $E = \mathbb{R} \times TM \approx J^1\pi_0$.

A *semispray*, or a second order differential equation field on the evolution space E is a (global) vector field $W \in \mathcal{X}(J^1\pi_0)$ on $J^1\pi_0$ such that each integral curve of the field is just the 1-jet of some section of the projection π_0. Locally, a semispray on $J^1\pi_0$ takes the form

$$W = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + \Gamma^i \frac{\partial}{\partial y^i}. \tag{3}$$

Any system of ODEs (1) defines (locally) by (3) a semispray with $\Gamma^i = f^i$.

A 1-form on E, $\omega \in \Lambda^1(E)$, is *contact* if it annihilates natural lifts of curve graphs, $\omega(j^1\gamma) = 0$ for any (local) section γ of π_0. That is, contact forms annihilate vector fields whose integral curves are naturally lifted from $\mathbb{R} \times M$. As a basis of the distribution of contact forms, we can take $\langle \omega^1, \dots, \omega^n \rangle$ where $\omega^i = dx^i - y^i dt$. In particular, $\omega^i(W) = 0$ for any semispray W (SODE) on E.

[3] $[\ ]$ denotes the Fröhlicher-Nijenhius bracket [14],

The canonical vertical endomorphism on E (arising again as a pull-back, a vertical lift, of identity endomorphism on $\mathbb{R}\times M$) is denoted by the same symbol S. In local coordinates, using contact forms,

$$S=\sum_k V_k\otimes\omega_k=\frac{\partial}{\partial y^k}\otimes(dx^k-y^k dt)=-\Delta\otimes dt+\frac{\partial}{\partial y^k}\otimes dx^k.$$

Now a semispray is equivalently characterized by $S(W)=0$ and $dt(W)=1$.

Given an evolution space E for a system of equations (1), we can construct a particular decomposition of its tangent bundle $TE\approx\mathbb{R}\times T^2M$ induced by the system. In TE, there is an intrinsic vertical subbundle $V(E)$ consisting of the vectors tangent to the fibres of the fibration $\pi_{1,0}$, $V(E)=\operatorname{Ker}\pi_{1,0}$, and a basis for $V(E)$ is formed by "vertical" vector fields $\frac{\partial}{\partial y^i}=V_i$,

$$\mathrm{V}(E)=\operatorname{span}\{\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n}\}.$$

Take the semispray $W=\frac{\partial}{\partial t}+y^i\frac{\partial}{\partial x^i}+f^i\frac{\partial}{\partial y^i}$ corresponding to (1), the distribution $D=\operatorname{span}\{W\}$, and introduce the functions

$$\Gamma^j_i=-\frac{1}{2}V_i(f^j)=-\frac{1}{2}\frac{\partial f^j}{\partial y^i}.$$

Consider the Lie derivative $\mathcal{L}_W S$. The eigenspaces corresponding to the eigenvalues $1,0,-1$ of the vector-valued 1-form $\mathcal{L}_W S$ are just $V(E)$, D, and $H(E)$ where $H(E)=\operatorname{span}\{H_1,\dots,H_n\}$, $H_i=\frac{\partial}{\partial x^i}-\Gamma^j_i V_i$. For $\mathcal{X}(E)$, we have a distinguished frame $\langle V_i,W,H_i\rangle$, with the dual co-frame $\langle\psi^i,dt,\omega^i\rangle$ where $\psi_i=dy^i-f^i dt+\Gamma^i_j\omega^j$.

We have the decomposition

$$J^2\pi_0\approx T(E)=V(E)\oplus D\oplus H(E). \tag{4}$$

Denote by $\mathcal{P}_V$, $\mathcal{P}_D$, and $\mathcal{P}_H$ projectors to the distributions from decomposition (4), $\mathcal{P}_V+\mathcal{P}_D+\mathcal{P}_H=I_E$ where I_E is the identity type $(1,1)$-tensor field on E. The so-called *Douglas tensor*, or *Jacobi endomorphism* Φ, arises as the Lie derivative of the horizontal projector P_H by W, composed with the vertical projector, $\mathcal{P}_V\circ\mathcal{L}_W\mathcal{P}_H$. To give the local expressions, denote $B^i_j=-\frac{\partial f^i}{\partial x^j}$ and

$$\Phi^i_j=W\left(\frac{1}{2}\frac{\partial f^i}{\partial y^j}\right)-\frac{\partial f^i}{\partial x^j}-\frac{1}{4}\frac{\partial f^i}{\partial y^k}\frac{\partial f^k}{\partial y^j}=B^i_j-\Gamma^i_k\Gamma^k_j-W\left(\Gamma^i_j\right). \tag{5}$$

Local expressions of the projectors and the Jacobi endomorphism are

$$\mathcal{P}_V=V_k\otimes\psi^k,\ \mathcal{P}_D=W\otimes dt,\ \mathcal{P}_H=H_k\otimes\omega^k,\ \Phi=\Phi^i_j\,V_j\otimes\omega^i. \tag{6}$$

3. Connections

A general (also non-linear, or jet, eventually Ehresmann, in the complete case, [14]) connection on a fibered bundle $p\colon Y \to X$ can be introduced as a section $\Gamma\colon Y \to J^1Y$ of the projection $\pi_{1,0} = J^1p\colon J^1Y \to Y$ ("a jet field", [23, p. 146]). It generalizes the concept of classical linear (affine) connection on a manifold. For our purpose, we use a jet connection on J^1Y, a section $\Gamma\colon J^1Y \to J^2Y$ of the projection $\pi_{2,1}\colon J^2Y \to J^1Y$, in the particular case $Y := \mathbb{R} \times M \to \mathbb{R}$.

Taking into account the identifications introduced above, under a *semispray connection* we will understand a section $\Gamma\colon \mathbb{R} \times TM \to \mathbb{R} \times T^2M$ of the projection $\pi_{2,1}\colon \mathbb{R} \times T^2M \to \mathbb{R} \times TM$, $\pi_{2,1} \circ \Gamma = \mathrm{id}$. In local coordinates, $(t, x^i, y^i, z^i) \circ \Gamma = (t, x^i, y^i, \Gamma^i)$ where the functions $\Gamma^i(t, x, y)$ on $\mathbb{R} \times TM$ are *components* of Γ (under coordinate transformations, they transform similarly as second derivatives). *Geodesics* (paths, integral sections) of the given semispray connection Γ are local sections γ of π_0 such that $\Gamma \circ j^1\gamma = j^2\gamma$ holds. In adapted coordinates, this property is expressed by a system of n second order ODEs for components of the corresponding local curves $c\colon (-\varepsilon, \varepsilon) \to M$, $c(t) = (c^i(t))$

$$\frac{d^2c^i}{dt^2} = \Gamma^i\left(t, c(t), \frac{dc}{dt}\right), \quad i = 1, \dots, n, \tag{7}$$

that is, we get just (2) with $f^i = \Gamma^i$.

Note that each Grifone connection $\hat{\Gamma}$ on TM induces a particular semispray connection on $\mathbb{R} \times TM$ if we put $\Gamma^i = -\hat{\Gamma}^i_k(x, y)y^k$.

4. The Inverse Problem

Under a first order Lagrangian on $(\mathbb{R}\times M, \pi_0, \mathbb{R})$ we understand a π_1-horizontal[4] n-form λ on $\mathbb{R} \times TM$. Locally, $\lambda = L dt$ where $L(t, x^k, y^k)$ is (an analytic, or at least C^2) function on $\mathbb{R} \times TM$, called a Lagrangian function on $\mathbb{R} \times TM$. The mappings $L \to \mathcal{E}_i(L) = \frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i}$ are sometimes referred to as *Euler-Lagrange operators* corresponding to L, and

$$\left(\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i}\right)(t, x(t), \dot{x}^i(t)) = 0 \tag{8}$$

are *Euler-Lagrange equations* (along parametrized curves), in fact necessary conditions for curves γ to be extremals of $I(\gamma) = \int_a^b L(t, x(t), \dot{x}(t))\, dt$. It appears in many branches of physics that the solution of some problems can be simplified if the basic equations can be expressed in terms of a variational principle[5]. So it is important to know which systems of fields or forces can be treated by Lagrange's method.

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[4]Recall that a vector field ξ on $\mathbb{R}\times TM$ is π_1-vertical if $T\pi_1(\xi) = 0$. A q-form η on $\mathbb{R}\times TM$ is π_1-horizontal if for any π_1-vertical ξ on $\mathbb{R} \times TM$, $i_\xi\eta = 0$ holds.

[5]Hamilton's principle in mechanics and Fermat's principle in optics are well-known examples.

The strong inverse problem of Lagrangian dynamics (of the calculus of variations) is to give necessary and sufficient conditions for the given second order differential equations $\mathcal{E}_i\left(t, x^k, \dot{x}^k, \ddot{x}^k\right) = 0$, $i = 1, \ldots, n$ ("as they stand") to be the Euler-Lagrange equations for some (regular) Lagrangian function, that is, to characterize Euler-Lagrange operators, [1] (technics of the variational bicomplex are used at present). The first investigation of this question is due to H. von Helmholtz [11] who found necessary integrability conditions for the second order ordinary differential operator $\mathcal{E}_i$. That is why the conditions are referred to as the *Helmholtz-type conditions* in the present, although their (local) sufficiency has been proved a bit later by A. Mayer [17].

On the other hand, the so-called weak inverse problem, also referred to as a variational multiplier problem or variational integrating factors problem means something a bit different. According to [10], Euler-Lagrange expressions $\mathcal{E}_i(L)$ may contain second derivatives only linearly. Therefore without loss of generality, Euler-Lagrange equations can be written (i, j, k range over $1, \ldots, n$) $g_{ij}(t, x^k, \dot{x}^k) \cdot \ddot{x}^j + h_i = 0$, the second derivatives are put in evidence. We say that Euler-Lagrange equations are normally expressed when they read[6]

$$g_{ij}\left(\ddot{x}^j - f^j\right) = 0, \tag{9}$$

eventually $g_{ij}\ddot{x}^j + h_i = 0$. Now we can formulate the variational multiplier problem as follows. We ask when the solutions of a system $\ddot{x}^j - f^i = 0$ are the solutions of $\mathcal{E}_i(L) = 0$ for some L (when we can rearrange the given equations so that they become Euler-Lagrange equations), more precisely, we should decide when there exist a non-degenerate functional matrix $(g_{ij}(t, x^k, y^k))$ and a (regular) Lagrangian function $L(t, x^k, y^k)$ on $\mathbb{R} \times TM$ such that

$$\sum_j g_{ij}\mathcal{E}_j = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \mathcal{E}_i(L). \tag{10}$$

If the answer is affirmative we should find all such pairs $((g_{ij}), L)$. In terms of the functions f^i, the corresponding Helmholtz conditions are given explicitely as follows ([2]). There should exist functions g_{ij} such that

$$\begin{gathered}
\det(g_{ij}) \neq 0,\ g_{ij} = g_{ji},\ \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \\
\frac{d}{dt}(g_{ij}) + \frac{1}{2}\frac{\partial f^k}{\partial y^j} g_{ik} + \frac{1}{2}\frac{\partial f^k}{\partial y^i} g_{jk} = 0, \\
g_{ik}\left[\frac{d}{dt}\frac{\partial f^k}{\partial y^j} - 2\frac{\partial f^k}{\partial x^j} - \frac{1}{2}\frac{\partial f^s}{\partial y^j}\frac{\partial f^k}{\partial y^s}\right] = g_{jk}\left[\frac{d}{dt}\frac{\partial f^k}{\partial y^i} - 2\frac{\partial f^k}{\partial x^i} - \frac{1}{2}\frac{\partial f^s}{\partial y^i}\frac{\partial f^k}{\partial y^s}\right].
\end{gathered} \tag{H}$$

[6]Here $g_{ij}(t, x, y)$, $h_i(t, x, y) = -g_{ij}f^j$, $f^j(t, x, y)$ can be viewed as functions on $\mathbb{R} \times TM$.

5. Helmholtz-type conditions

The solution of the problem is still open, the answer (formulated in terms of the given functions) is known only in particular cases (e.g. equations for geodesics of a pseudo-Riemannian space, or Finsler space, respectively, are always variational in weak sense), despite of the fact that many authors interested in the problem gave various necessary and sufficient Helmholtz-type conditions. (There are many different ways of deriving the Helmholtz conditions, and their concrete form for a given system might varry from author to author.) If the system of SODEs takes the general form

$$\mathcal{E}_i\left(t, x^k, \dot{x}^k, \ddot{x}^k\right) = 0, \quad i = 1, \dots, n \tag{11}$$

we have the following necessary and sufficient conditions [10]:

Theorem 5.1. *A set of functions $\mathcal{E}_i\colon \mathbb{R}\times T^2M \to \mathbb{R}$, $i = 1, \dots, n$ take the form of Euler-Lagrange operators of some first order Lagrangian $\lambda = Ldt$ on $\mathbb{R}\times TM$ if and only if the following conditions are satisfied identically*[7] *in any adapted chart:*

$$\frac{\partial \mathcal{E}_i}{\partial z^k} - \frac{\partial \mathcal{E}_k}{\partial z^i} = 0, \tag{12}$$

$$\frac{\partial \mathcal{E}_i}{\partial y^k} + \frac{\partial \mathcal{E}_k}{\partial y^i} = \frac{d}{dt}\left(\frac{\partial \mathcal{E}_i}{\partial z^k} + \frac{\partial \mathcal{E}_k}{\partial z^i}\right), \tag{13}$$

$$\frac{\partial \mathcal{E}_i}{\partial x^k} - \frac{\partial \mathcal{E}_k}{\partial x^i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \mathcal{E}_i}{\partial y^k} - \frac{\partial \mathcal{E}_k}{\partial y^i}\right). \tag{14}$$

The Helmholtz conditions for the system (1), (9), consisting of a mixture of algebraic and differential conditions, appear in some form in [4] and can be written (following Sarlet's matrix notation, [22]) as follows:

$$\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j},\ g = g^T,\ W(g) = g\Gamma + \Gamma^T g,\ g\Phi = \Phi^T g. \tag{15}$$

We suppose $\det(g_{ij}) \neq 0$. In components, the last three conditions from (15) read

$$g_{ij} = g_{ji},\ W(g_{ij}) = -\frac{1}{2}\left(g_{ik}\frac{\partial f^k}{\partial y^j} + \frac{\partial f^k}{\partial y^i} g_{kj}\right),\ g_{ik}\Phi^k_j = \Phi^k_i g_{kj}.$$

To express existence conditions for a multiplier in terms of the given functions f^i only (e.g. to eliminate the multiplier g from (15)) is the hard problem. In dimension $n = 2$, a complete answer for the inverse problem in the real analytic class was given by Jesse Douglas in [4], and reinterpreted in [3] geometrically. Very briefly, the Douglas's cases I–IV are distinguished according to how many of the tensors from $\{I, \Phi, \nabla\Phi, \nabla^2\Phi\}$ are linearly independent of the previous onces (here ∇ denotes the so-called dynamical covariant derivative).

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[7] Here(x^i, y^i, z^i) denote local coordinates on T^2M, see section 2.

6. The metrizability of linear connections

The Fundamental Lemma of (pseudo-) Riemannian geometry states that given a (non-degenerate) metric g on M there is a unique connection ∇ on M that is symmetric and "compatible" with the metric in the sense that the metric is covariantly constant with respect to the connection, $\nabla g = 0$ (geometrically, the scalar product is preserved under parallel transport). Such a connection is called Riemannian, or Levi-Civita.

The metrizability problem is the "reverse" question: given a connection ∇ on M, find necessary and sufficient conditions for ∇ (formulated in terms of the given connection) to be just the Levi-Civita connection of some metric, eventually find all such metrics. A system of integrability conditions was given in [5], cf. [26], many particular answers are known, an equivalent formulation in terms of geodesic mappings can be given [19], and a system of differential equations that controls (locally) this question was found in [18]. But even this problem is far from being completely solved.

Note that a manifold (M, ∇) endowed with a linear connection with components Γ^k_{ij} (particularly, any Riemannian manifold (M, g) with the canonical Levi-Civita connection) determines a special semispray connection on the fibered manifold π_1 with components as quadratic forms in the first derivatives, $\Gamma^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$. The graphs of geodesics of (M, ∇) coincide just with geodesics of Γ.

Remark that if the curvature tensor is distinct from 0 in one point then from continuity, it is does not vanish on an open neighborhood. The union of non-flat regions is open. On flat regions, the connection is always metrizable.

If ∇ is a symmetric[8] linear connection on an n-manifold M then equations for geodesics (parametrized by arc length) $\nabla_{\dot{c}}\dot{c} = 0$ take in local coordinates the well-known form

$$\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0, \tag{16}$$

the most familiar particular subcase of (1) with $f^i(x, y) = -\Gamma^i_{jk}(x)y^j y^k$. The question, when does there exist a multiplier $g(x)$ (depending on positions only, with $\det(g_{ij}(x)) \neq 0$ for any $x \in M$) that makes the system (16) variational, comes quite naturally, [15], [16]. If this is the case then the system (16) is equivalent to

$$g_{ij}(\ddot{x}^j + \Gamma^j_{rs}(x)\dot{x}^r\dot{x}^s) = 0. \tag{17}$$

[8]Without loss of generality, we can suppose that the connection is torsion-free, $\Gamma^i_{jk}(x) = \Gamma^i_{kj}(x)$, since the antisymmetric part of the connection has no influence on the form of the autoparallel equations for geodesics.

Using the notation and constructs introduced above we find in our case

$$\Gamma = (\Gamma^i_j(x,y)) = (\Gamma^i_{js}(x)y^s),\ W(\Gamma^i_j) = \left((\frac{\partial \Gamma^i_{js}}{\partial x^r} - \Gamma^i_{jk}\Gamma^k_{rs})y^r y^s \right),$$

$$(B^i_j) = \left(\frac{\partial \Gamma^i_{rs}}{\partial x^r} y^r y^s \right),\ \left(\Gamma^i_k \Gamma^k_j\right) = \left(\Gamma^i_{kr}\Gamma^k_{js}\right) y^r y^s.$$

The components of the Jacobi endomorphism Φ are

$$\Phi^i_j(x,y) = R^i_{hjk}(x)y^h y^k = -\sum_{h<k} R^i_{(hk)j} y^h y^k \tag{18}$$

where

$$R^i_{hjk} = \frac{\partial \Gamma^i_{kh}}{\partial x^j} - \frac{\partial \Gamma^i_{jh}}{\partial x^k} + \sum_s \left(\Gamma^i_{js}\Gamma^s_{kh} - \Gamma^i_{ks}\Gamma^s_{jh}\right)$$

are components of the Riemann curvature tensor[9] R of ∇. So suppose there exists a solution $g(x)$ to the Helmholtz conditions (15). The first condition holds trivially, and we are assuming symmetry of g; the third condition reads $\frac{\partial g_{ij}}{\partial x^h} y^h = (g_{is}\Gamma^s_{jh} + g_{js}\Gamma^s_{ih})y^h$ which is equivalent to[10] $\nabla g = 0$. Hence we checked:

Proposition 6.1. *If there is a (local) solution $g(x)$ to the Helmholtz conditions (15) for (1) then g is a (local) metric compatible with ∇, and ∇ is just the Levi-Civita connection for the pseudo-Riemannian manifold (M,g).*

Moreover, if we introduce the curvature tensor in type $(0,4)$ by $\tilde{R}(X,Y,Z,W) = g(W, R(X,Y)Z)$, with components $R_{ihjk} = g_{is}R^s_{hjk}$, the last condition from (15) is just $R_{ihkj}y^h y^k = R_{jhik}y^h y^k$, or equivalently $R_{i(hk)j} = R_{j(hk)i}$.

Remark 6.1. A similar problem can be formulated for Finsler case, [15] etc. From (15), a complete hierarchy of algebraic equations on the variational multiplier (and hence on the Riemannian metric) cen be obtained, involving the covariant derivatives of the Jacobi endomorphism and the curvature tensor, [20].

7. Variational connections

We say that a linear connection $\nabla(\Gamma^i_{jk})$ on M is *variational in the restrictive sense*, in short *restrictively variational* if there exists a (Lagrangian) function[11] $L\colon \mathbb{R} \times TM \to \mathbb{R}$ and a non-singular type $(0,2)$ tensor field $g\colon M \to T^0_2(M)$ on M such that with respect to any local fibre chart, the functions

$$-\mathcal{E}_i(x,y,z) = g_{ik}(x)(z^k + \Gamma^k_{rs}(x)y^r y^s), \qquad i = 1, \dots, n \tag{19}$$

[9] $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for $X,Y,Z \in \mathcal{X}(M)$, $R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^h} = R^i_{hjk}\frac{\partial}{\partial x^i}$.

[10] It means that components of g are related to components of the connection by the well-known formula $\Gamma^\ell_{ik} = \frac{1}{2}g^{\ell j}\left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}\right)$.

[11] sufficiently differentiable, at least C^2

coincide with Euler-Lagrange expressions $\mathcal{E}_i(L)$ of L. Note that g is not a priori supposed to be symmetric (g could be a "generalized metric" in the sense of Eisenhart), but we find that symmetry follows in this case. The relationship between restrictive variationality and metrizability for a linear connection on M can be expressed as follows (cf. [15]):

Theorem 7.1. *Given a manifold (M, ∇) with linear connection, the following conditions are equivalent:*

(i) ∇ is variational in the restrictive sense;

(ii) the symmetric part $\tilde{\nabla}$ of ∇ is metrizable;

(iii) there is a non-singular symmetric type $(0,2)$ tensor field g on M such that

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right);$$

$\tilde{\Gamma}^i_{jk}$ are components of the (symmetric) connection $\tilde{\nabla}$, and $g^{i\ell}$ are components of the tensor g^* dual to g in a natural pairing;

(iv) there is a non-singular symmetric type $(0,2)$ tensor field g on M such that the equations $g_{ik}(x)(z^k + \Gamma^k_{rs}(x) y^r y^s) = 0$ are variational.

Proof. By definition, (i) $\Leftrightarrow$ (iv); (ii) $\Leftrightarrow$ (iii) is classical: if a symmetric $\tilde{\nabla}$ is metrizable with a compatible metric g then $\tilde{\nabla}$ is just the Levi-Civita connection of the (M, g); $\tilde{\nabla} g = 0$ is equivalent to the formula in (iii) for any symmetric connection. (i) $\Leftrightarrow$ (ii): let ∇ be a linear connection on M, $\tilde{\Gamma}^i_{jk} = \tilde{\Gamma}^i_{kj}$ being components of its metrizable symmetric part $\tilde{\nabla}$, with compatible metric g. Let us introduce (global) function $L(x,y) = \frac{1}{2} g_x(y,y)$, $y \in T_x M$ by local coordinate expressions $L = \frac{1}{2} g_{rs}(x) y^r y^s$. We evaluate the corresponding expressions $\mathcal{E}_i(L)$ and verify that they obey the Helmholtz conditions; hence L is a Lagrangian function:

$$\mathcal{E}_i(L) = -\left(g_{is} z^s + \tilde{\Gamma}_{irs} y^r y^s \right) = -\left(g_{is} z^s + g_{i\ell} \tilde{\Gamma}^\ell_{rs} y^r y^s \right) \tag{20}$$

with $2\tilde{\Gamma}_{irs} = 2 g_{i\ell} \tilde{\Gamma}^\ell_{rs} = \partial_s g_{ir} + \partial_r g_{is} - \partial_i g_{rs}$; due to symmetry of g, $\tilde{\nabla} g = 0$, and (20),

$$\frac{\partial \mathcal{E}_i}{\partial z^k} - \frac{\partial \mathcal{E}_k}{\partial z^i} = -g_{is} \delta^s_k + g_{ks} \delta^s_i = -g_{ik} + g_{ki} = 0; \tag{21}$$

$$\begin{aligned} &\frac{\partial \mathcal{E}_i}{\partial u^k} + \frac{\partial \mathcal{E}_k}{\partial y^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{E}_i}{\partial z^k} + \frac{\partial \mathcal{E}_k}{\partial z^i} \right) \\ &\quad = 2 \left(\tilde{\Gamma}_{iks} + \tilde{\Gamma}_{kis} + \frac{\partial g_{ki}}{\partial x^s} \right) y^s = 2 (\tilde{\nabla} g) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}; y^s \frac{\partial}{\partial x^j} \right) = 0; \end{aligned} \tag{22}$$

$$\frac{\partial \mathcal{E}_i}{\partial x^k} = \frac{\partial g_{is}}{\partial x^k} z^s + \left(\frac{\partial^2 g_{is}}{\partial x^k \partial x^r} - \frac{1}{2} \frac{\partial^2 g_{rs}}{\partial x^k \partial x^r} \right) y^r y^s, \tag{23}$$

and finally (14). Hence (12)–(14) hold. Vice versa, if the connection is variational in the restrictive sense, then the Helmholtz conditions are satisfied, and the symmetry of g follows according to (12); (13) together with (12) give $\tilde{\nabla} g = 0$; if g is symmetric and covariantly constant with respect to $\tilde{\nabla} g$ then (14) gives no new condition. Hence (i) $\Rightarrow$ (ii), which completes the proof. $\square$

Given a system of SODE's of a particular form (16) we can sometimes use Theorem 7.1 for deciding whether the system is (locally) derivable from a Lagrangian: $\Gamma^k_{rs}(x)$ are viewed as components of a symmetric linear connection ∇ in some neighborhood $U \subset \mathbb{R}^n$; if ∇ is (locally) metrizable, $g_{ij}(x)$ (with $\det(g_{ij}(x)) \neq 0$ at any $x \in U$) being components of some non-degenerate metric g compatible with ∇ on U then the system (16) is equivalent to the system (17), hence the functions $g_{ik}(x)$ are the desired variational multipliers.

On the other hand, given a system of SODE's (17), if there are Lagrange multipliers independent of time and velocities then they are just components of a metric with geodesics given exactly by (16).

8. Metrizability in dimension two

For (M_2, ∇), a kind of answer to metrizability question can be obtained from the Helmholtz conditions for nowhere flat regions.

In dimension two, the curvature tensor of a (symmetric) linear connection is completely determined by its Ricci tensor; here we take[12]

$$\operatorname{Ric}(X, Z) = \operatorname{trace}(Y \mapsto R(X, Y)Z), \quad R_{hj} = \operatorname{Ric}\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^h}\right) = \sum\nolimits_s R^s_{hjs}.$$

Since $R^i_{hjj} = 0$ we get $R^i_{hjk} = \delta^i_k R_{hj} - \delta^i_j R_{hk}$. Particularly, $R = 0$ if and only if $\operatorname{Ric} = 0$.

On a Riemannian 2-manifold (M_2, g), the sectional curvature is reduced to Gaussian curvature $K = R_{1212}/\det(g_{ij})$, and $R^i_{hjk} = K(\delta^i_j g_{hk} - \delta^i_k g_{hj})$ holds [21]. We find immediately $\operatorname{Ric} = -K \cdot g$ (in fact, $R_{hj} = \Sigma_s R^s_{hjs} = K(\delta^s_j g_{hs} - \delta^s_s g_{hj}) = -K g_{hj}$). Hence the Ricci tensor is proportional to the metric tensor, and (M_2, g) is always an Einstein space [21]. Introducing $R^i_j = g^{is} R_{sj}$ (where g^{ij} are components of the dual tensor to g in natural pairing) and the scalar curvature $\varrho = \mathit{trace}\operatorname{Ric} = R^s_s = g^{rs} R_{rs}$ we can write[13] $\operatorname{Ric} = -\frac{1}{2}\varrho \cdot g$; $K = \varrho = 0$ iff $\operatorname{Ric} = 0$. Ric is recurrent if and only if R is recurrent. For a nowhere flat (M_2, g), the Ricci tensor is symmetric, recurrent and non-degenerate, $\det|R_{ij}| \neq 0$. Hence by "Riemannian" arguments (cf. [24]), for (local) (pseudo-)Riemannian metrizability of a nowhere flat symmetric connection on M_2, symmetry, recurrency and non-degeneracy of Ric are necessary conditions, and they are also sufficient.

[12] The alternative possibility, used frequently, differs by sign.

[13] Remark that neither ϱ nor K are available for an arbitrary linear connection.

An analogous answer (for $n = 2$) comes also from the Helmholtz conditions as follows.

Assume there is a solution $g(x)$ to (15) for the equations (16) in dimension 2. Then by Prop. 6.1, g is a metric compatible with ∇. On non-flat parts of M_2, $g_{ij} = -\frac{2R_{ij}}{\varrho}$, so the second Helmholtz condition means symmetry of Ric, the third one can be equivalently written as

$$-\frac{\partial}{\partial x^h}(\ln|\varrho(x)|) \cdot R_{ij} = \nabla_h R_{ij}$$

which means $\nabla \mathrm{Ric} = \omega \otimes \mathrm{Ric}$, recurrency, where $\omega = \omega_h dx^h$ is the 1-form with local components $\omega_h = \frac{\partial}{\partial x^h}(-\ln|\varrho(x)|)$. That is, locally, $\omega = df$ for the function $f = -\ln|\varrho|$. Let us analyze the fourth condition: if we assume symmetry of g we get a single condition

$$g_{11}\Phi^1_2 + g_{12}(\Phi^2_2 - \Phi^1_1) - g_{22}\Phi^2_1 = 0 \tag{24}$$

where

$$\Phi^1_1 = -R_{12}y^1y^2 - R_{22}(y^2)^2,\ \Phi^1_2 = R_{12}(y^1)^2 + R_{22}y^1y^2,$$
$$\Phi^2_2 = -R_{11}(y^1)^2 - R_{21}y^1y^2,\ \Phi^2_1 = R_{11}y^1y^2 + R_{21}(y^2)^2.$$

Since (24) should hold for all y^1, y^2 we find an equivalent homogeneous system of three linear algebraic equations in three unknowns g_{11}, g_{12}, g_{22},

$$\begin{aligned} R_{12}g_{11} - R_{11}g_{12} &= 0,\\ R_{22}g_{11} + (R_{12} - R_{21})g_{12} - R_{11}g_{22} &= 0,\\ R_{22}g_{12} - R_{21}g_{22} &= 0. \end{aligned} \tag{25}$$

The system (25) was analyzed in [24], [28] (there exists a non-trivial solution iff the determinant of its matrix vanishes, $(R_{12} - R_{21}) \cdot \det(R_{ij}) = 0$). If Ric is already assumed to be symmetric the system is simplified, the solution is one-dimensional if $R(x) \neq 0$, $g_{11} : g_{12} : g_{22} = R_{11} : R_{12} : R_{22}$, and three-parameter whenever $R = 0$ holds.

Theorem 8.1. *For a now-where flat symmetric linear connection ∇ on a two-manifold, the following conditions are equivalent:*

(i) There exists a solution (local solution, respectively) $g(x)$ to the Helmholtz condition (H).

(ii) ∇ is (locally) metrizable.

(iii) The Ricci tensor is non-degenerate, symmetric, and there exists a closed (exact, respectively) 1-form ω such that $\nabla \mathrm{Ric} = \omega \otimes \mathrm{Ric}$ holds.

(iv) ∇ is (locally) variational in the restrictive sense.

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Department of Mathematics,
Faculty of Civil Engeneering,
Brno University of Technology,
Veveri 331/95, 602 00 Brno,
Czech Republic
and
Palacký University,
Olomouc, Czech Republic
E-mail address: vanzurova.a@fce.vutbr.cz
E-mail address: alena.vanzurova@upol.cz